G.I. Atabekov

# LINEAR NETWORK THEORY

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# FOREWORD

THE theory of linear electrical circuits is developing in two main fields: network analysis, i.e. the calculation of electrical values for given circuits and parameters; and network synthesis, i.e. determining electrical circuits with prescribed characteristics.

These two branches of modern circuit theory have a common physical and mathematical basis. The physical basis consists of the experimental laws of circuit theory, and the mathematical basis consists of operations in a complex plane. Based on common principles, they are complementary in the sense of ever extending our information about the most rational methods of calculating and designing diverse electrical circuits. The analysis and synthesis are therefore developing and becoming known, not separately, but interdependently.

The theory of linear electrical circuits is of exceptional significance for developing the scientific outlook of specialists in electrical and radio engineering. All the specialized disciplines of these two fields are based on it, and that is why many people are today extremely interested in the detailed study of the analysis and synthesis of electrical circuits.

Every year the organization of mathematical education in higher technical institutions pays greater recognition to the necessity of intensifying the mathematical preparation of students in the relevant branches of mathematical analysis (special functions, matrices, Fourier and Laplace transformations, analytic functions, etc.). The present situation is important in that it makes it possible to set forth and study special theoretical disciplines, including the theory of linear electrical circuits, at the proper scientific and theoretical level.

This theory covers an extremely wide field of knowledge. Its scientific content is expounded, in varying degrees of completeness, in present-day text-books on the theoretical principles of electrical and radio engineering and of line communication. More detailed treatment of questions relating to the theory of electrical circuits (electric filters, wave-shaping circuits, etc.) is given in specialized courses and monographs.

This book is devoted to the problems of linear network analysis and synthesis. It is intended to serve as an aid to students in specialized branches of radio and electrical engineering, post-graduates, and engineers extending their theoretical knowledge.

The author has endeavoured to devote maximum attention to combining mathematical conclusions with physical representations, the material being arranged as far as possible in order of increasing complexity.

As an aid to readers who are not very familiar with analytic functions, brief information concerning the theory of functions of a complex variable is given at the end of the book.

In writing this book, side by side with many Russian and foreign publications, the author has also made use of his books *Linear Electrical Circuits* (Oborongiz 1957) and *Harmonic Analysis and the Operator Method* (Oborongiz 1956), which were based on lectures given by the author to students and post-graduates of the radio and electrical engineering faculties of the Moscow Aviation Institute.

The author is grateful to the staff of the Theoretical Electrical Engineering Department of the Moscow Aviation Institute, and to Professors Ya. Z. Tsypkin and D. F. Kharazov, for the valuable comments which they made on reading through the manuscript.

G. I. ATABEKOV

# NOTATION

| | |
|---|---|
| $A$ | Parameter of a four-terminal network (general circuit constants) |
| $a$ | Image phase function |
| $b$ | Image attenuation function; susceptance* |
| $b_C$ | Capacitive susceptance |
| $b_L$ | Absolute value of inductive susceptance |
| $b_{\text{ins}}$ | Insertion power loss |
| $b_{\text{ef}}$ | Effective attenuation |
| $C$ | Capacitance |
| $\dot{E}$ | Complex e.m.f. (effective value of e.m.f. in complex form) |
| $e$ | Natural logarithmic base; instantaneous e.m.f. |
| $F(j\omega)$ | Spectrum (Fourier transform) |
| $f$ | Frequency of a periodic function |
| $f_c$ | Cut-off frequency |
| $g$ | Image propagation function; conductance (real part of complex admittance) |
| $H$ | Hybrid parameter of a four-terminal network |
| $I$ | Effective (r.m.s.) value of current; direct (constant) current |
| $I_m$ | Amplitude of alternating current |
| $\dot{I}$ | Current sinor (effective value of alternating current in complex form) |
| $\dot{I}_m$ | Complex current amplitude |
| $I_a$ | Resistive current |
| $I_r$ | Magnitude of reactive (inductive or capacitive) current |
| $I_s$ | Short-circuit current |
| Im | Imaginary part |
| $i$ | Instantaneous current |
| $j =$ | $\sqrt{(-1)}$ |
| $K_U$ | Voltage-ratio transfer function (voltage gain or loss) |
| $K_I$ | Current-ratio transfer function (current gain or loss) |
| $k$ | Coefficient of coupling; filter constant |
| $L$ | Self-inductance |
| $l$ | Length |
| $M$ | Mutual inductance |
| $m$ | Filter constant |

* Susceptance ($b$) used in Russian literature is the negative of that used in English (they differ in sign).

| | |
|---|---|
| $n$ | Transformer turns ratio; reflexion factor |
| $P$ | Active (average) power |
| $p$ | Instantaneous power; complex frequency |
| $Q$ | Reactive power; quality factor of circuit |
| $Q_C$ | Quality factor of capacitor |
| $Q_L$ | Quality factor of induction coil |
| $q$ | Electrical charge |
| Re | Real part |
| $r$ | Resistance (real part of complex impedance) |
| $S$ | Apparent power (magnitude of the complex power); standing-wave ratio |
| $T$ | Period of a periodic function |
| $t$ | Time |
| $U$ | Effective (r.m.s.) value of voltage or e.m.f.; constant voltage or e.m.f. |
| $U_m$ | Amplitude of alternating voltage or e.m.f. |
| $\dot{U}$ | Voltage sinor (effective value of alternating voltage or e.m.f. in complex form) |
| $\dot{U}_m$ | Complex voltage (or e.m.f.) amplitude |
| $U_a$ | Resistive voltage drop |
| $U_r$ | Magnitude of reactive (inductive or capacitive) voltage drop |
| $U_o$ | Open-circuit voltage |
| $u$ | Instantaneous voltage or e.m.f. |
| $v$ | Velocity |
| $W$ | Energy |
| $W_C$ | Energy stored in the electric field (instantaneous net energy storage in capacitance) |
| $W_L$ | Energy stored in the magnetic field (instantaneous net energy storage in self-inductance) |
| $w$ | Number of turns |
| $x$ | Reactance; coordinate of length |
| $x_C$ | Absolute value of capacitive reactance |
| $x_L$ | Inductive reactance |
| $x_M$ | Reactance of mutual inductance |
| $Y$ | Complex or phasor admittance; short-circuit admittance parameter of a four-terminal network |
| $Y_{ii}$ | Self-admittance at node $i$ |
| $Y_{ik}$ | Mutual admittance between nodes $i$ and $k$ (admittance common to nodes $i$ and $k$); transfer admittance |
| $Y(p)$ | Admittance function (generalized admittance) |
| $y$ | Magnitude of the admittance (scalar admittance) |
| $Z$ | Complex or phasor impedance; open-circuit impedance parameter of a four-terminal network |

| | |
|---|---|
| $Z_c$ | Characteristic (image) impedance |
| $Z_{ii}$ | Self-impedance of loop $i$ |
| $Z_{ik}$ | Mutual impedance between loops $i$ and $k$ (impedance common to loops $i$ and $k$); transfer impedance |
| $Z(p)$ | Impedance function (generalized impedance) |
| $Z_o$ | Open-circuit impedance |
| $Z_s$ | Short-circuit impedance |
| $Z_M$ | Complex impedance of mutual inductance; characteristic (image) impedance of a lattice network |
| $Z_\Pi$ | Characteristic (image) impedance of a pi-network |
| $Z_T$ | Characteristic (image) impedance of a $T$-network |
| $z$ | Magnitude of the impedance (scalar impedance) |
| $\alpha$ | Phase function of line |
| $\beta$ | Attenuation function of line |
| $\gamma$ | Propagation function |
| $\delta$ | Fractional deviation from resonant frequency (fractional detuning) |
| $\delta(t)$ | Impulse function (delta function) |
| $\Delta$ | Determinant |
| $\Delta_{ik}$ | Cofactor of $i$-th row and $k$-th column of a determinant |
| $\varepsilon$ | Permittivity (dielectric constant) |
| $\eta$ | Efficiency |
| $\lambda$ | Wavelength |
| $\mu$ | Permeability |
| $\varrho$ | Characteristic impedance of line (pure resistive) |
| $\tau$ | Time; time constant |
| $\Phi$ | Magnetic flux |
| $\varphi$ | Phase angle (phase difference); impedance angle |
| $\Psi$ | Flux linkages |
| $\psi$ | Initial phase angle |
| $\omega$ | Angular (radian) frequency |

CHAPTER I

# BASIC DEFINITIONS AND CONCEPTS

## 1. Circuit Theory and its Relation to Field Theory

In electrical and radio engineering there are two different approaches to the study of electrical and magnetic phenomena: by field theory and circuit theory.

*Field theory* examines those phenomena which occur in three-dimensional space, as for example the radiation of electromagnetic energy, radio propagation, distribution of space charges and current densities, etc.

The elements dealt with in field theory are elementary volumes of a medium—a non-conductor, conductor or semiconductor. The medium is characterized by permittivity and permeability, conductivity, a parameter determining dielectric loss, etc.

Vectors of the electrical field intensity and magnetic flux density, current density, etc. are used as electrical and magnetic quantities in field theory. These field quantities are related by Maxwell equations expressing the laws of electromagnetic phenomena in various media. The conditions for formulating actual problems in field theory are set in the form of boundary conditions on the surfaces of separation of these media.

*Circuit theory* is used in cases where the phenomena under examination are essentially confined to movement along a defined path.

Instead of elementary volumes of the medium, inductance, capacitance and resistance are recognized as the elements used in this case. These circuit elements take into account the magnetic and electrical fields around the conductors, and the conversion of electric energy into heat associated with the flow of the current through the conductor.

Circuit theory operates with the following electrical quantities: voltage, current and charge. These quantities, like circuit elements, are integral quantities with respect to those which are dealt with in field theory and which have a differential character.

The principal laws of circuit theory are the laws of the balance of currents at a junction (Kirchhoff's current law) and the balance of voltages around any closed path of a circuit (Kirchhoff's voltage law).

The conditions for formulating actual problems in circuit theory are determined by the mutual connexion of the resistances, inductances and capacitances, and are represented graphically with the aid of so-called cir-

cuit diagrams or equivalent circuits. These diagrams show the manner in which the circuit elements are interconnected.

Circuit theory makes it possible to determine by direct means the values of voltage and current along a line, depending on distance and time, without intermediate calculation and subsequent integration of the specific electrical and magnetic quantities used in field theory. For this reason calculations are simplified considerably.

Circuits with distributed and lumped parameters are examined in circuit theory.

Voltages and currents in circuits with *distributed parameters* are studied as functions of the spatial coordinate (length) and of time.

In circuits with *lumped parameters* the electrical and magnetic fields, and electric energy dissipation in the form of heat may be considered concentrated in separate elements of the circuit, with sufficient accuracy for practical engineering purposes.

Changes in voltages and currents in these circuits are examined only as time functions, independently, of the spatial coordinate.

It should be noted that the division of electrical circuits into circuits with lumped and distributed parameters is a conditional one. The same electrical or radio-engineering device is considered a lumped-element circuit at very low frequencies and a circuit with distributed parameters at very high frequencies.

The demarcation of the fields of application of circuit theory and field theory is also conditional. One or other conception of examining electrical and magnetic phenomena is valid depending on the actual problems. For example, line processes may be examined both by circuit theory methods and by field theory methods.

References (43) and (51) are devoted to extending circuit theory to electromagnetic wave systems and applying generalized theory to high-frequency and micro-wave oscillatory waveguide and radiating systems in radio engineering.

## 2. Elements of an Electrical Circuit

Research and calculation on electrical circuits are based on various assumptions and on a certain idealization of circuit elements. By "elements" in electrical circuit theory we usually understand not the physical components of any electrical device but the *idealized* elements of the circuit which, theoretically, possess particular electrical or magnetic properties and which together approximately reflect the phenomena occurring in the actual devices.

We distinguish between active and passive elements.

*Active elements* are sources of electrical energy. The process of converting chemical, mechanical or other forms of energy into electrical energy

takes place in them. Section 3 of this chapter is devoted to sources of electrical energy used in circuit theory.

The *passive elements* of electrical circuits are resistances, inductances and capacitances.

*Resistance r* is a circuit element in which an irreversible conversion of electric energy into heat energy takes place, when a current flows through it.

*Inductance L* denotes a circuit element capable of storing the energy of a magnetic field.

Finally, *capacitance C* is a circuit element capable of storing the energy of an electrical field.

Thus, each of these elements has its own particular electromagnetic properties.

It should be noted that each of these terms—resistance, inductance, capacitance and their corresponding conventional notations $r$, $L$, $C$—are used in electrical circuit theory, firstly to denote the element itself and secondly to denote its parameter, i.e. the electromagnetic value which describes the given element quantitively (henceforth it will be supposed that these parameters are not dependent on voltage or current—*vide* Section 5).

Thus the term *resistance* and its corresponding conventional symbol $r$ denote an idealized circuit element in which electrical energy is dissipated as heat and, at the same time, express the parameter of the given element equal to the ratio of voltage $u_r$ across its terminals to the current passing through it,

$$r = \frac{u_r}{i}. \tag{1.1}$$

It is assumed that the sign of the current is the same as the sign of the voltage across the resistance. This voltage is defined as a voltage drop (decrease in potential) in the positive current direction* (*vide* Section 6).

In the MKS system $u_r$ is measured in volts (V), $i$ in amperes (A) and $r$ in ohms ($\Omega$).

The term *inductance* and its corresponding conventional symbol $L$ likewise denote an idealized circuit element in which the energy of a magnetic field may be stored and also express a quantity equal to the ratio of the flux linkages† of the self-inductance to the current passing this element.

$$L = \frac{\Psi}{i}. \tag{1.2}$$

* Positive or negative current has meaning only by comparison with a specified direction called reference or positive direction. Actual (physical) direction of the current is taken as the direction of flow of positively charged particles or opposite to the direction of negatively charged particles, whichever is appropriate. When flowing in the reference direction the current is taken to be positive.

† The flux linkages are the product of the magnetic flux and the number of turns which the flux is linking.

In the MKS system $\Psi$ is measured in webers, or volt-seconds, and $L$ in henries (H).

The electromotive force (e.m.f.) of self-induction is equal to the rate of change of the flux linkages with time, i.e.

$$e_L = -\frac{d\Psi}{dt} \tag{1.3}$$

or, taking (2) into account

$$e_L = -L\frac{di}{dt}. \tag{1.4}$$

There is a minus sign in (4) because the positive directions of the self induced e.m.f. and the current are selected so as to coincide. According to Lenz's law an increasing current is associated with a self-induced e.m.f. opposing the current increase.

Quantity

$$u_L = -e_L = L\frac{di}{dt} \tag{1.5}$$

is known as the inductive voltage drop or, alternatively, the voltage across the inductance.

If a part of a magnetic flux linked with an inductive element is simultaneously linked with another inductive element, then these two inductive elements possess, besides the parameters $L_1$ and $L_2$, another parameter $M$ known as *mutual inductance*. When magnetic fluxes and, correspondingly, flux linkages are linear functions of current, this parameter is determined by the ratio of the flux linkages of the mutual induction of one element to the current through another element:

$$M = \frac{\Psi_{12}}{i_2} \tag{1.6}$$

and

$$M = \frac{\Psi_{21}}{i_1}. \tag{1.7}$$

Here $\Psi_{12}$ is the flux linkages of the first element resulting from the current of the second element; $\Psi_{21}$ is the flux linkages of the second element resulting from the current of the first element.

In this case an e.m.f. of mutual induction is induced in the first and second elements equalling, respectively

$$e_{1M} = -\frac{d\Psi_{12}}{dt} = -M\frac{di_2}{dt} \tag{1.8}$$

and

$$e_{2M} = -\frac{d\Psi_{21}}{dt} = -M\frac{di_1}{dt}. \tag{1.9}$$

If the e.m.f. current and time are measured in volts, amperes and seconds (sec) respectively, then $M$ is measured in henries (H).

Unlike the other parameters, mutual inductance is characterized not as an independent element of the electrical circuit but only as a magnetic coupling of inductive elements.

Finally, the term *capacitance* and its corresponding conventional symbol $C$ denote an idealized circuit element in which electrical energy may be stored and, in addition, they express a quantity equal to the ratio of the charge to the voltage on the element

$$C = \frac{q}{u_C}. \tag{1.10}$$

If $q$ and $u_C$ are measured in coulombs and volts, then $C$ is measured in farads. The sign of the charge $q$ on the capacitance is the same as the sign of the voltage $u_C$ across the capacitance.

Let us suppose that the capacitance is formed by two plates separated by an insulating material (dielectric). Under the effect of an applied voltage, equal quantities of electricity of opposite polarity are concentrated on the plates: the plate with the high potential has a positive charge, and the plate with the low potential has a negative charge.

When the voltage applied to the plates changes, charge $q$ changes in accordance with formula (10): an additional positive charge appears on the plate whose potential increases, and a similar negative charge appears on the plate whose potential decreases. The flow of negative charge to the second plate is none other than the positive current for the opposite reference direction—from the second plate to the external circuit. Thus, as the voltage across the capacitance varies, a current is produced in the circuit whose value is determined by the time rate of change in the electrical charge:

$$i = \frac{dq}{dt} \tag{1.11}$$

or

$$i = C\frac{du_C}{dt}. \tag{1.12}$$

This current is regarded as a conduction current in conductors connected in series to the capacitance element (current supported by moving charged particles) and as a displacement current in the dielectric between the capacitance plates. The latter concept, used in field theory, denotes a

current supported by a changing electromagnetic field. As a result of the introduction of this concept the current in a circuit with capacitance is considered to be closed through the dielectric, and in this sense we may speak of a current through a capacitance.

In accordance with (11) and (12), positive current for a capacitance is associated with an increasing charge on the capacitance. The polarity o the voltage developed across the capacitance as it is charged is always in opposition to the charging current. The circuit supplies electric energy to the capacitance in charging it, the current being positive. On discharge, stored energy is returned to the rest of the circuit, the current being negative.

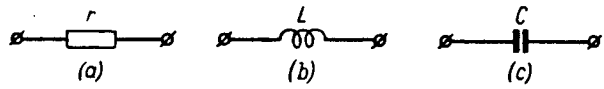



FIG. 1. Conventional symbols: (*a*) resistance; (*b*) inductance; (*c*) capacitance.

For resistance, inductance and capacitance, the pictorial symbols shown in Fig. 1 are used.

With the aid of these idealized circuit elements, electrical circuits can be represented as replacing various physical devices. It should be noted that concepts such as *inductor* (induction coil), *capacitor* (condenser device) and *resistor* (resistance device) denoting actual electrical devices, do not generally agree with the theoretical concepts of *inductance*, *capacitance* and *resistance*, which are conceived as separate circuit elements existing independently of each other.

Let us imagine the simplest inductor in the form of a circular turn of a conductor through which a current $i$ is flowing. When $i$ is direct current the voltage at the terminals of the circular turn is determined by the magnitude of the voltage drop in its resistance in accordance with (1), and the current is identical at all points in the turn.

Also, when $i$ is alternating current, the changing magnetic field induces an e.m.f. in the turn. Between the terminals, just as between other points of the turn, the electrical field is variable and a displacement current appears in the dielectric around the turn. As a result of this the current will not be identical at various points in the turn. The higher the frequency of the alternating current, the greater will be the induced e.m.f. and the displacement current.

For low frequencies the displacement current may be disregarded; and for high frequencies the displacement current caused by varying electrical field intensity may be commensurable in value with the current in the turn or may even exceed it. Thus, depending on the frequency range selected, the circuit diagram representing a turn or a coil with a large number of turns can contain either a resistance only (Fig. 2*a*), or an inductance

with a resistance connected in series (Fig. 2*b*), or an inductance and a resistance connected in parallel with a capacitance (Fig. 2*c*).

Let us now consider a parallel-plate capacitor. If the voltage is constant and the dielectric ideal, there will be no current in the circuit.

If the voltage is variable, an alternating current creating an alternating magnetic field is produced in the process of the electrical charge changing.

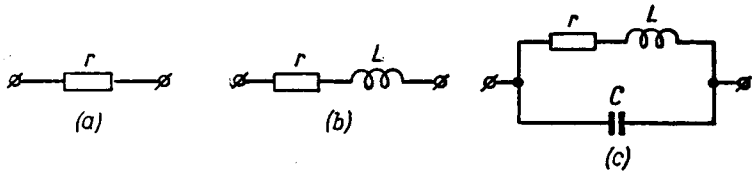



FIG. 2. Equivalent circuit diagrams for a turn or a coil: (*a*) with direct (constant) current; (*b*) in the low frequency band; (*c*) in the high frequency band.

The effect produced by the magnetic field can be taken into account in an equivalent circuit diagram by using a certain inductance connected in series with the capacitance. This inductance can generally be disregarded, owing to its small value.

Finally, owing to there being a certain conductance, heat losses occur in any non-conductor and increase with frequency. In addition heat is liberated in the capacitor plates. Heat losses are taken into account in the equivalent circuit of the capacitor by resistance *r* connected in parallel with capacitance *C* (Fig. 3).

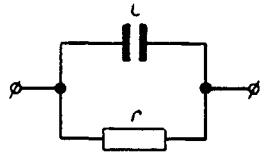



FIG. 3. Equivalent circuit diagram for a capacitor.

As was pointed out in the early part of this chapter the concept of resistance is associated with the conversion of electric energy into heat (or some other form of radiant energy). The energy dissipation by a conductor can be affected by various factors.

In resistors (rheostats and other resistance devices) inductive and capacitive effects occur to a greater or lesser extent, depending on frequency.

At low frequencies the resistance differs little from the resistance for a direct current. And as the frequencies increase the current is distributed non-uniformly over the cross-section of the conductors: the nearer to the centre of the conductor, the less the density of the current. The current is forced to the surface of the conductor, causing an increase in heat loss and

hence in resistance. This phenomenon, known as the *skin effect*, is usually taken into account at high frequencies and is studied in electromagnetic field theory.

Non-uniform current distribution over the cross-section of a conductor, and thereby the increase in electric energy dissipation as heat, can be caused by the variation of the current in a neighbouring conductor (proximity effect).

Moreover, eddy currents are induced in the body of any conducting mass by variation of magnetic flux, thereby causing an increase in resistance.

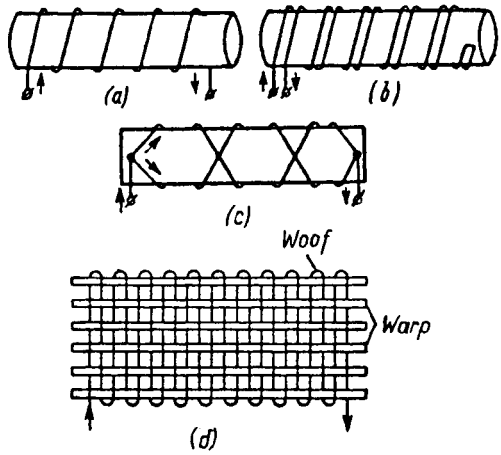



FIG. 4. Methods of constructing wire resistances with various amounts of inductance and capacitance: (*a*) inductance high, capacitance low; (*b*) inductance low, capacitance high; (*c*) inductance and capacitance low; (*d*) wattled resistance element.

Hysteresis in any magnetic materials located within the varying magnetic field, as well as dielectric loss and electromagnetic radiation, increase in their turn the time rate of electric energy dissipation as heat.

In view of the dependence of a resistance on frequency, we distinguish between resistances for direct current (ohmic resistance) and those for alternating current (effective resistance). Proper calculation, with due regard to the effects mentioned above causing quantitative discrepancy between ohmic and effective values of resistance, is one field problem.

The extent to which inductive and capacitive effects occur in wire resistors depends on the design features of the latter.

If the wire is wound in the form of a coil (Fig. 4*a*), the inductance is relatively high and the capacitance low. But if the wire has a bifilar winding (Fig. 4*b*), the inductance is very insignificant since the magnetic fluxes produced by the input and output currents are mutually compensating.

However, in view of the forward and reverse wires being arranged close to each other in this design the capacitance is relatively high.

The inductance and capacitance can be decreased simultaneously by winding wires on to the plate double and in opposite directions, as shown in Fig. 4*c*. With this type of winding the resulting magnetic field inside the coil approaches zero. Also, as a result of the difference in potentials of symmetrically arranged points in the parallel circuits being equal to zero, the electrical field is negligible.

In order to decrease inductance and capacitance a wattled winding is also used – a strip of cloth with a flexible insulating material as its warp and a high-resistance wire as its woof (Fig. 4*d*).

Magnetic fields, electrical fields and heat losses are thus interrelated in physical systems; "pure" inductance, capacitance and resistance can be obtained only hypothetically as a means for an approximate, but sufficiently accurate method of dealing with many electrical engineering problems.

The higher the frequency and the larger the linear dimensions of the systems themselves, the greater are the interdependence between the electrical and magnetic parameters and the inseparability of the electrical and magnetic fields. This happens particularly with lines or aerials which are as long as or longer than the electromagnetic wave.

It is not possible to make a strict demarcation of that range of frequencies for which a certain equivalent circuit will be valid, since this depends on a number of factors. As a rough guide it may be considered that it is permissible to regard the devices under examination in the form of separate elements with lumped parameters at frequencies at which the length of the electromagnetic wave does not exceed the largest linear dimension of the device by fewer than a 100 times (Ref. 89).

$$\lambda \geqq 100l. \tag{1.13}$$

The wavelength equals the phase velocity-to-frequency ratio

$$\lambda = \frac{v}{f}. \tag{1.14}$$

If $\lambda$ and $l$ are measured in centimetres then, equating $v$ with the speed of light ($3 \times 10^{10}$ cm/sec), on the basis of (13) and (14) we get

$$f \cdot l \cong 3 \cdot 10^8. \tag{1.15}$$

This approximate relationship is extremely conditional, since the range of frequencies in which the phenomena in the circuit are described by the simplest equation (5) or (12) depends on the arrangement of the devices.

### 3. Voltage Source and Current Source

The conventional devices—the e.m.f. source and the current source—are used as sources of electrical power in circuit theory.

The *e.m.f. source* (also called ideal voltage source) is defined as an active two-terminal element, which maintains the voltage across its terminals of a prescribed waveform independently of the current flowing through it. It is assumed that there are no passive elements in one such hypothetical source and consequently the current passing through does not cause any internal voltage drop. In other words, the voltage $u(t)$ on the terminals of this source equals its internal e.m.f.

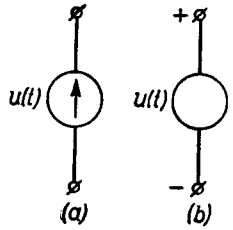



FIG. 5. E.m.f. source (ideal voltage source).

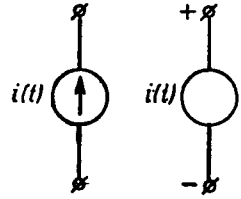



FIG. 6. Ideal current source.

The conventional symbols of the e.m.f. source are given in Fig. 5. Here the arrow or the + and − signs indicate the reference polarity or the positive direction of the e.m.f., i.e. the direction of the potential rise in the source at those instants when the function $u(t)$ is positive.

The magnitude of the current in an electrical circuit connected up to an e.m.f. source depends on the parameters of this circuit and on $u(t)$. If the terminals of the e.m.f. source are short circuited, the current, theoretically, should be infinitely great. Therefore the e.m.f. source is a source of infinite power (theoretical concept). In fact, when the terminals of an actually existing electrical power source (galvanic cell, accumulator or generator) are closed, the current can only have a finite value since the e.m.f. of the source is compensated by the voltage drop within the source.

The equivalent circuit of a physical source of electrical energy may be represented as an e.m.f. source connected in series with some passive elements, the latter to describe the internal circuit properly. Henceforth this type of source will be referred to as a *voltage source* to distinguish it from the e.m.f. source. Its internal parameters are usually small in comparison with the parameters of the external electrical circuit; they can be related to the latter or, in certain cases, not taken into account at all (depending on the relationship of the values and the accuracy of calculation required).

The ideal *current source* is defined as an active two-terminal element, which maintains a current of a prescribed waveform independently of the voltage appearing across its terminals.

The conventional symbols for an ideal current source are given in Fig 6. The arrow in the current source or the signs + and − indicate the positive (or reference) direction of the current $i(t)$, i.e. the actual (physical) current direction at those instants when the function $i(t)$ is positive.

With a d.c. source, the actual current direction coincides with the reference direction (current positive).

With an a.c. source, in one of the half-cycles, the current flowing in the reference direction is taken to be positive; in the other half-cycle the actual current direction is opposite to the reference direction, and the current is taken to be negative.

In proportion to an unlimited increase in the resistance of the external circuit connected to the current source, the voltage on its terminals and, correspondingly, the power developed by it, increase infinitely. Therefore the ideal current source, just like the e.m.f. source, is a source of infinite power. A passive element describing the internal parameters of the current source and connected to its terminals, limits the power of the latter.

The current source, a theoretical concept, as will be shown later (Chapters II–IV), is used in a number of cases to simplify the calculation of electrical circuits.

A device consisting of an accumulator connected in series with a large additional resistance is, in certain respects, similar to a current source

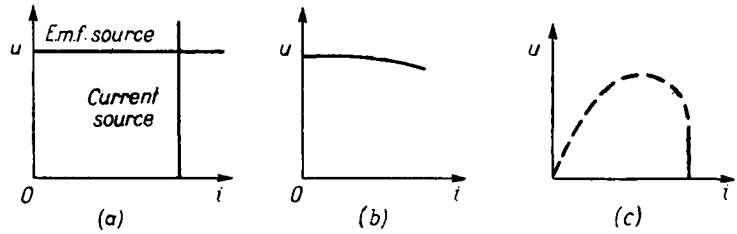



FIG. 7. Voltage–current relationship in sources: (*a*) idealized characteristics; (*b*) characteristic of d.c. generator with independent excitation; (*c*) characteristic of a series-connected d.c. generator.

Another example of a current source is the five-electrode amplifying electronic valve (pentode). These devices, having resistances incommensurably greater than the resistances of the external circuit, produce a current which is barely dependent on the external load changing within wide limits, and in this particular respect they are analogous to a current source.

The dependence of the voltage at the terminals of any device on the

current passing through it, or a graph of voltage against current is called the voltage–current or current–voltage characteristic of the given device.

The voltage–current characteristics of ideal sources of e.m.f. and current are straight, parallel axes (Fig. 7*a*).

The voltage–current characteristics of actual electrical power sources can approximate to the idealized characteristics. Thus, for example, in the significant part of the characteristic $u = f(i)$, the voltage $u$ on the terminals of a d.c. generator with independent excitation (Fig. 7*b*) and current $i$ of a series-connected d.c. generator (Fig. 7*c*) changes negligibly (this part of the characteristic is shown as a solid line in Fig. 7*c*).

## 4. Circuit Diagrams of Networks

A circuit diagram is a graphic representation of an electrical network*. It shows the sequence in which the elements of the electrical circuit are connected.

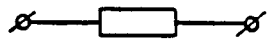


FIG. 8. Branch.

The "electrical" components of a circuit diagram are the active and passive elements contained in the given circuit. Their conventional symbols were given in Figs. 1, 5 and 6.

The "geometrical" components of a circuit diagram are the branches and nodes.

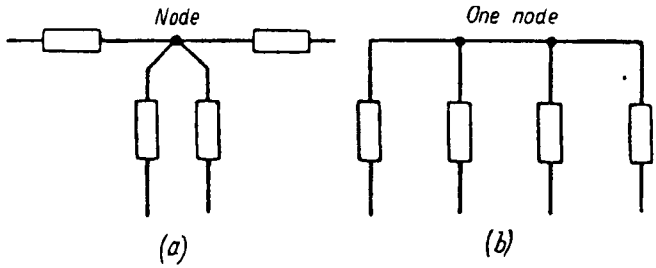



FIG. 9. Node.

A branch or arm is formed of one or several circuit elements connected in series. Generally, if the detailed nature of the branch is not important or is unknown it is represented as a rectangle (Fig. 8).

A node or junction is the point of connexion of three or more branches (Fig. 9). Lines connecting the branches in a diagram are considered to be

* This term is often used synonymously with the term circuit, yet having a somewhat more general connotation.

perfect conductors. Therefore, for example, diagrams *a* and *b* of Fig. 9 are identical in the electrical sense: they both contain one node.

Branches connected to one pair of nodes (Fig. 10) are called parallel.

As an example Fig. 11 illustrates the circuit diagram of an electrical network containing 5 branches and 3 nodes.

Any closed path traversed through an electrical circuit is called a loop or mesh. The arrow in Fig. 11 indicates one of the loops formed in the given circuit diagram.

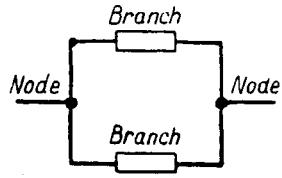



FIG. 10. Parallel connexion of two branches.

Depending on the number of loops in the circuit we distinguish between single-loop and multi-loop circuit. A single-loop circuit is the simplest. Using the rules for converting circuit diagrams (Chapter III) it is possible, in a number of cases, to replace a multi-loop circuit by a single-loop circuit, which simplifies calculations.

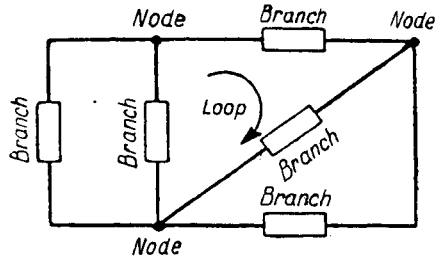



FIG. 11. Circuit diagram of a network.

## 5. Linear Electrical Circuits

The passive elements of an electrical circuit are called *linear* if the voltage and current in the element are related by a linear equation of the first order—algebraic or differential.

A linear resistance obeys Ohm's law which expresses the direct proportionality between the voltage at the terminals of a resistance and the current flowing through it.

As is seen from expression (1), the voltage and current in the resistance are directly proportional to each other provided that the value $r$ does not depend on the current or the voltage.

In the case of inductive or capacitive elements, the voltage and current are related by linear differential equations (5) or, correspondingly, (12). It is then assumed that values $L$ and $C$ are independent of current or voltage.

Thus the resistance and also the inductive and capacitive elements are linear on the condition that values $r$, $L$ and $C$ do not depend on electrical quantities.

With respect to the active elements of an electrical circuit, the condition of the linearity of the e.m.f. source is that the e.m.f. should be independent of the current flowing through the source. In its turn the condition of the linearity of the current source is that the current should be independent of the voltage at the terminals of the source.

An electrical circuit consisting of linear active and passive elements is called a linear circuit.

Actual electrical devices, strictly speaking, do not obey the law of linearity.

Thus, for example, when the current is varying in an induction coil with a ferromagnetic core, the relationship between the magnetic flux (and, correspondingly, the flux linkages) and the current does not remain constant, i.e. the value of $L$ in (12) generally depends on the current.

Moreover, when the current passes through a conductor heat is released, the conductor becomes warm and its resistance $r$ changes.

Depending on the dielectric material, the capacitance in the function of the charge (or of the applied voltage) can also change to a greater or lesser degree.

Non-linear devices, moreover, include electronic, ionic and semiconductor elements whose parameters depend on electrical quantities.

If the operating range on which a certain device is calculated, i.e. for given restricted limits of the change in the electrical quantities, preserves the law of linearity with a degree of accuracy sufficient for practical purposes, then such a system is considered linear.

The study and calculation of linear circuits, as a rule, entail smaller difficulties than the study and calculation of non-linear circuits. Therefore, in those cases where the linear law reflects the physical facts closely enough, the circuit is regarded as linear.

To facilitate the solving of problems, the conditions in which actual devices lose their linear properties (e.g. as a result of considerable overheating) are disregarded, and the electrical circuit is considered linear without restrictions.

## 6. Basic Laws of Electrical Circuits

The calculation of an electrical circuit is based on physical laws expressing the properties of electrical circuits. Among the basic laws of electrical circuits are Ohm's and Kirchhoff's laws.

## Ohm's Law

*Ohm's law for linear circuits states the direct proportionality between the voltage at the terminals of a resistance and the current flowing through the given resistance.*

If a current $i$ (Fig. 12) flows through resistance $r$, the potential of terminal 1, at which the current enters, is higher than the potential at terminal 2 from which it flows out. The difference in potential between

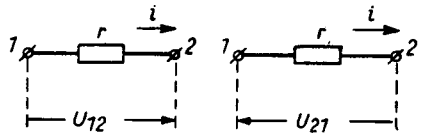



FIG. 12. Reference directions for current and voltage.

terminals 1 and 2, i.e. the voltage drop in the positive current direction, is expressed by the formula from Ohm's law

$$u_{12} = ri. \tag{1.16}$$

The voltage reckoned in the reverse direction (the voltage of point 2 with respect to point 1) has the opposite sign

$$u_{21} = -u_{12} = -ri. \tag{1.17}$$

If the double subscript notation is used, then the order of the subscripts indicates the reference polarity for potential or, what is the same, the positive direction selected for the voltage. When there are no subscripts, the specified voltage reference direction is indicated by an arrow.

Using the conductance value $g = 1/r$ it is possible to transform formula (16)

$$i = gu_{12}. \tag{1.18}$$

In accordance with (16) or (18) the resistance has linear voltage–current characteristics: a plot of the voltage across the resistance against the current through the resistance is a straight line through the origin (Fig. 13).

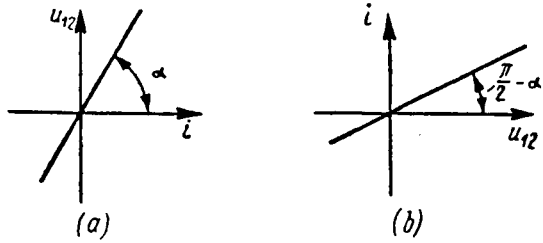



FIG. 13. Voltage–current relationship in a linear resistance.

The tangent of the angle of inclination of the characteristic $u_{12} = f_1(i)$ to the abscissa axis is proportional to $r$

$$\tan \alpha = r \cdot \frac{m_i}{m_u},$$

where $m_u$ and $m_i$ are the scales on which the voltages and currents are plotted.

In turn, the tangent of the angle of inclination of the characteristic $i = f_2(u_{12})$ to the abscissa axis is proportional to $g = 1/r$

$$\tan (90° - \alpha) = \cot \alpha = g \frac{m_u}{m_i}.$$

Ohm's law can be applied to a branch with an e.m.f. source for which a voltage–current characteristic can also be plotted.

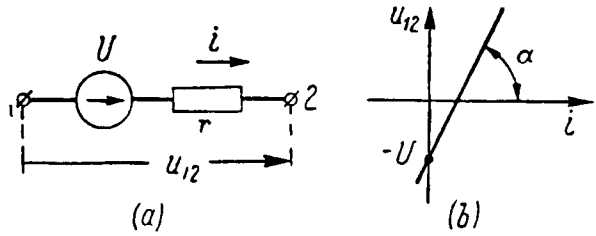



FIG. 14. (*a*) Series connexion of resistance $r$ and e.m.f. source; (*b*) voltage–current characteristic.

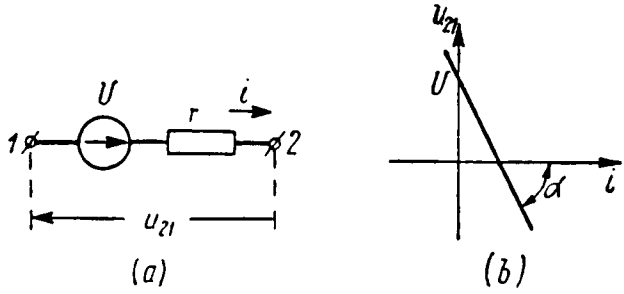



FIG. 15. (*a*) Series connexion of resistance $r$ and e.m.f. source; (*b*) voltage–current characteristic.

Such a branch, through which a current $i$ passes, is shown in Fig. 14*a*. The potential of terminal 1 is lower than the potential of terminal 2 by $U$, the value of the e.m.f., and at the same time higher than it by the value of the voltage drop in resistance $r$. Consequently, the voltage at the terminals of the branch is finally

$$u_{12} = -U + ri. \tag{1.19}$$

The voltage–current characteristic for this case is given in Fig. 14*b*.

The difference of potential taken from terminal 2 to terminal 1 (Fig. 15) is opposite in sign to expression (19), i.e. equals the e.m.f. of the source after deducting the voltage drop in resistance $r$

$$u_{21} = -u_{12} = U - ri. \tag{1.20}$$

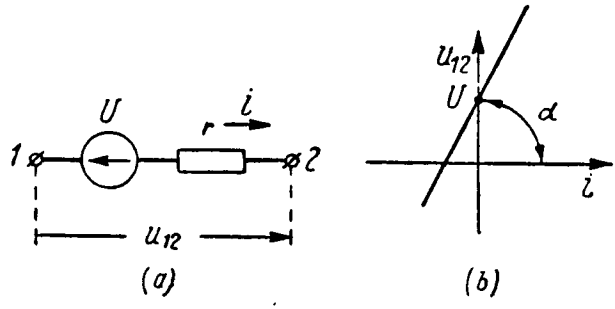



FIG. 16. (*a*) Series connexion of resistance $r$ and e.m.f. source; (*b*) voltage–current characteristic.

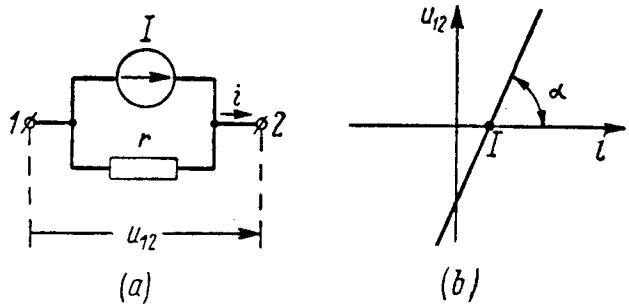



FIG. 17. Parallel connexion of resistance $r$ and current source; (*b*) voltage–current characteristic.

The voltage–current characteristic of this direction for reading the voltage at the terminals of the branch, also called the *external characteristic*, is given in Fig. 15*b*.

When terminals 1 and 2 are short-circuited, voltages $u_{12}$ and $u_{21}$ equal zero and the formula of Ohm's law has the form

$$i = \frac{U}{r}. \tag{1.21}$$

Figure 16 shows the case of a branch with an e.m.f. directed against the current and gives the corresponding voltage–current characteristic.

Finally, Fig. 17 shows the parallel connexion of resistance $r$ with a current source $I$ and the voltage–current characteristic corresponding to this case.

*Thus, the voltage–current characteristics of a section of a circuit consisting of a linear resistance r, connected in series with an e.m.f. source U, or in parallel with a current source I, are linear.*

Expression (20) shows that for the given value of current $i$ the functional relation $u_{21} = f(r)$ is represented by a straight line $AB$ (Fig. 18), the tangent

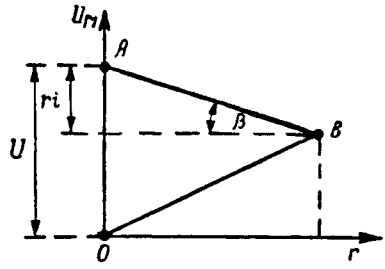



FIG. 18. Relationship between voltage and resistance (for given current).

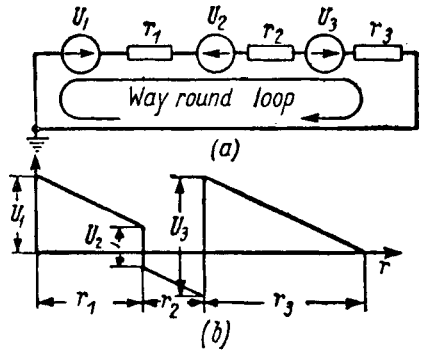



FIG. 19. Graph of the distribution of potentials throughout a circuit.

of the angle of inclination of the straight line to the abscissa axis being proportional to $i$

$$\tan \beta = i \frac{m_r}{m_u},$$

where $m_u$, $m_r$ are the scales on which the voltages and resistances are plotted.

The graph in Fig. 18 is constructed on the assumption that the e.m.f. $U$ is connected in series with the resistance $r$. Passing through the e.m.f. source (in the direction of the e.m.f.), the potential increases intermittently by the value $U$ and then, passing through resistance $r$ (in the direction of the current), decreases rectilinearly by the magnitude of the voltage drop.

If $r$ is regarded as the internal resistance of the voltage source, without separating it from the e.m.f. $U$, the graph of the change in potential will be in the form of a straight line $OB$.

By equating the potential of any point of the loop-circuit in Fig. 19*a* containing resistances and e.m.f. sources to an arbitrary numerical value, e.g. zero, it is possible to construct a graph of the distribution of potential throughout the loop. Starting off round the loop from any node in the circuit diagram, e.g. from the one whose potential is taken as zero, and completing the whole loop, we arrive at the initial potential. Such a graph of the distribution of potential is given as an example in the Fig. 19*b* network. It should be noticed that the potential decreases on passing through the e.m.f. source in the direction opposite to the e.m.f.

In the graph shown in Fig. 19*b* the inclination of the straight lines is identical since the current is exactly the same in all the resistances (single-loop circuit). In the case of a multi-loop circuit where the currents in the branches are different, the inclinations of the straight lines are not identical. Such a plot of the voltage against the resistance may be considered as a graphical illustration of Kirchhoff's voltage law.

**Kirchhoff's Laws**

Kirchhoff's first law (current law) states the fact that an electrical charge at a node is neither accumulated nor consumed: the sum of currents flowing into the node equals the sum of currents flowing out, i.e. *the algebraic sum of the currents at the node is zero:*

$$\sum i = 0. \tag{1.22}$$

The summation is over the currents $i$ in the branches connected at the node under examination. The signs of the currents are then determined by taking into account the positive current directions, which are usually indicated on the circuit diagram by arrows: in equation (22) an identical sign, e.g. positive, is ascribed to all the currents directed towards the node and, correspondingly, all the currents directed from the node appear in equation (22) with the opposite sign. In other words any current directed from the node can be regarded as a current directed towards the node, but with the opposite sign.

As an example, Fig. 20*a* shows a node at which four branches are connected. In this case equation (22) has the form:

$$i_1 + i_2 - i_3 - i_4 = 0.$$

Kirchhoff's current law holds not only for a node but for any closed circuit (a mesh contour). By writing down the equations from Kirchhoff's first law for all the nodes of a mesh contour and adding them, we discover that the sum of the currents flowing into and out of the given mesh contour

is zero. The law of signs referred to above is thus observed. For example, for the Fig. 20*b* diagram we have

$$i_1 - i_2 - i_3 = 0.$$

Kirchhoff's second law (voltage law) is formulated as follows: *the algebraic sum of the voltages around a closed circuit equals zero:*

$$\sum u = 0. \tag{1.23}$$

We can follow the circuit in any direction we choose, e.g. clockwise. The following rule of signs for the e.m.f. and the voltage drops included in

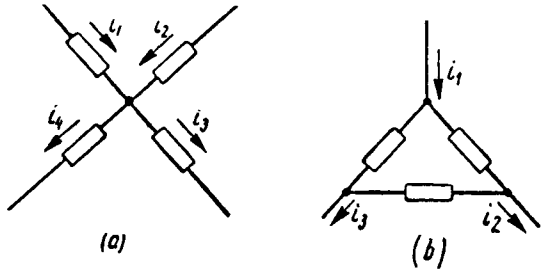



FIG. 20. Illustration of Kirchhoff's current law.

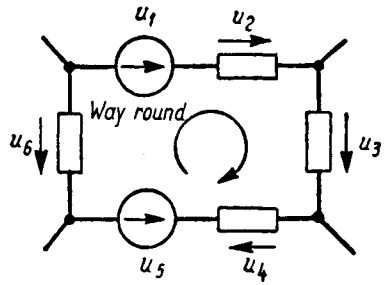



FIG. 21. Illustration of Kirchhoff's voltage law.

equation (23) is then observed: the e.m.f. and the voltage drops corresponding in direction to the direction of traversal are taken as having different signs. Thus Kirchhoff's second law establishes that the algebraic sum of the e.m.f. in a closed circuit equals the algebraic sum of the voltage drops in the elements of the circuit.

For example, for the Fig. 21 diagram we have:

$$u_1 - u_5 = u_2 + u_3 + u_4 - u_6$$

or, in accordance with (23)

$$u_1 - u_2 - u_3 - u_4 - u_5 + u_6 = 0.$$

The difference in potential or the voltage between two points of an electrical circuit is equal to the work done in moving a unit electrical charge from one point to another. Consequently, Kirchhoff's voltage law establishes that the work done in moving a unit positive "test" charge hypothetically along a closed circuit is equal to zero, in so far as the test charge is to possess no net surplus or deficiency of energy upon its return to the starting point: energy gains (potential rises) experienced by the test charge must be exactly balanced by equal energy losses (potential drops).

Kirchhoff's voltage law thus corresponds to the law of the conservation of energy.

Formulae (22) and (23), written in a general form in terms of instantaneous quantities, are valid for a.c. circuits as well as for d.c. circuits.

## 7. Energy and Power

Let us suppose that for an elementary interval of time $\mathrm{d}t$ an electrical charge $\mathrm{d}q$ passes through a network branch (energy receiver or load) under the effect of an applied voltage $u$. The *elementary work* done in this interval of time, or alternatively, the elementary energy input to the branch equals $\mathrm{d}W = u\,\mathrm{d}q = ui\,\mathrm{d}t$. In the MKS system energy is measured in joules, and power in watts (W).

*Instantaneous power* is defined as the time derivative of energy

$$p = \frac{\mathrm{d}W}{\mathrm{d}t} = ui, \tag{1.24}$$

i.e. is equal to the product of the instantaneous value of voltage and current.

Power $p$ is an algebraic quantity; it is positive when the signs of $u$ and $i$ are the same. If the reference direction for voltage is the same as that for current, then when $p > 0$ the network branch under examination will absorb energy, and when $p < 0$ it will return all or part of the absorbed energy to the rest of the network.

Let us examine separately the expressions of the instantaneous values of power and energy for various circuit elements.

In accordance with (24), the instantaneous power supplied to the resistance $r$ equals

$$p_r = ri^2 = gu^2 = ui. \tag{1.25}$$

The instantaneous power delivered to the inductance $L$ is

$$p_L = \frac{\mathrm{d}W_L}{\mathrm{d}t} = L\frac{\mathrm{d}i}{\mathrm{d}t} \cdot i. \tag{1.26}$$

The instantaneous power delivered to the capacitance $C$ is

$$p_C = \frac{dW_C}{dt} = C\frac{du}{dt} \cdot u. \tag{1.27}$$

According to the Joule–Lenz law, the energy required to maintain current $i$ in resistance $r$ equals

$$W_r = \int ri^2\,dt = \int gu^2\,dt. \tag{1.28}$$

The limits of the integral are determined by an examined interval of time in the course of which the energy is supplied to the given resistance.

The energy delivered to the inductance is stored in its magnetic field. If the current starts from zero at $t = 0$, then the magnetic energy is

$$W_L = \int_0^t p_L\,dt = \int_0^i Li\,di = \frac{1}{2}Li^2. \tag{1.29}$$

In turn, the energy delivered to the capacitance is stored in its electric field. If the initial voltage across the capacitance is zero at some time $t = 0$, the net energy storage in the capacitance at any later time $t$ is

$$W_C = \int_0^t p_C\,dt = \int_0^u Cu\,du = \frac{1}{2}Cu^2. \tag{1.30}$$

When $i$ and $u$ are variable, $W_L$ and $W_C$ are time functions (instantaneous values of the energy of the magnetic and electrical fields).

From the expressions given above it is clear that energy is a scalar quantity, always positive.

## 8. Periodic and Nonperiodic Processes in Linear Electrical Circuits

Processes in electrical circuits may be divided into *steady-state* and *transients* or, respectively, *periodic* and *nonperiodic processes.*

*Steady-state* conditions in an electrical circuit are characterized by the fact that the voltages and currents remain constant or vary periodically. Function $f(t)$, the argument of which physically denotes time, is a periodic function if, for any positive or negative value of $t$, the following equality is observed:

$$f(t + T) = f(t). \tag{1.31}$$

Here $T$ is a period, i.e. the time interval after which the function repeats. Geometrically this denotes that the ordinates of two arbitrary points on the graph $f(t)$ with abscissae differing by $T$, are identical.

The value which is the reciprocal of the period, i.e. the number of periods occurring in unit time, is known as *frequency.* In the MKS system

frequency has the dimensions $sec^{-1}$ and the hertz (the frequency equals 1 hertz if the period equals 1 sec) is its unit of measurement.

If we agree to regard a constant quantity as the limit of a periodic function with an infinite period or, the same thing, with a frequency equal to zero, then steady-state conditions can, in general, be defined as *periodic.*

Equation (31) expresses the infinite periodic recurrence of the phenomenon; the periodicity of the function denotes its existence at any instant of time preceding the instant under consideration.

In actual fact there is no infinite periodic recurrence of phenomena. Therefore, strictly speaking, a periodic function should be regarded as a certain mathematical abstraction used for practical calculations. With a degree of approximation quite sufficient for engineering practice, many processes in electrical circuits are considered periodic and, correspondingly, the operating conditions of the circuit are considered steady. Currents in steady-state conditions are regarded as forced oscillations produced by electrical energy sources, for example by e.m.f. sources operating in the circuit. The frequency of these oscillations coincides with the frequency of the exciting electromotive forces.

When examining a steady state in any electrical circuit containing $r$, $L$ and $C$ elements, it is supposed that, from the instant when the circuit is switched on to the electrical energy source until the instant of observation, so much (theoretically infinite) time passes that the natural oscillations of the system disappear and conditions set by the source are established. Therefore when a steady state in an electrical circuit is being examined, the beginning of the time reading (time reference) may be selected arbitrarily. The choice of the origin only has an influence on the value of the initial phase (Chapter II, Section 1) of the voltage and current in a steady state.

If some sudden change is made in the parameters or mode of operation of the circuit, then *transients* occur during which natural oscillations of the system take place together with the forced oscillations.

Transition to new steady-state conditions depends on the increase or decrease of the energy of the electrical and magnetic fields and requires time since the energy of these fields cannot change suddenly (Chapter XI, Section 1). In transient conditions electrical quantities produced by electrical sources have a *nonperiodic* character and do not conform to formula (31).

The transition from the initial conditions to new steady-state conditions theoretically takes an infinitely long time since the natural oscillations disappear only after an infinitely long time. But after a very short interval of time the natural oscillations reach such negligibly small values that they can, in practice, be disregarded, and the process can be considered steady.

Chapters II–X are devoted to periodic processes and Chapters XI–XIII to nonperiodic.

### 9. Effective Value and Average Value for a Periodic Function

The magnitude of a voltage or a current is usually judged by the *effective* or *root mean square* (r.m.s.) value for a period $T$. By the effective value of a periodically varying current $i$ we understand the quantity

$$I = \sqrt{\left(\frac{1}{T}\int_0^T i^2 \, \mathrm{d}t\right)}. \tag{1.32}$$

By squaring (32) and multiplying both parts of the expression obtained by $rT$, we find

$$I^2 rT = \int_0^T i^2 r \, \mathrm{d}t.$$

This equality shows that *the effective value of a periodically varying current is equal in magnitude to a constant current which would produce heat in a given resistance at the same average rate that it is produced by the varying one.*

Analogously, the effective value of the periodically varying voltage is

$$U = \sqrt{\left(\frac{1}{T}\int_0^T u^2 \, \mathrm{d}t\right)}. \tag{1.33}$$

By squaring (33) and multiplying both parts of the equality obtained by $T/r$ we find

$$\frac{U^2}{r} T = \int_0^T \frac{u^2}{r} \, \mathrm{d}t.$$

Consequently, *the effective value of a periodically varying voltage is equal in magnitude to the constant voltage which, when applied to a given resistance, would produce the same average heating effect.*

With a sinusoidal current

$$\int_0^T i^2 \, \mathrm{d}t = \int_0^T I_m^2 \sin^2 \omega t \, \mathrm{d}t = \frac{I_m^2}{2} \int_0^T (1 - \cos 2\omega t) \, \mathrm{d}t = \frac{I_m^2}{2} T,$$

since

$$\int_0^T \cos 2\omega t \, \mathrm{d}t = 0.$$

Consequently,

$$I = \frac{I_m}{\sqrt{2}} \cong 0{\cdot}707 \, I_m.$$

Analogously, the effective value of the sinusoidal voltage is

$$U = \frac{U_m}{\sqrt{2}} \cong 0{\cdot}707\ U_m.$$

In practice the concept of the *average* (average arithmetical) value of the periodic function is also used, corresponding to the positive half-wave,

$$I_{av} = \frac{2}{T}\int_0^{T/2} i\,dt$$

or

$$U_{av} = \frac{2}{T}\int_0^{T/2} u\,dt.$$

In the case of a sinusoidal current

$$I_{av} = \frac{2I_m}{T}\int_0^{T/2} \sin \omega t\,dt = \frac{2I_m}{\omega T}\Big| - \cos \omega t\Big|_0^{T/2} = \frac{2}{\pi} I_m \cong 0{\cdot}637\ I_m.$$

By analogy, the average value of the sinusoidal voltage is

$$U_{av} = \frac{2}{\pi} U_m \cong 0{\cdot}637\ U_m.$$

Voltages and currents can be measured by various kinds of instruments. Instruments based on electromagnetic, electrodynamic and thermal systems respond to the effective value; magnetoelectrical instruments with a rectifier respond to the average value; and amplitude electronic voltmeters respond to the maximum value of a function.

CHAPTER II

# BASIC RELATIONSHIPS IN SINUSOIDAL-CURRENT CIRCUITS

## 1. Representation of Sinusoidal Functions by Rotating Vectors

The predominant form of a periodic process in electrical circuits is *sinusoidal*, characterized by the fact that all the voltages and currents in the circuit vary sinusoidally. The sinusoid is the only periodic real function which has a derivative of the same shape as the function itself, and this particular property of sinusoidal functions makes them very useful in electrical engineering.

A *sinusoidal function* is the simplest form of periodic function. Every non-sinusoidal periodic function can be represented by the sum of a constant and sinusoidal components (Chapter X).

The sinusoidal function

$$u = U_m \sin(\omega t + \psi) \tag{2.1}$$

is characterized by amplitude, frequency and initial phase.

The argument of function (1) is the angle $\omega t$ (or, correspondingly, time $t$).

The *angular* or *radian frequency* $\omega$ in (1) is connected to the period and frequency in the relationship $\omega T = 2\pi$ or $\omega = 2\pi f$.

The dimension of quantity $\omega$ is $\sec^{-1}$ and it is calculated in radians per second (rad/sec).

The *initial phase* $\psi$ depends on the origin selected, which is arbitrary. It is calculated in the same units as the argument $\omega t$ (in radians*) and is determined by the magnitude of the shift of the sinusoid with respect to the origin, i.e. is measured by an abscissa corresponding to the transition point of the negative half-wave into a positive.

A cosine curve may be regarded as a sinusoid with an initial phase $\psi = \pi/2$. Thus, if the function is given in cosinusoidal form $u = U_m \cos(\omega t + \psi_1)$, it can be reduced in the form of (1) by substituting expression $\psi_1 = \psi - \pi/2$.

* In those cases where the angle $\psi$ is calculated in degrees, the argument $\omega t$ is also converted into degrees.

Distinction between sine and cosine functions is often unnecessary and the term sinusoidal function is used to describe either function or a linear combination of both.

The initial phase $\psi$ is an *algebraic* value. Angle $\psi$ is positive when the sinusoidal function is displaced to the left with respect to the origin (Fig. 1 *a*); being, accordingly, read to the left, it has a positive sign in this case.

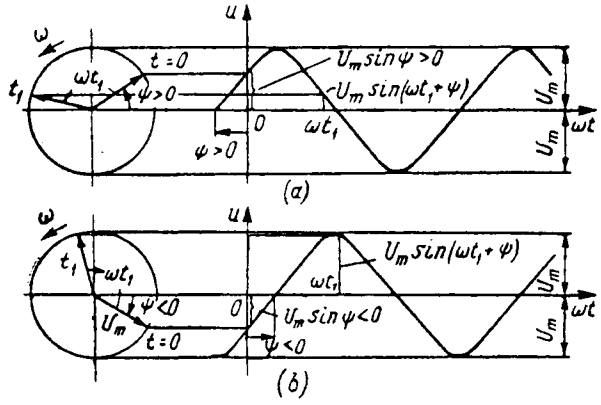



FIG. 1. Representation of a sinusoidal function in the form of the projections of a rotating vector on a vertical axis.

If the sinusoidal function is displaced to the right with respect to the origin (Fig. 1 *b*), angle $\psi$, read, accordingly, to the right, has a negative sign.

The given function turns to zero for values of the argument $\omega t = -\psi \pm k\pi$ where $k = 0, 1, 2...$

At an instant $t = 0$, argument $\omega t = 0$ and the value of the function is $U_m \sin\psi$. This value of the function equals the projection on to the vertical axis of the vector, the modulus of which equals amplitude $U_m$ of the given sinusoidal function. Depending on the sign of the initial phase $\psi$ this vector at the instant $t = 0$ is turned in a positive direction through the angle $\psi$ with respect to the horizontal axis—anticlockwise (Fig. 1 *a*) or in a negative direction—clockwise (Fig. 1 *b*).

If the vector is turned in a positive direction with an angular velocity $\omega$, then from the instant $t = 0$ to the instant $t_1$ it has turned through the angle $\omega t_1$ and at the instant $t_1$ has turned through an angle $\omega t_1 + \psi$ with respect to the reference axis. Here, as above, $\psi$ is an algebraic quantity.

The projection of the said vector on to the vertical axis when $t = t_1$ equals $U_m \sin(\omega t_1 + \psi)$. Comparing the left and right parts of Fig. 1 we see that *at every given instant t the value of the sinusoidal function equal to the projection of a rotating vector on to a vertical axis, corresponds to*

*argument* $\omega t$. Therefore the examination of sinusoidal functions can be replaced by the examination of *rotating vectors.*

If two sinusoidal functions have the same frequency, the vectors corresponding to these functions rotate together at the same angular frequency about their common origin and therefore the angles between them remain constant.

Figure 2 shows two sinusoidal functions

$$u_1 = U_{1m} \sin(\omega t + \psi_1) \quad \text{and} \quad u_2 = U_{2m} \sin(\omega t + \psi_2),$$

with the same angular frequency $\omega$. Curve $u_1$ displaced to the left with respect to $u_2$ increases from zero to its positive maximum sooner than curve $u_2$. It is therefore said that $u_1$ leads $u_2$ in phase, or, alternatively that $u_2$ lags behind $u_1$ in phase. The difference of the initial phases $\varphi = \psi_1 - \psi_2$ is called the *phase angle* or the phase difference by which $u_1$ leads $u_2$. This angle is also formed between the vectors on the vector diagram in Fig. 2*b*.

When the initial phases are equal, i.e. for a phase-angle difference equal to zero, it is said that the curves are *in phase.* The vectors corresponding to these curves are, in this case, in the same direction. For a phase

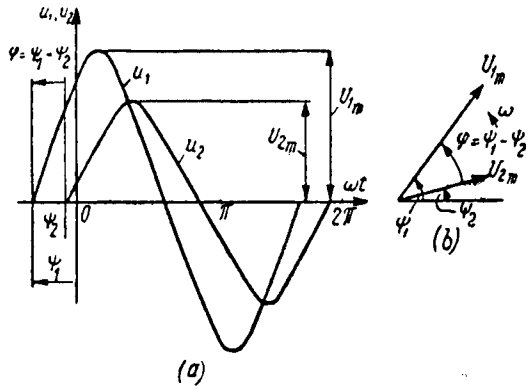



FIG. 2. Phase-angle difference: (*a*) between sine curves; (*b*) between vectors.

shift of 180° it is said that the curves are *opposite in phase*, or alternatively, are in antiphase. Their corresponding vectors are in diametrically opposed directions.

A vectorial representation of sinusoidal functions whose frequency is identical makes it easier to add and subtract these functions.

Let it be required to add the functions $u_1 = U_{1m} \sin(\omega t + \psi_1)$ and $u_2 = U_{2m} \sin(\omega t + \psi_2)$ represented in Fig. 3*a*. The ultimate curve $u = u_1 + u_2$, the ordinates of which at each given instant of time are determined by the algebraic sum of the corresponding ordinates of curves $u_1$ and $u_2$,

is a sinusoidal function with an amplitude $U_m$ and an initial phase $\psi$ (Fig. 3*a*).

In view of the fact that the sum of the projections of the two vectors equals the projection of the geometrical sum of these vectors, the amplitude and the initial phase of the resulting curve can be found from the *vector diagram* (Fig. 3*b*); the vector represented by the diagonal of a parallelogram whose sides equal $U_{1m}$ and $U_{2m}$ corresponds to the required curve.

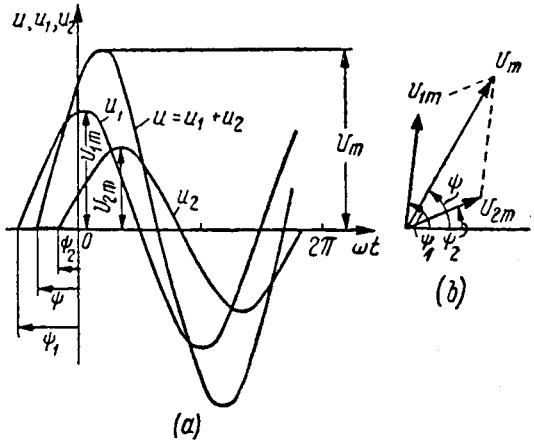



FIG. 3. Addition: (*a*) of sine curves; (*b*) of vectors.

According to Fig. 3*b*

$$U_m = \sqrt{[U_{1m}^2 + U_{2m}^2 + 2U_{1m}U_{2m}\cos(\psi_1 - \psi_2)]}. \qquad (2.2)$$

Formula (2) can be obtained analytically, starting with the equation $u = u_1 + u_2$:

$$U_m \sin(\omega t + \psi) = U_{1m} \sin(\omega t + \psi_1) + U_{2m} \sin(\omega t + \psi_2)$$

or

$$U_m \cos\psi \sin\omega t + U_m \sin\psi \cos\omega t$$

$$= (U_{1m}\cos\psi_1 + U_{2m}\cos\psi_2)\sin\omega t + (U_{1m}\sin\psi_1 + U_{2m}\sin\psi_2)\cos\omega t.$$

From this it follows that

$$U_m \cos\psi = U_{1m}\cos\psi_1 + U_{2m}\cos\psi_2, \qquad (2.3)$$

$$U_m \sin\psi = U_{1m}\sin\psi_1 + U_{2m}\sin\psi_2. \qquad (2.4)$$

Equations (3) and (4) contain two unknown quantities ($U_m$ and $\psi$) which can be found by a joint solution.

Squaring (3) and (4), we eliminate $\psi$ by adding the values obtained

$$U_m^2 = U_{1m}^2 + U_{2m}^2 + 2U_{1m}U_{2m}\cos(\psi_1 - \psi_2),$$

which corresponds to (2).

Dividing (4) by (3) we eliminate $U_m$

$$\tan\psi = \frac{U_{1m}\sin\psi_1 + U_{2m}\sin\psi_2}{U_{1m}\cos\psi_1 + U_{2m}\cos\psi_2}. \tag{2.5}$$

Angle $\psi$ can also be obtained directly from the vector diagram of Fig. 3*b*.

Subtracting function $u_2 = U_{2m}\sin(\omega t + \psi_2)$ from the function $u_1 = U_{1m}\sin(\omega t + \psi_1)$ is equivalent to adding $u_1$ and $-u_2 = U_{2m}\sin(\omega t + \psi_2 \pm \pi)$.

Therefore in the case of subtraction to substitute the initial phase in formulae (2) and (5) by $\psi_2 + \pi$, or equivalently, $\psi_2 - \pi$.

The computation of the angle $\psi$ with the aid of expression (5) involves the inverse tangent function. This function is multivalued. And this depends not only on the value of the complete fraction (5) but also on the signs of the numerator and denominator individually.

If the numerator and denominator are positive the angle lies in the first quadrant $0 < \psi < \pi/2$.

If the numerator is positive and the denominator is negative then the angle terminates in the second quadrant $\pi/2 < \psi < \pi$.

If the numerator and denominator are negative then the angle terminates in the third quadrant $\pi < \psi < 3\pi/2$.

Finally, if the numerator is negative and the denominator is positive then the angle terminates in the fourth quadrant $3\pi/2 < \psi < 2\pi$.

Instead of a positive angle $\psi$ exceeding $\pi$, a negative angle $\psi - 2\pi$ can also be taken.

When a vector diagram is used to establish phase difference angles or the amplitudes of sinusoidal waves of the same frequency, the vector diagram may be considered stationary (when the frequencies are equal, the angles between the vectors do not depend on time).

Constructing vector diagrams does not usually entail determining instantaneous values of sinusoidal functions; in such cases vector diagrams are constructed not for amplitudes but for effective values, i.e. the moduli of the vectors decrease in comparison with the amplitudes by $\sqrt{2}$ times*. The vector diagram is then considered stationary (it does not rotate).

* Such a vector diagram is called a *sinor diagram* (for term "sinor", see Section 5 of this chapter).

## 2. Current–Voltage Relations in Basic Circuit Elements

By using the general dependences of the electrical values given in expression (2) Chapter I, it is possible to obtain relationships between the sinusoidal voltages and the currents for the three simplest cases.

*Resistance.* When a sinusoidal current $i_r = I_m \sin(\omega t + \psi)$ flows through a resistance $r$ a voltage drop

$$u_r = ri_r = rI_m \sin(\omega t + \psi) = U_m \sin(\omega t + \psi). \tag{2.6}$$

occurs in it.

Consequently, *the voltage $u_r$ across the resistance $r$ and the current $i_r$ passing through this resistance are in phase:* They reach their maxima $U_m$ and $I_m$ and pass through zero together (Fig. 4a); the phase angle between the current $i_r$ and the voltage $u_r$ equal zero $\varphi = 0$.

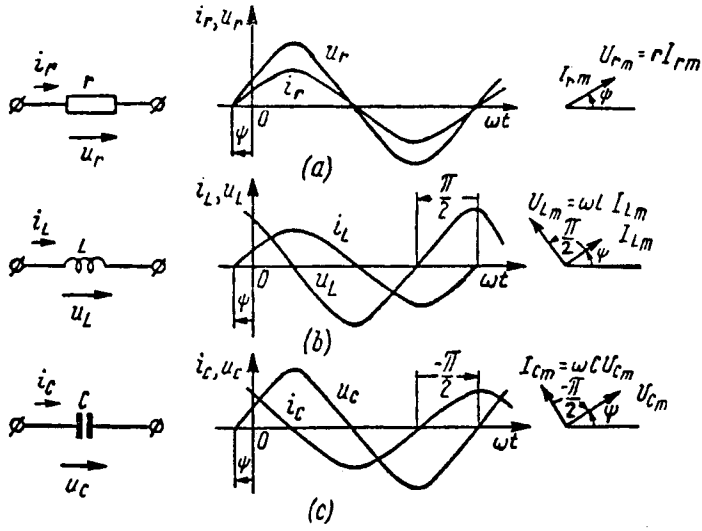



FIG. 4. Current–voltage relations in basic circuit elements: (*a*) $r$; (*b*) $L$; (*c*) $C$.

*Instantaneous amplitude and effective values of voltage and current are related by Ohm's law*

$$u = r \cdot i, \quad U_m = rI_m \quad \text{and} \quad U = rI. \tag{2.7}$$

By using conductance $g = 1/r$ we obtain

$$i = gu, \quad I_m = gU_m \quad \text{and} \quad I = gU. \tag{2.8}$$

*Inductance.* Taking current $i_L$ flowing through an inductance as sinusoidal, on the basis of expression (5) Chapter I we find

$$u_L = L\frac{di_L}{dt} = \omega L I_m \cos(\omega t + \psi) = U_m \sin\left(\omega t + \psi + \frac{\pi}{2}\right). \quad (2.9)$$

The expression obtained shows that voltage $u_L$ leads current $i_L$ by angle $\pi/2$: the maximum voltage $u_L$ is displaced to the left with respect to the maximum current $i_L$ by $\pi/2$ (Fig. 4*b*). When the current passes through zero, voltage $u_L$ reaches a positive or a negative maximum since it is proportional to the rate of change of current $di_L/dt$ which is maximum at the instant the current passes through zero (the sine curve of the current has maximum steepness at this instant); when the current reaches maximum its time rate of change and, consequently, the voltage $u_L$ become zero. The vector diagram given in Fig. 4*b* corresponds to this phenomenon.

By the *phase-angle difference* $\varphi$ between voltage and current we understand, by convention, the *difference between the initial phases* of voltage and current. Consequently, in this case $\varphi = \pi/2$.

*The relation of the amplitudes or the effective values of voltage and current is similar to Ohm's law*

$$U_m = \omega L I_m, \quad U = \omega L I. \quad (2.10)$$

The quantity $x_L = \omega L$, measured like a resistance, in ohms, is called *inductive reactance* and the reciprocal $b_L = 1/\omega L$ is called *inductive susceptance**

$$I_m = b_L U_m, \quad I = b_L U. \quad (2.11)$$

*Capacitance.* If the voltage across a capacitance is sinusoidal, i.e. $u_C = U_m \sin(\omega t + \psi)$ then, on the basis of expression (12) Chapter I

$$i_C = C\frac{du_C}{dt} = \omega C U_m \cos(\omega t + \psi) = I_m \sin\left(\omega t + \psi + \frac{\pi}{2}\right). \quad (2.12)$$

The electrical charge changes in accordance with a sinusoidal law, corresponding to the applied voltage $u_C$. The alternate storage of positive and negative charges on the plates of the capacitive element allows a sinusoidal current $i_C$ to flow into the circuit in which the capacitance is connected. Its value is determined by the time rate of change of the charge of the capacitance $dq/dt$.

Expression (12) shows that current $i_C$ leads the applied voltage $u_C$ by angle $\pi/2$ (Fig. 4*c*). The maximum (positive and negative) values of voltage $u_C$ correspond to the zero values of current. This is explained by the fact that when the current passes through zero the electrical charge $q$

* The term "inductive susceptance" used in Russian literature has a meaning different in sign from that commonly used in English.

of the capacitance reaches a maximum value (positive or negative) while voltage $u_C$, in accordance with expression (10) Chapter I, is directly proportional to the charge.

As in the previous case, by the angular phase difference between the voltage and the current we understand the difference between the initial phases of voltage and current, i.e. $\varphi = -\pi/2$.

Thus angle $\varphi$ represents an algebraic quantity: it is positive in the case of inductance and negative in the case of capacitance.

*The relation of the amplitudes, correspondingly, the effective values of voltage and current is similar to Ohm's law*

$$U_m = \frac{1}{\omega C} I_m, \quad U = \frac{1}{\omega C} I. \tag{2.13}$$

The quantity $x_C = 1/(\omega C)$, measured like a resistance, in ohms, is called *capacitive reactance** and its reciprocal value $b_C = \omega C$ is called *capacitive susceptance.* Expression (13) is the same as the following:

$$I_m = b_C U_m, \quad I = b_C U. \tag{2.14}$$

Thus, there are various parameters in sinusoidal current circuits, depending on the types of elements: resistance $r$, inductive reactance $x_L = \omega L$ and capacitive reactance $x_C = 1/(\omega C)$.

The current in an inductance lags behind the voltage and the current in a capacitance leads the applied voltage by an angle $\pi/2$.

It should be noted that only in the case of resistance $r$ is Ohm's law applicable to the instantaneous values of voltage and current. In the remaining cases the ratio of the instantaneous values of $u$ and $i$ do not represent reactance but reflect a certain time function of no practical interest and no use.

### 3. Series Connexion of $r$, $L$ and $C$

When a current $i$ flows through a circuit consisting of passive $r$, $L$ and $C$ elements connected in series (Fig. 5) a voltage drop is produced on the terminals of this circuit equal, in accordance with Kirchhoff's voltage law, to the algebraic sum of the voltage drops on the individual elements

$$u = u_r + u_L + u_C, \tag{2.15}$$

or, in differential form

$$u = ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i\,\mathrm{d}t. \tag{2.16}$$

* The term "capacitive reactance" used in Russian literature is the negative of that commonly used in English.

In view of the fact that for the moment we are considering steady-state conditions for an electrical circuit with a sinusoidal current, let us take

$$i = I_m \sin(\omega t + \alpha). \tag{2.17}$$

Voltage $u_r$ is in phase with current $i$, voltage $u_L$ leads, and voltage $u_C$ lags behind $i$, by $\pi/2$ (Fig. 6).

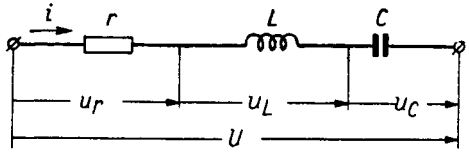



FIG. 5. A.c. circuit with series connexion of $r$, $L$ and $C$.

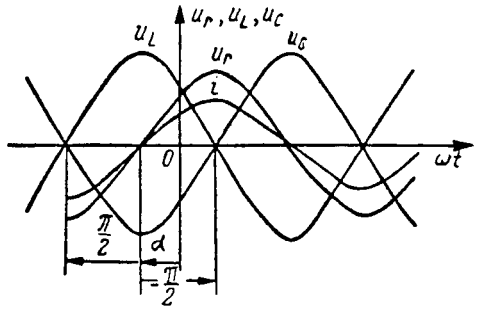



FIG. 6. Curves of voltage drop in $r$, $L$ and $C$.

Consequently, the voltage at the terminals of the circuit

$$u = rI_m \sin(\omega t + \alpha) + \omega L I_m \sin\left(\omega t + \alpha + \frac{\pi}{2}\right) + \frac{1}{\omega C} I_m \sin\left(\omega t + \alpha - \frac{\pi}{2}\right).$$

Bearing in mind that

$$\sin\left(\omega t + \alpha - \frac{\pi}{2}\right) = -\sin\left(\omega t + \alpha + \frac{\pi}{2}\right),$$

we find

$$u = U_m \sin(\omega t + \psi) = rI_m \sin(\omega t + \alpha) + \left(\omega L - \frac{1}{\omega C}\right) I_m \sin\left(\omega t + \alpha + \frac{\pi}{2}\right). \tag{2.18}$$

Equation (18) is a trigonometrical form of writing Kirchhoff's voltage law for a circuit with $r$, $L$ and $C$ connected in series.

The quantity $x = x_L - x_C = \omega L - 1/(\omega C)$ which appears in it is the *reactance* of the given circuit which, depending on the sign, may be inductive ($x > 0$) or capacitive ($x < 0$).

The vector diagram given in Fig. 7*b* corresponds to equation (18) for the case when $x > 0$, and Fig. 8*b* for the case when $x < 0$.

The voltage drops across the resistance and reactance are the sides of a right-angled voltage triangle, the hypotenuse of which is equal to the

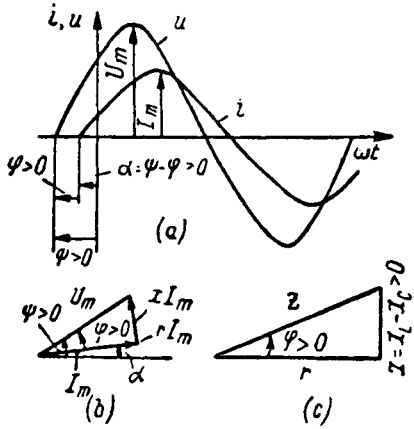



FIG. 7. Current lags behind voltage ($x > 0$): (*a*) sine curves of voltage and current; (*b*) vector diagram; (*c*) impedance triangle.

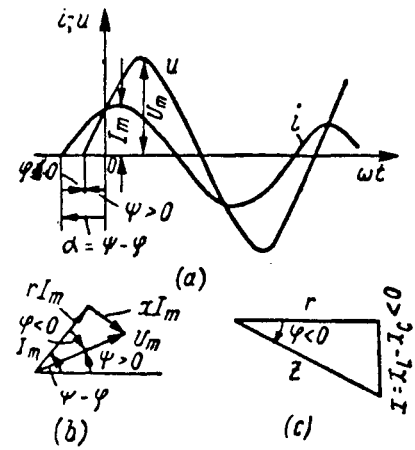



FIG. 8. Current leads voltage ($x < 0$): (*a*) sine curves of voltage and current; (*b*) vector diagram; (*c*) impedance triangle.

total drop across the circuit. Hence

$$U = \sqrt{\left[(rI)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 I^2\right]}$$

or

$$U = \sqrt{(r^2 + x^2)}\, I. \tag{2.19}$$

The expression obtained shows that the effective values, like the amplitude of the voltage and current, obey Ohm's law for alternating current

$$U = zI, \quad U_m = zI_m, \tag{2.20}$$

where

$$z = \sqrt{(r^2 + x^2)} = \sqrt{\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]} \tag{2.21}$$

is the magnitude of the impedance of the given circuit.

From the vector diagrams it follows that the angular phase difference between voltage $u$ and current $i$ (equal to the difference between the initial phases of the voltage and current), called *impedance angle*, is

$$\varphi = \psi - \alpha = \arctan \frac{x}{r} = \arctan \frac{\omega L - \dfrac{1}{\omega C}}{r}. \tag{2.22}$$

With a voltage $u = U_m \sin(\omega t + \psi - \varphi)$ at the terminals of a $r, L$ and $C$ circuit, the current is determined by the formula

$$i = \frac{U_m}{z} \sin(\omega t + \psi - \varphi), \tag{2.23}$$

where the angle $\varphi$ is read off the vector diagram in the direction from the current vector to the voltage vector, and is an acute or right angle: $|\varphi| \leqq \pi/2$.

Angle $\varphi$ is positive when the circuit reactance is inductive, i.e. when $x > 0$; the current then lags behind the voltage (Fig. 7).

Angle $\varphi$ is negative when the circuit reactance is capacitive, i.e. when $x < 0$; the current then leads the voltage (Fig. 8).

Thus, *angle $\varphi$ is positive for a lagging and negative for a leading current.*

The current is in phase with the voltage ($\varphi = 0$) when $x = x_L - x_C = 0$, i.e. when the inductive and capacitive reactances are equal. Such operating conditions in an electrical circuit are known as *series resonance* (Chapter VIII, Section 1).

By dividing the sides of a right-angled voltage triangle by $I_m$ we obtain a right-angled *impedance triangle* (Figs. 7*b* and 8*b*) which satisfies condition (21). Here impedance angle $\psi$ is read off from $r$ to $z$ which corresponds to reading from $I$ to $U$.

As the constructions show, resistance and reactance are related to the magnitude of impedance by the relationships

$$r = z \cos \varphi, \quad x = z \sin \varphi. \tag{2.24}$$

For the characteristic of inductance coils representing a circuit with $r$ and $L$ connected in series, the concept of the *quality factor* of the coil $Q_L = \omega L$, which is equivalent to the tangent of the impedance angle of the coil, is used in the theory of resonance circuits. The lower the resistance $r$, the higher the quality factor of the coil, all else being equal.

The quality factor of the inductance coils used in radio engineering is not usually higher than $Q_L = 200$ to 300.

In order to obtain higher quality factors so-called piezoelectric crystals are used (Chapter VIII, Section 7).

## 4. Parallel Connexion of $r$, $L$ and $C$

When a current $i$ flows through an electrical circuit consisting of $r$, $L$ and $C$ elements connected in parallel (Fig. 9), current $i$, in accordance with Kirchhoff's current law, is equal to the algebraic sum of the currents in the individual elements

$$i = i_r + i_L + i_C \tag{2.25}$$

or, in differential form

$$i = gu + C \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{L} \int u \, \mathrm{d}t. \tag{2.26}$$

Here $u$ is the voltage on the terminals of the said circuit. For a sinusoidal voltage $u = U_m \sin (\omega t + \psi)$ current $i_r$ is in phase with voltage $u$, current $i_L$ lags behind and current $i_C$ leads $u$ by an angle $\pi/2$.

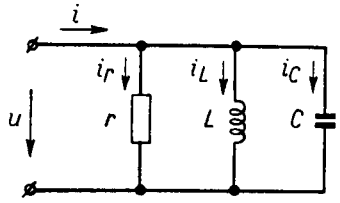



FIG. 9. A.c. circuit with parallel connexion of $r$, $L$ and $C$.

Consequently, the total current in the circuit equals

$$i = I_m \sin (\omega t + \alpha) = gU_m \sin (\omega t + \psi)$$

$$+ \frac{1}{\omega L} U_m \sin \left( \omega t + \psi - \frac{\pi}{2} \right) + \omega C U_m \sin \left( \omega t + \psi + \frac{\pi}{2} \right)$$

or

$$i = gU_m \sin(\omega t + \psi) + \left(\frac{1}{\omega L} - \omega C\right) U_m \sin\left(\omega t + \psi - \frac{\pi}{2}\right). \tag{2.27}$$

Equation (27) is the trigonometrical form of writing Kirchhoff's current law for a circuit with $r$, $L$ and $C$ connected in parallel. The quantity $b = b_L - b_C = (\omega L)^{-1} - \omega C$ which is contained in it is the *susceptance*

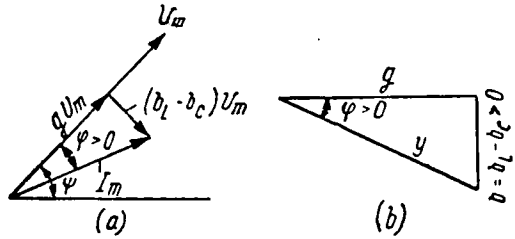



FIG. 10. When $b > 0$: ($a$) vector diagram; ($b$) admittance triangle.

of the given circuit, which, depending on the sign, may be either inductive ($b > 0$) or capacitive ($b < 0$).

The right-angled current triangles shown in Fig. 10, for $b > 0$, and Fig. 11 for $b < 0$, correspond to equation (27).

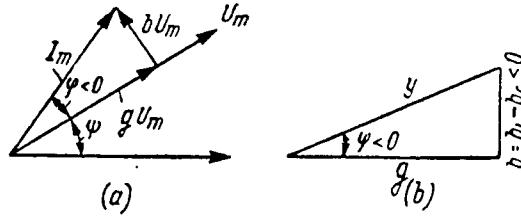



FIG. 11. When $b < 0$: ($a$) vector diagram; ($b$) admittance triangle.

It follows from the current triangle that

$$I = \sqrt{\left[(gU)^2 + \left(\frac{1}{\omega L} - \omega C\right)^2 U^2\right]}$$

or

$$I = yU, \quad I_m = yU_m. \tag{2.28}$$

Here

$$y = \sqrt{(g^2 + b^2)} = \sqrt{\left[g^2 + \left(\frac{1}{\omega L} - \omega C\right)^2\right]} \tag{2.29}$$

is the magnitude of the admittance of the given circuit.

The angular phase difference between voltage $u$ and current $i$ is equal to the difference between the initial phases of voltage and current

$$\varphi = \psi - \alpha = \arctan \frac{b}{g} = \arctan \frac{\dfrac{1}{\omega L} - \omega C}{g}. \tag{2.30}$$

Correspondingly, current $i$ is determined from the formula

$$i = yU_m \sin (\omega t + \psi - \varphi), \tag{2.31}$$

where angle $\varphi$ is read on the vector diagram in the direction from the current vector to the voltage vector, and is an acute or right angle $|\psi| \leqq \pi/2$.

Angle $\varphi$ is positive when the susceptance of the circuit is inductive, i.e. when $b > 0$; the current then lags behind the voltage (Fig. 10*a*).

Angle $\varphi$ is negative when the susceptance of the circuit is capacitive, i.e. when $b < 0$; the current then leads the voltage (Fig. 11*a*).

The current is in phase with the voltage when $b = b_L - b_C = 0$, i.e. when the inductive and capacitive susceptances are equal. These operating conditions in an electrical circuit are known as *parallel resonance* (Chapter VIII, Section 1).

Dividing the sides of the right-angled current triangle by $U_m$ we obtain a right-angled *admittance triangle* (Figs. 10*b* and 11*b*) which satisfies condition (29). Here admittance angle $\varphi$ is read off from $y$ to $g$, which corresponds to reading $\varphi$ from $I$ to $U$.

As the constructions show, conductance and susceptance are related to the magnitude of admittance by the relationships

$$g = y \cos \varphi, \quad b = y \sin \varphi. \tag{2.32}$$

For the characteristic of capacitors representing a circuit with $r$ and $C$ connected in parallel, the concept of the *quality factor of the capacitor* $Q_C = \omega Cr$, which is equivalent to the tangent of the angle $|\varphi|$ is used in circuit theory. The reciprocal is called the *loss-angle tangent* of the capacitor $\tan \delta = 1/Q_C$ (the loss angle $\delta$ supplements the angle $|\varphi|$ to 90°). The larger resistance $r$, the higher the quality factor of the capacitor and the less loss, other things being equal.

The quality factor of the capacitors used in radio engineering runs into hundreds and thousands (the magnitude of $\tan \delta$ for various frequencies and dielectrics may be in the range of $10^{-1}$ to $10^{-4}$).

Generally, if a branch or a section of a circuit has a conductance $g$ and a susceptance $b$, the resistance and the reactance of the given branch are determined from the expressions

$$r = \frac{g}{y^2} = \frac{g}{g^2 + b^2}, \quad x = \frac{b}{y^2} = \frac{b}{g^2 + b^2}, \tag{2.33a}$$

which are based on (24) taking into account the fact that

$$z = \frac{1}{y}, \quad \cos\varphi = \frac{g}{y}, \quad \sin\varphi = \frac{b}{y}.$$

If the section of a circuit, in its turn, has a resistance $r$ and a reactance $x$, the conductance and the susceptance of the given section are determined from the expressions

$$g = \frac{r}{z^2} = \frac{r}{r^2 + x^2}, \quad b = \frac{x}{z^2} = \frac{x}{r^2 + x^2}, \tag{2.33b}$$

which are based on (32), taking into account the fact that

$$y = \frac{1}{z}, \quad \cos\varphi = \frac{r}{z}, \quad \sin\varphi = \frac{x}{z}.$$

Expressions (33a) and (33b) show that the resistance and the reactance of the section depend on $g$ and $b$ simultaneously since the conductance and the susceptance depend on $r$ and $x$ simultaneously, the signs of $x$ and $b$ being identical, and $r$ and $g$ being positive.

It should be remembered that the relationships $r = 1/g$ and $x = 1/b$ hold only for each circuit element (resistance, inductance or capacitance) taken separately. For the entire section, expressions (33a) and (33b) are valid from which it will be seen that the resistance and the conductance of the whole section of the circuit, like the reactance and the susceptance of the section, are not mutually reciprocal.

## 5. Applications of Complex Quantities

In practice, electrical a.c. circuits are calculated in trigonometrical form or with the aid of vector diagrams only in the case of relatively simple circuits which do not contain a large number of loops and sources, magnetic couplings, etc.

As circuits become more complex, considerable difficulties occur in making calculations in trigonometrical form or with the aid of vector diagrams, and there arises the acute need of a method of calculation which makes it possible to calculate a.c. electrical circuits algebraically. The *complex amplitude method* (complex or symbolic method) based on the use of complex numbers is one such convenient method. This method is essentially the "algebra" of present-day electrical and radio engineering, whereas the vector diagram method is their "geometry".

It is generally known that a complex number is represented in a complex plane by a point, a real axis acting as the abscissa axis and an

imaginary axis as the ordinate axis in a rectangular system. Thus, in Fig. 12*a* the point with coordinates $A_1$ and $A_2$ represents a complex number $A_1 + jA_2$, where $j = \sqrt{(-1)}$ (in electrical engineering it is less convenient to use the notation $i = \sqrt{(-1)}$ since the letter $i$, as a rule, denotes current).

A complex number may be denoted conventionally by

$$\dot{A} = A_1 + jA_2. \tag{2.34}$$

Expression (34) is an *algebraic* form of writing a complex number.

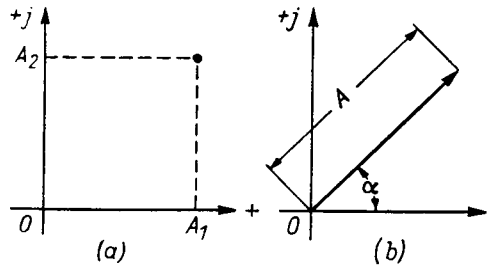



FIG. 12. Geometrical representation of a complex number in a complex plane: (*a*) by a point; (*b*) by a vector.

Each point in a complex plane is determined by the radius-vector of the point, i.e. a vector whose origin coincides with the origin of the coordinates and which terminates in a point corresponding to the given complex number (Fig. 12*b*).

Using a polar system of coordinates, we write the complex number in *polar* form

$$\dot{A} = A \angle \alpha \tag{2.35}$$

or, the same thing, in *exponential* form

$$\dot{A} = Ae^{j\alpha}. \tag{2.36}$$

Here $A$ is the modulus and $\alpha$ is the argument or phase.

Taking into account that $A_1 = A \cos \alpha$, $A_2 = \sin \alpha$, we get

$$\left.\begin{aligned} A &= \sqrt{(A_1^2 + A_2^2)}, \\ \alpha &= \arctan \frac{A_2}{A_1}. \end{aligned}\right\} \tag{2.37}$$

Correspondingly, the *trigonometrical* form of writing the complex number is given by

$$\dot{A} = A(\cos \alpha + j \sin \alpha). \tag{2.38}$$

Two complex numbers are considered equal if their real and imaginary parts are separately equal. Geometrically this denotes that the vectors representing the complex numbers are equal.

Thus, if $\dot{A} = A\angle\alpha$, $\dot{B} = B\angle\beta$, the equality $\dot{A} = \dot{B}$ is equivalent to the equalities $A = B$, $\alpha = \beta$.

The concepts of "larger" and "smaller" are applicable only to the coordinates of complex numbers, moduli and phases, these concepts not existing for the complex numbers themselves.

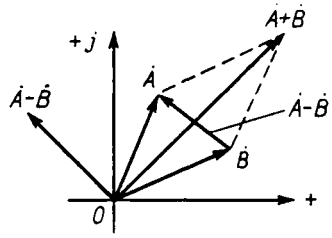



FIG. 13. Adding or subtracting complex numbers.

The convenience of using one or other form of writing a complex number depends in each individual case on the mathematical operations which should be carried out on the complex number.

Thus, when complex numbers are being added or subtracted the algebraic (or trigonometrical) form of writing a complex number is used:

$$\left.\begin{aligned}\dot{A} + \dot{B} &= (A_1 + B_1) + j(A_2 + B_2),\\ \dot{A} - \dot{B} &= (A_1 - B_1) + j(A_2 - B_2).\end{aligned}\right\} \tag{2.39}$$

In a geometrical interpretation for obtaining a vector representing the sum or the difference of complex numbers, vectors representing these numbers should be added or subtracted according to the rule of operations governing the vectors (Fig. 13).

When complex quantities are being multiplied or divided, it is most convenient to use the exponential (polar) form of writing a complex number:

$$\dot{A}\dot{B} = ABe^{j(\alpha+\beta)} = AB\angle\alpha + \beta, \tag{2.40}$$

$$\frac{\dot{A}}{\dot{B}} = \frac{A}{B}e^{j(\alpha-\beta)} = \frac{A}{B}\angle\alpha - \beta. \tag{2.41}$$

As can be seen from (40) and (41), the modulus of the product is equal to the product of the moduli of the factors, and the argument of the product is equal to the sum of their arguments; the modulus of the quotient

is equal to the quotient of the moduli of the dividend and divisor, and the argument of the quotient is equal to the difference of their arguments.

In the geometrical interpretation, the vector representing the product of $\dot{A}$ and $\dot{B}$ is obtained by rotating vector $\dot{A}$ anticlockwise through an angle $\beta$ (the argument of vector $\dot{B}$) and multiplying it by $B$ (Fig. 14*a*).

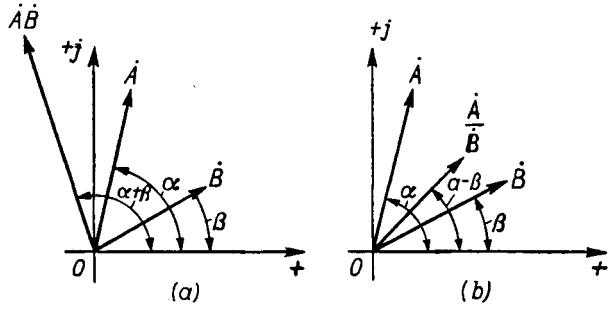



FIG. 14. Multiplying or dividing complex numbers.

Correspondingly, the vector representing the quotient of $\dot{A}$ by $\dot{B}$ is obtained by rotating vector $\dot{A}$ clockwise through an angle $\beta$ and dividing it by $B$ (Fig. 14*b*).

Two complex numbers (or vectors) are called *mutually conjugate* if their moduli are equal and the arguments are equal in magnitude and

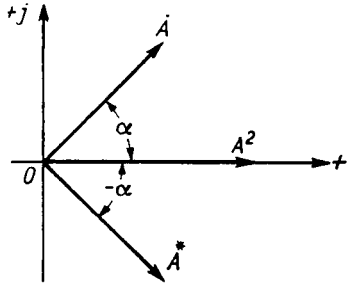



FIG. 15. Conjugate complex numbers and their product.

reversed in sign; in other words, conjugate complex numbers differ from each other only with respect to the sign of the imaginary part:

$$\left.\begin{aligned} \dot{A} &= Ae^{j\alpha} = A \angle \alpha = A_1 + jA_2, \\ \dot{A}^* &= Ae^{-j\alpha} = A \angle -\alpha = A_1 - jA_2. \end{aligned}\right\} \tag{2.42}$$

In the geometrical interpretation, the points representing conjugate numbers are arranged symmetrically with respect to the real axis (Fig. 15).

Conjugate complex numbers possess the important property that their product gives a real number equal to the square of the moduli of the factors:

$$\dot{A}\overset{*}{\dot{A}} = A^2. \tag{2.43}$$

In other words, the points representing the product of conjugate complex numbers are arranged along the real axis (Fig. 15).

When the algebraic form of writing complex numbers is used, the product of the two complex numbers has the form

$$\begin{aligned}\dot{A}\dot{B} &= (A_1 + jA_2)(B_1 + jB_2)\\ &= (A_1B_1 - A_2B_2) + j(A_2B_2 + A_1B_2).\end{aligned} \tag{2.44}$$

Correspondingly, dividing two complex numbers by multiplying and dividing the numerator and denominator by the conjugate denominator, gives

$$\begin{aligned}\frac{\dot{A}}{\dot{B}} &= \frac{(A_1 + jA_2)(B_1 - jB_2)}{(B_1 + jB_2)(B_1 - jB_2)}\\ &= \frac{A_1B_1 + A_2B_2}{B_1^2 + B_2^2} + j\frac{A_2B_1 - A_1B_2}{B_1^2 + B_2^2}.\end{aligned} \tag{2.45}$$

Expression (44) is the same as expression (40), differing only in its greater number of operations. The same should be said about expressions (45) and (41).

The raising of a complex number to a power $-1$ (finding the reciprocal of a complex number) is a particular case of division:

$$\begin{aligned}(\dot{A})^{-1} &= \frac{1}{A} = \frac{1}{A}e^{-j\alpha} = \frac{1}{A}\angle -\alpha\\ &= \frac{1}{A_1 + jA_2} = \frac{A_1}{A_1^2 + A_2^2} - j\frac{A_2}{A_1^2 + A_2^2}.\end{aligned} \tag{2.46}$$

The power and roots of complex numbers are found very conveniently in the exponential (polar) form:

$$\left.\begin{aligned}(\dot{A})^n &= A^n e^{jn\alpha} = A^n \angle n\alpha,\\ \sqrt[n]{\dot{A}} &= \sqrt[n]{A}e^{j\frac{\alpha}{n}} = \sqrt[n]{A}\angle\frac{\alpha}{n}.\end{aligned}\right\} \tag{2.47}$$

In Section 1 it was shown that it was possible to represent sinusoidal functions with rotating vectors (Fig. 1).

A vector whose end rotates in a positive direction (anticlockwise) with an angular velocity $\omega$ can be expressed analytically in the following way:

$$Ae^{j(\omega t+\alpha)} = \dot{A}e^{j\omega t}, \tag{2.48}$$

where $\dot{A} = Ae^{j\alpha} = A \angle \alpha$ is a *complex amplitude* representing the given vector at the instant $t = 0$ (Fig. 16).

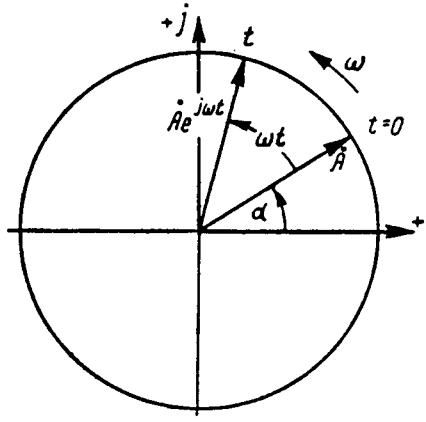



FIG. 16. Representation of a rotating vector in a complex plane.

The factor $e^{j\omega t}$ is a rotational operator: multiplying a complex amplitude by $e^{j\omega t}$ denotes that the vector $\dot{A}$ turns through an angle $\omega t$ in a positive direction.

A complex function can be expressed in trigonometrical form

$$Ae^{j(\omega t+\alpha)} = A\cos(\omega t + \alpha) + jA\sin(\omega t + \alpha), \tag{2.49}$$

i.e. each sinusoidal function $A\sin(\omega t + \alpha)$ may be regarded as the imaginary part of a complex function taken without the factor $j$, which, by mathematical convention, is written thus:

$$A\sin(\omega t + \alpha) = \operatorname{Im} Ae^{j(\omega t+\alpha)} = \operatorname{Im} \dot{A}e^{j\omega t}. \tag{2.50}$$

The symbol Im denotes that the imaginary part of the complex function without factor $j$ is taken.

Analogously, the cosinusoidal function can, where necessary, be represented as the real part of a complex function

$$A\cos(\omega t + \alpha) = \operatorname{Re} \dot{A}e^{j\omega t}, \tag{2.51}$$

where the symbol Re denotes that the real part of the complex function is taken.

The addition, subtraction, differentiation and integration of functions can be replaced, in accordance with the above, by the same mathematical operations on the imaginary parts of the complex functions. The operations on the imaginary parts of the complex functions can be replaced, in their turn, by operations on the complex functions themselves, subsequently extracting the imaginary part of the result obtained. This is explained by the commutability of the operations of addition, subtraction, differentiation and integration with respect to the symbolic operation Im or Re.

The given proposition follows from the following obvious identities:

$$\operatorname{Im}\left(\dot{A}_1 e^{j\omega_1 t}\right) \pm \operatorname{Im}\left(\dot{A}_2 e^{j\omega_2 t}\right) = \operatorname{Im}\left(\dot{A}_1 e^{j\omega_1 t} \pm \dot{A}_2 e^{j\omega_2 t}\right), \quad (2.52)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im}\left(\dot{A} e^{j\omega t}\right) = \operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{A} e^{j\omega t}\right), \quad (2.53)$$

$$\int \operatorname{Im}\left(\dot{A} e^{j\omega t}\right) \mathrm{d}t = \operatorname{Im} \int \dot{A} e^{j\omega t}\, \mathrm{d}t. \quad (2.54)$$

It should be noted that the rule of the commutability of the said operations with respect to the symbolic operation Im or Re does not extend to the operation of multiplication and division:

$$\operatorname{Im}\left(\dot{A}_1 e^{j\omega_1 t}\right) \cdot \operatorname{Im}\left(\dot{A}_2 e^{j\omega_2 t}\right) \neq \operatorname{Im}\left(\dot{A}_1 e^{j\omega_1 t} \cdot \dot{A}_2 e^{j\omega_2 t}\right), \quad (2.55)$$

$$\frac{\operatorname{Im} \dot{A}_1 e^{j\omega_1 t}}{\operatorname{Im} \dot{A}_2 e^{j\omega_2 t}} \neq \operatorname{Im} \frac{\dot{A}_1 e^{j\omega_1 t}}{\dot{A}_2 e^{j\omega_2 t}}. \quad (2.56)$$

In electrical engineering applications, quantities having two dimensions (complex numbers or vectors in a complex plane) are referred to as *phasors*. It has been suggested that the term *sinor* be applied to that phasor which represents a time-varying sinusoidal function. It is convenient to scale the modulus of a sinor to the effective value of a sinusoidal wave. In this book, therefore, the term sinor will be used synonymously with the term effective sinor, i.e. a r.m.s.-valued complex quantity which symbolically represents a sinusoid. Other complex numbers not associated with sinusoids are referred to as phasors.

## 6. Ohm's and Kirchhoff's Laws in Complex Notation

Let us examine the application of the complex method to a case of $r$, $L$ and $C$ elements connected in series and in parallel.

### Series Connexion of $r$, $L$ and $C$

Let us suppose that in the equation of voltages

$$u = ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i\,\mathrm{d}t \quad (2.57)$$

the parameters of $r$, $L$ and $C$ and the sinusoidal voltage $u = U_m \sin(\omega t + \psi)$ on the terminals of the circuit are given, and current $i$ is the value to be sought. In view of the fact that steady conditions in a circuit with a sinusoidal current are being considered here, the solution of the given differential equation will be found in the form of a sinusoidal function

$$i = I_m \sin(\omega t + \psi - \varphi).$$

In accordance with the previous paragraph, let the given sinusoidal voltage be symbolized by the complex function $\dot{U}_m e^{j\omega t}$ and the required sinusoidal current $i$ by the complex function $\dot{I}_m e^{j\omega t}$. The complex voltage and current amplitudes are, correspondingly, equal to $\dot{U}_m = U_m e^{j\psi}$ and $\dot{I}_m = I_m e^{j(\psi-\varphi)}$.

Differential equation (57) is then transcribed as follows:

$$\operatorname{Im} \dot{U}_m e^{j\omega t} = r \operatorname{Im} \left(\dot{I}_m e^{j\omega t}\right) + L \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \left(\dot{I}_m e^{j\omega t}\right) + \frac{1}{C} \int \operatorname{Im} \left(\dot{I}_m e^{j\omega t}\right) \mathrm{d}t. \tag{2.58}$$

Using the rules of the commutability of the operations of addition, differentiation and integration with respect to the symbolic operation Im, we convert equation (58) as follows:

$$\operatorname{Im} \dot{U}_m e^{j\omega t} = \operatorname{Im} \left( r\dot{I}_m e^{j\omega t} + L \frac{\mathrm{d}}{\mathrm{d}t} \dot{I}_m e^{j\omega t} + \frac{1}{C} \int \dot{I}_m e^{j\omega t} \mathrm{d}t \right). \tag{2.59}$$

Equation (59) is satisfied on condition* that the complex expressions on which operation Im is carried out in the given equation are equal to each other, i.e.

$$\dot{U}_m e^{j\omega t} = r\dot{I}_m e^{j\omega t} + L \frac{\mathrm{d}}{\mathrm{d}t} \dot{I}_m e^{j\omega t} + \frac{1}{C} \int \dot{I}_m e^{j\omega t} \mathrm{d}t$$

or, equivalently:

$$\dot{U}_m e^{j\omega t} = r\dot{I}_m e^{j\omega t} + j\omega L \dot{I}_m e^{j\omega t} + \frac{1}{j\omega C} \dot{I}_m e^{j\omega t}. \tag{2.60}$$

It should be noted that when the function $e^{j\omega t}$ is integrated, the integration constant is dropped since in the steady conditions being examined here the electrical charges or voltages on the capacitances represent sinusoidal functions not containing constant items.

* In view of the fact that equation (59) is true for an arbitrary instant $t$, the equality of the imaginary parts also causes the equality of the real parts of the complex expressions on which the operation Im is carried out.

As a result of reducing all parts of equation (60) by factor $e^{j\omega t}$ we obtain an algebraic complex equation expressing *Kirchhoff's voltage law* for complex amplitudes

$$\dot{U}_m = r\dot{I}_m + j\omega L\dot{I}_m + \frac{1}{j\omega C}\dot{I}_m. \tag{2.61}$$

After taking $\dot{I}_m$ out of the bracket and introducing the conventional notation for a phasor quantity

$$Z = r + j\omega L + \frac{1}{j\omega C} = r + j\left(\omega L - \frac{1}{\omega C}\right) \tag{2.62}$$

called *complex impedance* of the electrical circuit under examination, we obtain the equation

$$\dot{U}_m = Z\dot{I}_m, \tag{2.63}$$

expressing the a.c. version of *Ohm's law* for complex amplitudes. Dividing both parts of equation (63) by $\sqrt{2}$, we get Ohm's law for effective sinors

$$\dot{U} = Z\dot{I}. \tag{2.64}$$

Analogously, by dividing both parts of equation (61) by $\sqrt{2}$ we get Kirchhoff's voltage law in terms of sinors

$$\dot{U} = r\dot{I} + j\omega L\dot{I} + \frac{1}{j\omega C}\dot{I}. \tag{2.65}$$

Complex impedance $Z$ is represented in algebraic form by expression (62). The same quantity expressed in trigonometrical and exponential (polar) form is given by

$$Z = z\cos\varphi + jz\sin\varphi, \tag{2.66}$$

$$Z = ze^{j\varphi} = z\angle\varphi. \tag{2.67}$$

The magnitude of the impedance $z$ and impedance angle $\varphi$ are determined in accordance with (21) and (22), i.e.

$$z = \sqrt{\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}, \quad \varphi = \arctan\frac{\omega L - \frac{1}{\omega C}}{r}.$$

From (63), the complex amplitude of the current

$$\dot{I}_m = \frac{\dot{U}_m}{Z} = \frac{U_m}{z}e^{j(\psi-\varphi)}, \tag{2.68}$$

where $\psi - \varphi$ is the initial current phase. Consequently, the required current in trigonometrical form (i.e. instantaneous current) is

$$i = \operatorname{Im}\left(\dot{I}_m e^{j\omega t}\right) = \frac{U_m}{z}\sin(\omega t + \psi - \varphi), \Bigg\} \tag{2.69}$$

which corresponds to the result obtained in Section 3.

It should be noted that equation (65) can be regarded as the algebraic interpretation of the sinor diagram in Fig. 17 drawn in a complex plane,

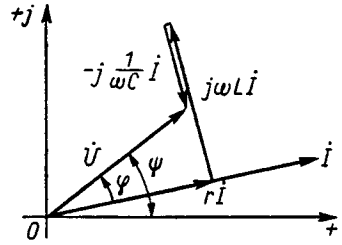



FIG. 17. Sinor diagram for the case of series connexion of $r$, $L$ and $C$.

where $\dot{U}_r = r\dot{I}$ is the voltage drop across the resistance $r$ (in phase with current $\dot{I}$); $\dot{U}_L = \omega L\dot{I}$ is the voltage drop across the inductance (leads the current by $\pi/2$); $\dot{U}_C = -j(\omega C)^{-1}\dot{I}$ is the voltage drop across the capacitance (lags behind the current by $\pi/2$).

**Parallel Connexion of $r$, $L$ and $C$**

For reasons analogous to those given above, Ohm's and Kirchhoff's laws for an a.c. circuit consisting of $r$, $L$ and $C$ elements connected in parallel can be put into a complex form.

Limiting ourselves to expressing current sinors we get, in accordance with *Kirchhoff's current law*

$$\dot{I} = g\dot{U} + \frac{\dot{U}}{j\omega L} + j\omega C\dot{U} = \dot{I}_r + \dot{I}_L + \dot{I}_C. \tag{2.70}$$

Here $\dot{I}_r = g\dot{U}$ is the current through resistance $r$ (in phase with voltage $\dot{U}$);

$\dot{I}_L = -j\dot{U}/\omega L$ is the current through the inductance (lags behind the voltage by $\pi/2$);

$\dot{I}_C = j\omega C\dot{U}$ is the current through the capacitance (leads the voltage by $\pi/2$).

The sinor diagram of Fig. 18 is a geometrical interpretation of equation (70).

A phasor quantity

$$Y = g + \frac{1}{j\omega L} + j\omega C = g - j\left(\frac{1}{\omega L} - \omega C\right) \tag{2.71}$$

is called *complex admittance* of the electrical circuit under examination. The equation

$$\dot{I} = Y\dot{U} \tag{2.72}$$

expresses the a.c. version of *Ohm's law* in complex notation.

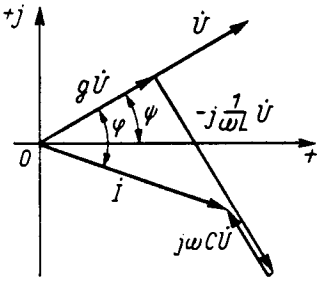



FIG. 18. Sinor diagram for the case of parallel connexion of $r$, $L$ and $C$.

The trigonometrical and exponential (polar) forms of complex admittance are given by

$$Y = y\cos\varphi - jy\sin\varphi \tag{2.73}$$

and

$$Y = ye^{-j\varphi} = y \angle -\varphi. \tag{2.74}$$

The magnitude of the admittance $y$ and angle $\varphi$ are determined in accordance with (29) and (30).

According to (72), the current sinor is

$$\dot{I} = yUe^{j(\psi-\varphi)}, \tag{2.75}$$

which symbolically represents a sinusoidal current

$$i = \mathrm{Im}\left(\dot{I}_m e^{j\omega t}\right) = yU_m \sin(\omega t + \psi - \varphi). \tag{2.76}$$

Generally, for the same section of circuit the relationship between resistance and reactance on the one hand, and conductance and susceptance on the other, is expressed by formulae (33). Using the complex form with the given impedance $Z = r + jx$, we find the admittance*

$$Y = \frac{1}{r + jx} = \frac{r}{r^2 + x^2} - j\frac{x}{r^2 + x^2} = g - jb. \tag{2.77}$$

* In English literature it is the custom to write $Y = g + jb$, so that susceptance $b$ used in this book is the negative of that used in English.

If, in its turn, the admittance $Y = g - jb$ is given, then the impedance is

$$Z = \frac{1}{g - jb} = \frac{g}{g^2 + b^2} + j\frac{b}{g^2 + b^2} = r + jx. \tag{2.78}$$

## 7. Power in Sinusoidal-current Circuits

If voltage and current are sinusoidal, i.e. $u = U_m \sin \omega t$ and $i = I_m \sin(\omega t - \varphi)$, then in accordance with expression (24) Chapter I, the instantaneous power

$$p = 2UI \sin \omega t \sin(\omega t - \varphi) = UI[\cos \varphi - \cos(2\omega t - \varphi)]. \tag{2.79}$$

Expression (79) shows that the instantaneous power consists of two parts: a constant value $UI \cos \varphi$ and a sinusoidal value $UI \cos(2\omega t - \varphi)$ with a frequency twice that of the voltage and current. Consequently, the rate at which energy enters a sinusoidal-current circuit is not constant.

The average power during a period, called *active power*, is equal to the constant item of expression (79), since the average value of the sinusoidal item completing two cycles during period $T$ is equal to zero, i.e.

$$P = \frac{1}{T}\int_0^T ui \, dt = UI \cos \varphi. \tag{2.80}$$

Active power is measured in watts (W). The factor $\cos \varphi$ is called the *power factor*. As is seen from (80), the active power is equal to the product of the effective values of the voltage and the current with the power factor. The power factor of an electrical energy receiver depends on the impedance angle of the given receiver; the nearer the angle $\varphi$ to zero, the nearer $\cos \varphi$ to unity and, consequently, the greater is the active power transmitted by the source to the receiver, for the given values of $U$ and $I$.

Increasing the power factor of industrial electrical plants is an important technical and economical problem.

Taking into account (20), (24), (28) and (32), let us convert the expression of active power:

$$\left.\begin{aligned} P &= zI^2 \cos \varphi = rI^2, \\ P &= yU^2 \cos \varphi = gU^2. \end{aligned}\right\} \tag{2.81}$$

Active power can also be expressed by a *resistive current* ($I_a = I \cos \varphi$) or a *resistive voltage drop* ($U_a = U \cos \varphi$):

$$P = UI_a \tag{2.82}$$

or

$$P = U_a I. \tag{2.83}$$

Three typical cases are considered below: a pure resistive circuit; a pure reactive circuit; a mixed circuit.

### Circuit with Pure Resistance ($\varphi = 0$)

When $\cos\varphi = 1$, in accordance with (79)

$$p = UI[1 - \cos 2\omega t], \tag{2.84}$$

i.e. the instantaneous power oscillates with a doubled angular frequency ($2\omega$) about the average value $P = UI$ (Fig. 19).

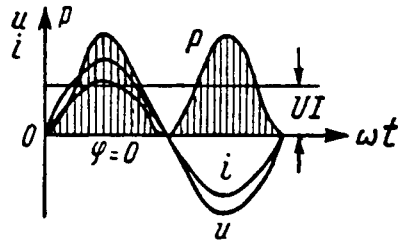



FIG. 19. Instantaneous and average power in a pure resistive circuit.

The instantaneous power remains positive: energy enters the receiver from the source and does not return to the source. All the energy supplied to the load is converted into heat.

### Pure Reactive Circuit ($\varphi = \pm \pi/2$)

When $\cos\varphi = 0$, in accordance with (79)

$$p = \mp UI \sin 2\omega t; \tag{2.85}$$

the upper sign corresponds to the case of an inductive circuit ($\varphi = \pi/2$), and the lower sign to the case of a capacitive circuit ($\varphi = \pi/2$). After each

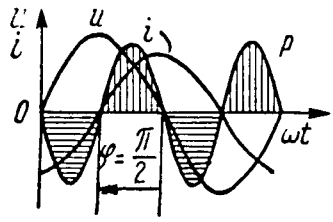



FIG. 20. Instantaneous power in a pure inductive circuit.

quarter of a period the sign of instantaneous power changes: the receiver first stores energy ($p > 0$), then expends it, returning it to the source ($p < 0$).

Energy entering from source to receiver is temporarily stored in a magnetic or electrical field and then returns to the source when the field disappears (Figs. 20 and 21).

In the case of an inductive circuit, the energy of the magnetic field reaches a maximum $LI_m^2/2$ at the instant the current through the inductance is transferred through its amplitude value; then it decreases and becomes zero when the current equals zero. Correspondingly, in the case of a capacitive circuit, the energy of the electrical field reaches a maximum $CU_m^2/2$ when there is an amplitude value of voltage across the capacitance; then it decreases and becomes zero when the voltage equals zero.

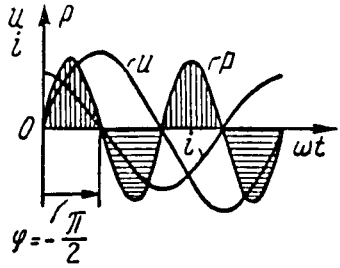



FIG. 21. Instantaneous power in a pure capacitive circuit.

Thus energy oscillations occur between the source and the receiver, the electromagnetic energy not being converted into other forms of energy, e.g. heat, and the active power being zero ($P = 0$).

In an inductive circuit the instantaneous power, in accordance with expression (26), Chapter I, is equal to the rate of change of the energy of the magnetic field, and, in accordance with expression (27), Chapter I, the instantaneous power in a capacitive circuit equals the rate of change of the energy of the electrical field.

**Circuit with an Impedance**

As an example, let us examine the case of an impedance circuit ($0 < \varphi < \pi/2$). On the basis of (79)

$$p = UI\cos\varphi - UI\cos(2\omega t - \varphi), \tag{2.86}$$

with

$$1 > \cos\varphi > 0.$$

As Fig. 22 shows, the instantaneous power oscillates with a double angular velocity ($2\omega$) about an axis displaced relative to the abscissa axis by $P = UI\cos\varphi$.

Although the instantaneous power takes negative values when $u$ and $i$ have different signs, during the greater part of the period it is positive and,

accordingly, the positive areas of curve predominate over the negative. As a result, the average power for the period, i.e. the active power $P > 0$.

In electrical systems in which the a.c. generators are the sources of electrical energy, power is obtained from engines rotating the generators. In radio engineering, where sinusoidal oscillations are created by electronic devices, power is obtained from d.c. sources supplying electron-tube oscillators or transistor oscillators.

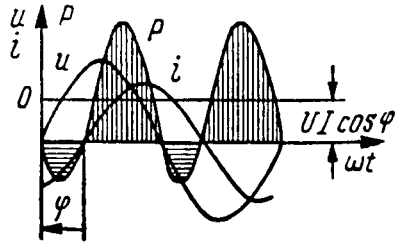



FIG. 22. Instantaneous and average power in a circuit with resistance and inductance.

The amplitude of the sinusoidal component of instantaneous power is equal to the product of the effective values of voltage and current

$$S = UI. \tag{2.87}$$

This quantity bears the name of *apparent power* and is measured in volt-amperes (VA).

On the basis of (80) and (87), the power factor equals the ratio of active power to apparent power

$$\cos \varphi = \frac{P}{S}. \tag{2.88}$$

In electrical circuit calculations the concept of *reactive power* is also used

$$Q = \sqrt{(S^2 - P^2)} = \sqrt{[(UI)^2 - (UI \cos \varphi)^2]} = UI \sin \varphi, \tag{2.89}$$

measured in reactive volt-amperes (VAR).

Obviously,

$$\sin \varphi = \frac{Q}{S}, \quad \tan \varphi = \frac{Q}{P}. \tag{2.90}$$

Taking (20), (24), (28) and (32) into account, let us convert the expression of reactive power:

$$\left.\begin{aligned} Q &= zI^2 \sin \varphi = xI^2, \\ Q &= yU^2 \sin \varphi = bU^2. \end{aligned}\right\} \tag{2.91}$$

Reactive power can also be expressed by the value of the reactive (inductive or capacitive) current ($I_r = I \sin \varphi$) or the value of the reactive voltage drop ($U_r = U \sin \varphi$):

$$Q = UI_r \tag{2.92}$$

or

$$Q = U_r I. \tag{2.93}$$

In accordance with the previously accepted rule of signs for angle $\varphi$, the value of the reactive power is positive when current lags (inductive load) and negative when current leads (capacitive load).

For inductance and capacitance, reactive power may be represented as follows:

$$\left.\begin{aligned} Q_L &= UI \sin \frac{\pi}{2} = UI = \omega L I^2 = \omega \frac{LI_m^2}{2} = \omega W_{L\max}, \\ Q_C &= UI \sin\left(-\frac{\pi}{2}\right) = -UI \\ &= -\omega C U^2 = -\omega \frac{CU_m^2}{2} = -\omega W_{C\max}, \end{aligned}\right\} \tag{2.94}$$

where $W_{L\max}$ and $W_{C\max}$ are the maximum values of energy stored periodically in the magnetic and electrical fields. The reactive power of the passive circuit equals the algebraic sum of the reactive powers for the inductances and capacitances contained in the given circuit

$$Q = \sum (Q_L + Q_C) = \omega\left(\sum W_{L\max} - \sum W_{C\max}\right). \tag{2.95}$$

Generally, $\sum W_{L\max}$ and $\sum W_{C\max}$ are twice the average values with respect to which the total instantaneous values of the energy of the magnetic and, correspondingly, electrical fields perform sinusoidal oscillations at a frequency of $2\omega$ (Ref. 80).

In a complex electrical circuit consisting of resistances, inductances and capacitances, the currents in the inductances, like the voltages across the capacitances, cannot be in phase. Therefore, in this kind of circuit $\sum W_{L\max}$ and $\sum W_{C\max}$ exceed the maximum values of energy periodically stored in the magnetic and electrical fields.

On the basis of the above dependences, the reactance and susceptance of the circuit can be expressed as the functions of $\sum W_{L\max}$ and $\sum W_{C\max}$

$$x = \frac{Q}{I^2} = \frac{\omega}{I^2}\left(\sum W_{L\max} - \sum W_{C\max}\right),$$

$$b = \frac{Q}{U^2} = \frac{\omega}{U^2}\left(\sum W_{L\max} - \sum W_{C\max}\right).$$

In the case of the equality $\sum W_{L\max} = \sum W_{C\max}$, reactance and susceptance, like the reactive power at the terminals of the circuit, equal zero, i.e. a resonance phenomenon takes place in the circuit: a continuous redistribution of the energy of the electrical and magnetic fields occurs, and all the energy from the source is consumed in the resistance of the circuit, i.e. is converted into heat energy (Chapter VIII, Section 1).

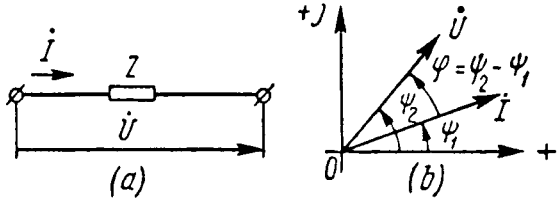



FIG. 23. (*a*) Reference directions; (*b*) current and voltage sinors.

Apparent power can be written in complex form. Let us suppose that a sinusoidal current is flowing through an electrical circuit, the positive directions of the current and the voltage at the circuit terminals being chosen so as to coincide (Fig. 23).

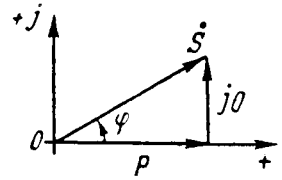



FIG. 24. Power triangle in a complex plane.

The effective current and voltage sinors are, respectively

$$\dot{I} = Ie^{j\psi_1} \quad \text{and} \quad \dot{U} = Ue^{j\psi_2}.$$

The angular phase difference between voltage and current equals the difference between their initial phases

$$\varphi = \psi_2 - \psi_1.$$

Let us multiply voltage sinor $\dot{U}$ by the current sinor $\overset{*}{\dot{I}}$, conjugate with $\dot{I}$, i.e.

$$\dot{U}\overset{*}{\dot{I}} = UIe^{j\psi_2}e^{-j\psi_1} = UIe^{j(\psi_2-\psi_1)}.$$

Hence it follows that

$$\dot{U}\overset{*}{\dot{I}} = UI\cos\varphi + jUI\sin\varphi$$

or, the same thing,

$$\dot{S} = \dot{U}\dot{I}^* = P + jQ. \tag{2.96}$$

Thus, the complex value $\dot{S}$ has a real part equal to the active power and an imaginary part equal to the reactive power; it is known as *complex power* and, being a phasor quantity, it can be represented graphically in a complex plane (Fig. 24).

Modulus $S$ equals apparent power.

Instead of product $\dot{U}\dot{I}^*$ several writers take product $\dot{U}^*\dot{I}$. In this case the imaginary part of the complex power is obtained with a minus sign, i.e. for a lagging current the reactive power is negative. The accepted procedure in the U.S.S.R. is to use the expression for complex power according to formula (96).

## 8. Maximum Power Transfer

In practice it is often necessary to select a load impedance so that, for a given source impedance, *maximum active power* is absorbed by the load. We shall denote the impedance of the voltage source in Fig. 25 by $Z_0 = r_0 + jx_0$ and the load impedance by $Z = r + jx$.

The active power consumed by the load is

$$P = I^2 r = \frac{U^2 r}{|Z_0 + Z|^2} = \frac{U^2 r}{(r_0 + r)^2 + (x_0 + x)^2}. \tag{2.97}$$

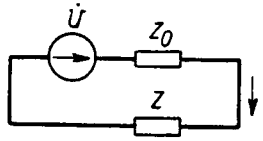



FIG. 25. Power transfer from a source to a receiver.

If the reactance $x$ is changed then, obviously, for any value of $r$ the current and, correspondingly, the active power reach their greatest value when

$$x = -x_0.$$

Then

$$P = \frac{U^2 r}{(r_0 + r)^2}. \tag{2.98}$$

We now find the condition of the maximum of function (97) on the assumption that $r$ is a variable quantity, i.e. from the condition $dP/dr = 0$

$$(r_0 + r)^2 - 2r(r_0 + r) = 0 \quad \text{or} \quad r_0 + r - 2r = 0.$$

Consequently, we get the additional condition

$$r = r_0.$$

On the basis of the equalities which have been discovered we can conclude that the condition for maximum power transfer from a source to a receiver is the equality

$$Z = \overset{*}{Z}_0, \tag{2.99}$$

where $\overset{*}{Z}_0$ is a complex impedance conjugate with $Z_0$.

When this condition is observed the active power to the load is

$$P_{\max} = \frac{U^2}{4r_0}, \tag{2.100}$$

and the efficiency of transmission is 0·5: the load receives half the power output of the source, the other half being consumed in the source resistance.

If the reactance of the source is inductive, on the basis of (99) the reactance of the receiver should be capacitive.

In practice, the reactance of the circuit is compensated by using capacitors connected to the load in series or in parallel.

Let us now examine the condition for maximum power transfer when the magnitude of the load impedance can be changed, but the angle of the load impedance is held constant.

The apparent power to the load is

$$S = I^2 z = \frac{U^2 z}{|Z_0 + Z|^2} = \frac{U^2 z}{z_0^2 \left|1 + \frac{z}{z_0} e^{j(\varphi - \varphi_0)}\right|^2},$$

where $\varphi$ and $\varphi_0$ are the angles of the impedances $Z$ and $Z_0$.

After converting

$$S = \left(\frac{U}{z_0}\right)^2 \frac{z}{1 + 2\frac{z}{z_0}\cos(\varphi - \varphi_0) + \left(\frac{z}{z_0}\right)^2}. \tag{2.101}$$

By taking $z$ as a variable quantity, it is possible to write, from the maximum of function $S$ (i.e. $dS/dz = 0$), that

$$1 + 2\frac{z}{z_0}\cos(\varphi - \varphi_0) + \left(\frac{z}{z_0}\right)^2 - z\left[2\frac{1}{z_0}\cos(\varphi - \varphi_0) + 2\frac{z}{z_0^2}\right] = 0$$

or

$$1 + \left(\frac{z}{z_0}\right)^2 - 2\left(\frac{z}{z_0}\right) = 0,$$

hence

$$z = z_0. \tag{2.102}$$

Substituting (102) in (101) gives the maximum value of apparent power

$$S_{\max} = \frac{U^2}{2z_0[1 + \cos(\varphi - \varphi_0)]}. \tag{2.103}$$

Thus, when the angle of a load impedance is held constant maximum power is absorbed by the load when the magnitude of the load impedance is equal to the magnitude of the source impedance. Then, the greater the absolute value of the difference between the angles of impedances $Z$ and $Z_0$, the greater the power transmitted.

The conditions for maximum power transfer are widely used in radio engineering, line communication, electrical automation and instrument making.

Electrical systems generating and consuming great power and striving to obtain high efficiency in the generators, achieve this when the load impedances are considerably larger in magnitude than the generator impedances.

## 9. Balance of Power

It follows from the law of conservation of energy that in any electrical circuit the law of the balance of active power is observed: the active power generated by the sources equals the total active power consumed by the load.

It can be shown, in turn, that the total output of reactive power is equal to the total reactive power consumed.

If complex expressions for currents, voltages and power are used, the said propositions will follow from these considerations.

For an electrical circuit containing $q$ nodes, it is possible to write, in accordance with Kirchhoff's current law, $q - 1$ equations in the form

$$\overset{*}{I}_{k1} + \overset{*}{I}_{k2} + \cdots + \overset{*}{I}_{k,q-1} + \overset{*}{I}_{kq} = 0,$$

where the positive directions of all the currents are taken from node $k$ to nodes 1, 2, etc. Let us multiply each of these equations by the voltage read from the corresponding node to the reference node $q$ and add up these products:

$$\dot{U}_{1q}\left(\overset{*}{I}_{12} + \overset{*}{I}_{13} + \cdots + \overset{*}{I}_{1q}\right) + \dot{U}_{2q}\left(\overset{*}{I}_{21} + \overset{*}{I}_{23} + \cdots + \overset{*}{I}_{2q}\right)$$
$$+ \cdots + \dot{U}_{q-1,q}\left(\overset{*}{I}_{q-1,1} + \overset{*}{I}_{q-1,2} + \cdots + \overset{*}{I}_{q-1,q}\right) = 0,$$

thus, taking into account that

$$\dot{U}_{kq} - \dot{U}_{mq} = \dot{U}_{km} \quad \text{and} \quad \overset{*}{\dot{I}}_{mk} = -\overset{*}{\dot{I}}_{km},$$

we obtain

$$\dot{U}_{12}\overset{*}{\dot{I}}_{12} + \dot{U}_{13}\overset{*}{\dot{I}}_{13} + \cdots + \dot{U}_{q-1,q}\overset{*}{\dot{I}}_{q-1,q} = 0.$$

Thus, the total complex power consumed in all the branches of the electrical circuit equals zero; consequently, the algebraic sums of the real and imaginary parts of the complex power, taken separately, also equal zero.

In other words, both the algebraic sum of the active power consumed in all the branches of the circuit and the algebraic sum of the reactive power consumed equal zero. In so far as negative consumed power is power output, the law of balance of both active and reactive power follows.

CHAPTER III

# NETWORK REDUCTION AND TRANSFORMATION

## 1. Simplification of Series–Parallel Networks

When solving electrical networks it is often advisable to reduce their circuit diagrams to simpler ones. Thus, for example, in a number of cases when there is one source or several sources of electrical energy an electrical network may be converted into a single-loop circuit or into a circuit with two nodes, which makes it considerably easier to find the quantities sought for.

One of the main forms of converting circuit diagrams often used in practice, is the reduction of a series–parallel network to a single equivalent impedance.

Electrical circuits with a series–parallel connexion can be reduced to a single-loop circuit by replacing the parallel branches by one branch and,

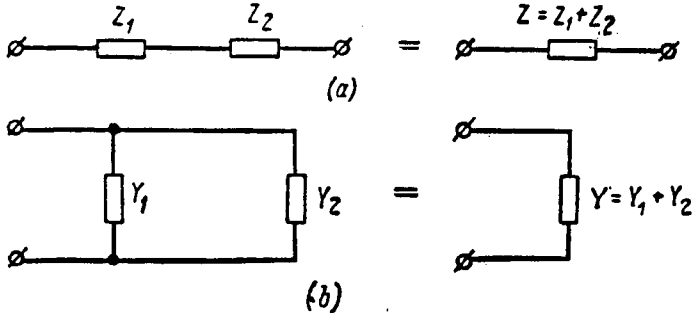



FIG. 1. Connexions: (*a*) series; (*b*) parallel.

accordingly, by replacing the impedances connected in series by one impedance.

It should then be borne in mind that in a series circuit (Fig. 1 *a*) individual complex impedances are added algebraically; and in a parallel circuit (Fig. 1 *b*) complex admittances are added algebraically.

The first proposition follows from the condition of equality of the currents in the impedances connected in series. If we denote the complex

impedances by $Z_1, Z_2, \ldots, Z_n$, then the total voltage on the whole circuit is determined from the formula

$$\dot{U} = Z_1\dot{I} + Z_2\dot{I} + \cdots + Z_n\dot{I} = (Z_1 + Z_2 + \cdots + Z_n)\dot{I} = Z\dot{I},$$

from which the net complex impedance of the circuit

$$Z = \sum_{k=1}^{n} Z_k. \tag{3.1}$$

If all the $Z_k = r_k + jx_k$ represent similar impedances, either inductive or capacitive (Fig. 2), then impedance $Z$ is found as a result of arithmetically

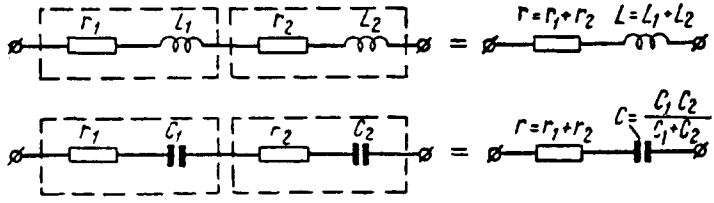



FIG. 2. Series circuit.

adding, individually, the resistances $r_k$ and inductances $L_k$ or inverse capacitances $1/C_k$:

$$Z = r + j\omega L \quad \text{or} \quad Z = r + \frac{1}{j\omega C},$$

where

$$r = \sum_{k=1}^{n} r_k, \quad L = \sum_{k=1}^{n} L_k, \quad \frac{1}{C} = \sum_{k=1}^{n} \frac{1}{C_k}.$$

The second proposition follows from the condition of the equality of the voltages at the terminals of the branches connected in parallel; if the complex admittances of these branches are denoted by $Y_1, Y_2, \ldots, Y_n$, the total current of the whole circuit is determined from the formula

$$\dot{I} = Y_1\dot{U} + Y_2\dot{U} + \cdots + Y_n\dot{U} = (Y_1 + Y_2 + \cdots + Y_n)\,\dot{U} = Y\dot{U},$$

from which the net admittance of the circuit

$$Y = \sum_{k=1}^{n} Y_k. \tag{3.2}$$

If all the $Y_k = g_k - jb_k$ represent similar admittances, either inductive or capacitive (Fig. 3), then admittance $Y$ is found as a result of arithme-

tically adding, individually, the conductances $g_k$ and capacitances $C_k$ or the inverse inductances $1/L_k$:

$$Y = g + j\omega C$$

or

$$Y = g - j\frac{1}{\omega L},$$

where

$$g = \sum_{k=1}^{n} \frac{1}{r_k}, \quad C = \sum_{k=1}^{n} C_k, \quad \frac{1}{L} = \sum_{k=1}^{n} \frac{1}{L_k}.$$

In accordance with the above, a series–parallel network is converted into a single-loop circuit in the following sequence: the net complex admittance of the parallel branches is determined; the reciprocal of this admittance is found (i.e. the complex impedance of the parallel branches); the impedance thus found is added to the complex impedance of the element connected in series. The resulting impedance is equivalent to the impedance of the initial series–parallel network.

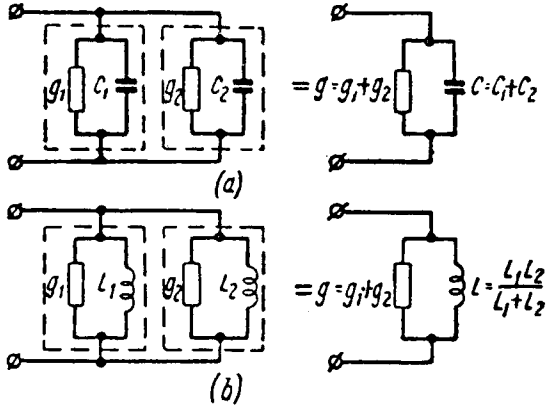



FIG. 3. Parallel circuit.

For example, in the case of the Fig. 4 circuit diagram, the admittance of the branches connected in parallel on the basis of (2) is equal to $Z_2^{-1} + Z_3^{-1}$ as a result of which the impedance of the whole circuit is in accordance with (1):

$$Z = Z_1 + \frac{1}{\frac{1}{Z_2} + \frac{1}{Z_3}} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}.$$

The total current in the circuit

$$\dot{I}_1 = \frac{\dot{U}}{Z} = \frac{\dot{U}}{Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}}.$$

This current is distributed along the parallel branches as follows: through $Z_2$ flows a current

$$\dot{I}_2 = \dot{I}_1 \frac{Y_2}{Y_2 + Y_3} = \dot{I}_1 \frac{Z_3}{Z_2 + Z_3} = \dot{U} \frac{Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3},$$

and through $Z_3$ a current

$$\dot{I}_3 = \dot{I}_1 \frac{Y_3}{Y_2 + Y_3} = \dot{I}_1 \frac{Z_2}{Z_2 + Z_3} = \dot{U} \frac{Z_2}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}.$$

Generally, if $n$ branches are connected in parallel the currents in them are in the same ratio as their complex admittances: the current in the $k$-th

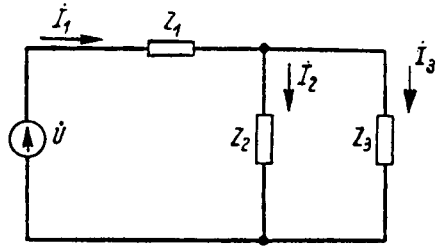



FIG. 4. Series–Parallel network.

branch is equal to the product of the total current in all the branches and the ratio of the admittance of the $k$-th branch to the sum of the admittances in all branches

$$\dot{I}_k = \dot{I} \frac{Y_k}{\sum_{k=1}^{n} Y_k}. \tag{3.3}$$

When $n$ elements are connected in series the voltages across them are in the same ratio as their complex impedances

$$\dot{U}_k = \dot{U} \frac{Z_k}{\sum_{k=1}^{n} Z_k}. \tag{3.4}$$

## 2. Series-to-Parallel Transformation

Let us denote the impedance of a circuit consisting of two elements connected in series by $Z = r + jx$. The admittance of the given circuit is equal to $Y = g - jb$, the resistance and susceptance being respectively

$$g = \frac{r}{r^2 + x^2} \quad \text{and} \quad b = \frac{x}{r^2 + x^2}.$$

If two elements with conductance $g$ and susceptance $b$ calculated by these formulae are coupled in parallel, the total admittance of the circuit

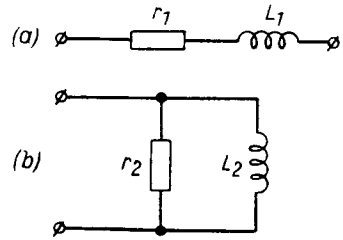



FIG. 5. Equivalent *rL* circuits, at fixed frequency: (*a*) series; (*b*) parallel.

will equal $Y$ and, correspondingly, the impedance at the terminals of the circuit will equal $Z$.

Two such circuits with series and parallel connexions and identical impedances at the terminals are called equivalent.

In view of the fact that the reactance appearing in the calculation formulae depends on frequency, the condition of the equivalence of these circuits is fulfilled only for that frequency at which reactance $x$ is calculated.

Let us take a circuit with a resistance $r_1$ and an inductance $L_1$ connected in series (Fig. 5*a*). We shall convert it into a circuit with the elements connected in parallel (Fig. 5*b*).

The conductance and susceptance of the initial circuit

$$g = \frac{r_1}{r_1^2 + (\omega L_1)^2}, \quad b = \frac{\omega L_1}{r_1^2 + (\omega L_1)^2}.$$

It follows from the condition of equivalence of the circuits in Fig. 5*a* and *b* that

$$r_2 = \frac{1}{g} = \frac{r_1^2 + (\omega L_1)^2}{r_1}, \quad \omega L_2 = \frac{r_1^2 + (\omega L_1)^2}{\omega L_1}. \tag{3.5}$$

By selecting frequency $\omega$ and computing parameters $r_2$ and $L_2$ according to formulae (5), we get a circuit equivalent to the original circuit for a

*single frequency* $\omega$ only; for other values of $\omega$ the circuits will have unequal complex impedances, i.e. the equivalence will be violated.

If the Fig. 5*b* circuit is the initial circuit and $r_2$, $L_2$ and $\omega$ are the given parameters, then parameters $r_1$ and $L_1$ of the equivalent circuit (Fig. 5*a*) are determined from expressions (33), Chapter II:

$$r_1 = \frac{g}{y^2} = \frac{\dfrac{1}{r_2}}{\left(\dfrac{1}{r_2}\right)^2 + \left(\dfrac{1}{\omega L_2}\right)^2},$$

$$\omega L_1 = \frac{b}{y^2} = \frac{\dfrac{1}{\omega L_2}}{\left(\dfrac{1}{r_2}\right)^2 + \left(\dfrac{1}{\omega L_2}\right)^2}. \tag{3.6}$$

It is also clear from these expressions that the numerical values of $r_1$ and $L_1$ of the equivalent circuit depend on the fixed frequency $\omega$.

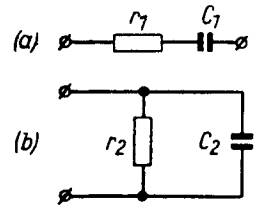



FIG. 6. Equivalent *rC* circuits, at fixed frequency: (*a*) series; (*b*) parallel.

Conditions of equivalence for circuits with a resistance and a capacitance connected in series and in parallel (Fig. 6) have the form:

$$r_2 = \frac{r_1^2 + \left(\dfrac{1}{\omega C_1}\right)^2}{r_1}, \qquad \frac{1}{\omega C_2} = \frac{r_1^2 + \left(\dfrac{1}{\omega C_1}\right)^2}{\dfrac{1}{\omega C_1}},$$

$$r_1 = \frac{\dfrac{1}{r_2}}{\left(\dfrac{1}{r_2}\right)^2 + (\omega C_2)^2}, \qquad \frac{1}{\omega C_1} = \frac{\omega C_2}{\left(\dfrac{1}{r_2}\right)^2 + (\omega C_2)^2}.$$

### 3. Star-to-Mesh Transformation

In the calculation of electrical circuits it is sometimes necessary to replace a part of a network connected in the form of a star by an equivalent mesh circuit, without affecting the rest of the network, i.e. keeping the currents and voltages in the remainder of the network the same.

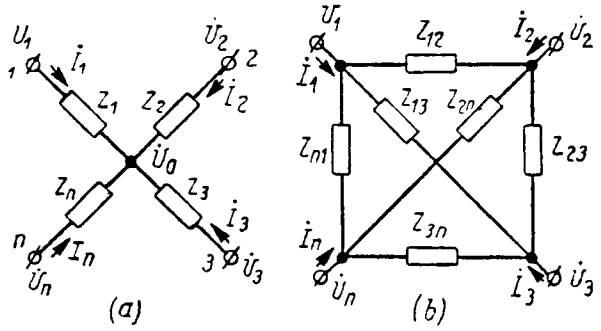



FIG. 7. Star-mesh transformation.

Figure 7 represents an equivalent $n$-ray star and $n$-sided mesh on the condition that the complex impedances of the branches of the $n$-sided mesh satisfy the following star-mesh transformation relations:

$$Z_{kl} = Z_k Z_l \sum_{k=1}^{n} \frac{1}{Z_k} \tag{3.7}$$

or, the same thing,

$$Y_{kl} = Y_k Y_l \frac{1}{\sum_{k=1}^{n} Y_k}. \tag{3.8}$$

Here the *equivalence* of the circuits connected by an $n$-ray star and an $n$-sided mesh is taken to mean that, for identical potentials on the like terminals, the currents flowing through the like terminals are identical. This is equivalent to saying that the amount of power consumed in each circuit is equal.

Formula (8) shows that the admittances of the sides and diagonals of the mesh are equal to the products of the admittances of the rays of the star adjoining the same terminals, divided by the sum of the admittances of all the rays.

For proof we shall denote the voltages at the nodes of the star with respect to a certain reference node selected outside the circuit, by $\dot{U}_0$, $\dot{U}_1$,

$\dot{U}_2, \ldots, \dot{U}_n$ and, correspondingly, the currents contained in the external nodes (terminals) of the star by $\dot{I}_1, \dot{I}_2, \ldots, \dot{I}_n$.

Then

$$\dot{I}_1 = Y_1\dot{U}_{10}, \quad \dot{I}_2 = Y_2\dot{U}_{20}, \ldots, \quad \dot{I}_n = Y_n\dot{U}_{n0}, \tag{3.9}$$

where

$$\dot{U}_{10} = \dot{U}_1 - \dot{U}_0, \quad \dot{U}_{20} = \dot{U} - \dot{U}_0, \ldots, \quad \dot{U}_{n0} = \dot{U}_n - \dot{U}_0$$

is the voltage between the external and internal nodes of the star; and $Y_1$, $Y_2, \ldots, Y_n$ are the admittances of the rays of the star.

On the basis of Kirchhoff's current law

$$\dot{I}_1 + \dot{I}_2 + \cdots + \dot{I}_n = 0 \quad \text{or} \quad Y_1\dot{U}_{10} + Y_2\dot{U}_{20} + \cdots + Y_n\dot{U}_{n0} = 0,$$

from which

$$\dot{U}_0 = \frac{Y_1\dot{U}_1 + Y_2\dot{U}_2 + \cdots + Y_n\dot{U}_n}{Y_1 + Y_2 + \cdots + Y_n}. \tag{3.10}$$

Substituting (10) in (9) yields

$$\left.\begin{aligned}
\dot{I}_1 &= \frac{1}{\sum_{k=1}^{n} Y_k}\left[Y_1Y_2\dot{U}_{12} + Y_1Y_3\dot{U}_{13} + \cdots + Y_1Y_n\dot{U}_{1n}\right], \\
\dot{I}_2 &= \frac{1}{\sum_{k=1}^{n} Y_k}\left[Y_2Y_1\dot{U}_{21} + Y_2Y_3\dot{U}_{23} + \cdots + Y_2Y_n\dot{U}_{2n}\right], \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{I}_n &= \frac{1}{\sum_{k=1}^{n} Y_k}\left[Y_nY_1\dot{U}_{n1} + Y_nY_2\dot{U}_{n2} + \cdots + Y_nY_{n-1}\dot{U}_{n,n-1}\right].
\end{aligned}\right\} \tag{3.11}$$

Here $\dot{U}_{12}$, $\dot{U}_{13}$, ... are the voltages between the nodes 1–2, 1–3, etc.

In the mesh network the external currents contained in the nodes are related to the voltages between the nodes by the formulae

$$\left.\begin{aligned}
\dot{I}_1 &= Y_{12}\dot{U}_{12} + Y_{13}\dot{U}_{13} + \cdots + Y_{1n}\dot{U}_{1n}, \\
\dot{I}_2 &= Y_{21}\dot{U}_{21} + Y_{23}\dot{U}_{23} + \cdots + Y_{2n}\dot{U}_{2n}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{I}_n &= Y_{n1}\dot{U}_{n1} + Y_{n2}\dot{U}_{n2} + \cdots + Y_{n,n-1}\dot{U}_{n,n-1}.
\end{aligned}\right\} \tag{3.12}$$

Equations (11) and (12) agree when conditions (8) are fulfilled, i.e. in that case the electrical circuits connected by a star and a mesh are equivalent.

For the particular case of a *three-ray star* ($n = 3$), or $Y$, the formulae of the impedances of the equivalent *three-terminal mesh*, or *delta* $\Delta$ (Fig. 8), based on equations (7) and (8), have the form:

$$Z_{12} = \frac{Z_1 Z_2}{Z}, \quad Z_{23} = \frac{Z_2 Z_3}{Z}, \quad Z_{31} = \frac{Z_3 Z_1}{Z}, \tag{3.13}$$

where

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \tag{3.14}$$

or

$$Y_{12} = \frac{Y_1 Y_2}{Y_1 + Y_2 + Y_3}, \quad Y_{23} = \frac{Y_2 Y_3}{Y_1 + Y_2 + Y_3},$$

$$Y_{31} = \frac{Y_3 Y_1}{Y_1 + Y_2 + Y_3}. \tag{3.15}$$

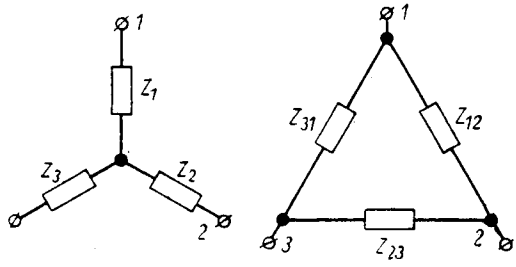



FIG. 8. Y-delta transformation.

## 4. Mesh-to-Star Transformation

The problem of converting a mesh into an equivalent $n$-ray star is solved with $n = 3$ for any values of the admittances of the mesh, and with $n > 3$ only for certain relationships between the said values (Ref. 24).

The formulae for replacing a delta network by an equivalent $Y$ (Fig. 8) ($n = 3$) is easily reduced on the basis of (13) and (14):

$$Z_{12} + Z_{23} + Z_{31} = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z} = \frac{Z_1 Z_2 Z_3}{Z_2},$$

$$Z_{12} Z_{23} = \frac{Z_1 Z_2^2 Z_3}{Z^2}.$$

Dividing the product $Z_{12}Z_{23}$ by the sum $Z_{12} + Z_{23} + Z_{31}$ gives

$$Z_2 = \frac{Z_{12}Z_{23}}{Z_{12} + Z_{23} + Z_{31}}. \tag{3.16}$$

By cyclic interchange of the subscripts of equation (16) we get

$$Z_1 = \frac{Z_{31}Z_{12}}{Z_{12} + Z_{23} + Z_{31}}, \tag{3.16a}$$

$$Z_3 = \frac{Z_{23}Z_{31}}{Z_{12} + Z_{23} + Z_{31}}. \tag{3.16b}$$

Consequently, in a delta-$Y$ transformation, the impedance of any $Y$ branch equals the product of the two adjacent delta-branch impedances divided by the sum of the three delta-branch impedances.

## 5. Equivalent Voltage and Current Sources

The two different sources of electrical energy—the voltage source and the current source—are considered *equivalent* if, when one source is substituted for the other, the currents and voltages in the external network to which these sources are connected remain unchanged.

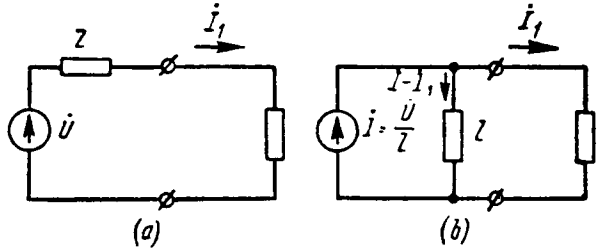



FIG. 9. Equivalent voltage and current sources.

Figure 9 shows equivalent voltage and current sources sending a current $\dot{I}_1$ into the external network and maintaining an identical voltage $\dot{U}_1$ at their terminals.

The following relationship between the e.m.f. of $\dot{U}$ and the current $\dot{I}$ of the sources serves as the condition for the equivalence of the sources and will henceforth be referred to as the rule of equivalent voltage and current sources:

$$\dot{U} = Z\dot{I}, \tag{3.17}$$

where $Z$ is the internal impedance both of the voltage source and of the current source.

In fact a voltage $\dot{U}_1$ at the terminals of the voltage source is obtained as a result of subtracting the voltage drop of current $\dot{I}_1$ in the source impedance $Z$ from the e.m.f. $\dot{U}$ (Fig. 9*a*).

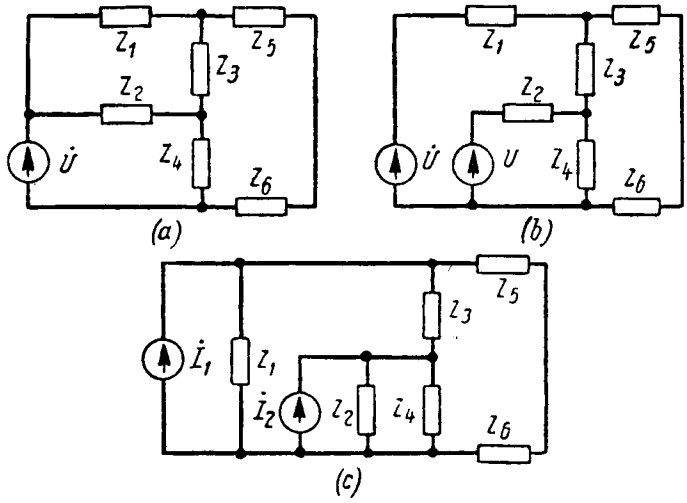



FIG. 10. Conversion of a zero-impedance voltage source into a current source.

Correspondingly, the voltage $\dot{U}_1$ at the terminals of the current source for a given value of current $\dot{I}_1$ fed into the external network is equal to the voltage drop of the current $\dot{I} - \dot{I}_1$ in impedance $Z$ of the source (Fig. 9*b*).

In both cases with the same impedance in the external network, the voltages at the terminals are:

$$\dot{U} - Z\dot{I}_1 = Z(\dot{I} - \dot{I}_1) = Z\dot{I} - Z\dot{I}_1,$$

i.e. condition (17) is obtained.

It should be noted that the values of the power inside the equivalent voltage and current sources are not equal. In the first place the apparent power consumed in the source is $zI_1^2$; in the second $z\,|\dot{I} - \dot{I}_1|^2$. For example, when the sources are disconnected from the external circuit ($\dot{I}_1 = 0$), the power in the source is not consumed in the first case, and is $zI^2$ in the second case.

Therefore the equivalence of the sources should be understood only in the sense of the constancy of the currents, voltages and power in the external network connected to the sources.

If the internal impedance of the voltage source is zero, then formula (17), for finding the equivalent current source for a given value of the source's e.m.f., cannot be applied directly. In such cases the impedance of the external circuit connected in series with the e.m.f. should be regarded as an internal impedance of the source, which makes it possible to determine the required current source for the given e.m.f. from condition (17).

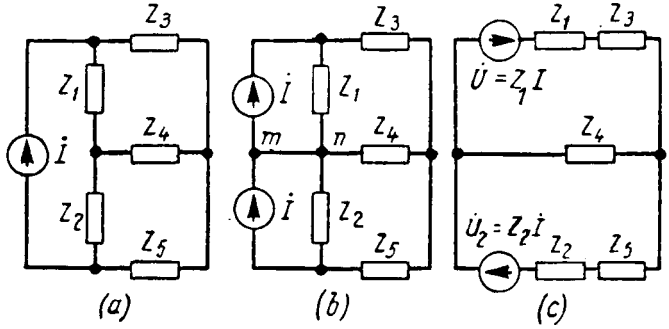



FIG. 11. Conversion of an infinite-impedance current source into a voltage source.

In a case where a branch contains an e.m.f. only, i.e. the impedance of the branch is zero, which happens, for example, in the Fig. 10*a* circuit, then it is impossible to replace the given source of e.m.f. $\dot{U}$ by an equivalent current source. In that case, an artificial procedure is used for replacing the Fig. 10*a* circuit by the equivalent circuit with two identical e.m.f. $\dot{U}$ sources shown in Fig. 10*b*, as a result of which these e.m.f.s are connected in series with impedances $Z_1$ and $Z_2$. Regarding impedances $Z_1$ and $Z_2$ as internal impedances of the voltage sources and using the rule for replacing the voltage source by an equivalent current source, we get the Fig. 10*c* circuit with two current sources

$$\dot{I}_1 = \frac{\dot{U}}{Z_1} \quad \text{and} \quad \dot{I}_2 = \frac{\dot{U}}{Z_2}.$$

An analogous obstacle with respect to the direct use of formula (17) also arises in the case of converting a given current source into a voltage source if the internal impedance of the current source equals infinity, which takes place, for example, in Fig. 11*a*.

In this case the artificial procedure for converting the Fig. 11*a* circuit into the Fig. 11*b* circuit can be used, the distribution of currents and voltages in the latter not differing from the given in so far as the current in section *mn* equals zero. If we then regard $Z_1$ and $Z_2$ as internal impedances of both current sources, then, on the basis of (17), the Fig. 11*c* circuit, containing two current sources, is obtained.

In the case of complex networks, replacing the voltage source by an equivalent current source or vice versa may sometimes simplify calculations.

## 6. Conversion of Circuits with Two Nodes*

Let us apply the rule on equivalent voltage and current sources to converting a circuit with $n$ branches coupled in parallel and containing voltage sources (Fig. 12*a*).

Replacing the given voltage sources by current sources, we obtain the Fig. 12*b* circuit. The current sources collectively form one equivalent current source $\dot{I}$ (Fig. 12*c*), with

$$\dot{I} = \dot{I}_1 + \dot{I}_2 + \cdots + \dot{I}_n = \frac{\dot{U}_1}{Z_1} + \frac{\dot{U}_2}{Z_2} + \cdots + \frac{\dot{U}_n}{Z_n} = \sum_{k=1}^{n} \dot{U}_k Y_k. \quad (3.18)$$

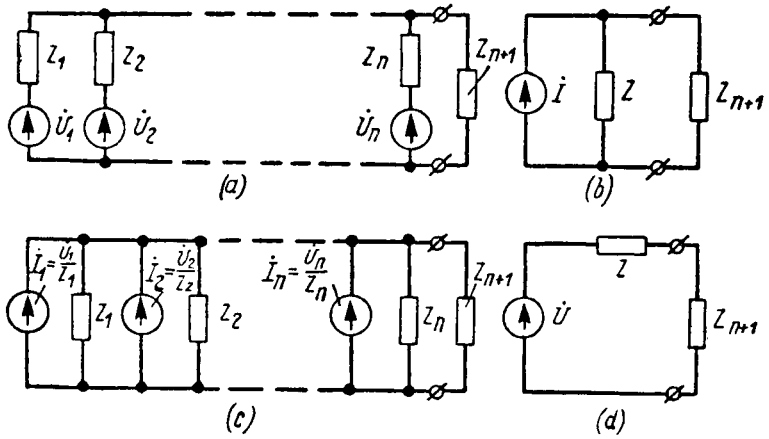



FIG. 12. Parallel combination of voltage sources.

By using the correlation (17), it is finally possible to proceed from the Fig. 12*c* circuit to the Fig. 12*d* circuit which is equivalent to the initial Fig. 12*a* circuit. Here

$$\dot{U} = \frac{\sum_{k=1}^{n} \dot{U}_k Y_k}{\sum_{k=1}^{n} Y_k}, \quad (3.19)$$

$$Z = \frac{1}{\sum_{k=1}^{n} Y_k}. \quad (3.20)$$

* Known as Millman's theorem.

Thus, $n$ parallel branches with voltage sources between two nodes can be replaced by one current source (Fig. 12*c*) or a voltage source (Fig. 12*d*).

The current in the branch with impedance $Z_{n+1} = 1/Y_{n+1}$ equals

$$\dot{I}_{n+1} = \dot{I} \frac{Y_{n+1}}{\sum_{k=1}^{n+1} Y_k} = \frac{\dot{U}}{Z + Z_{n+1}}. \tag{3.21}$$

The voltage between the two nodes is found by the formula

$$\dot{U}_{12} = \dot{I} \frac{1}{\sum_{k=1}^{n+1} Y_k} = \frac{\sum_{k=1}^{n} \dot{U}_k Y_k}{\sum_{k=1}^{n+1} Y_k}. \tag{3.22}$$

Formulae (18)–(22) are widely used in practice for calculating electrical circuits with *two nodes* and also for calculating more complicated networks, reduced to a circuit with two nodes.

## 7. Mobility of Sources

The equations of Kirchhoff's voltage law (Chapter IV, Sections 1 and 2) show that the currents in a circuit are determined by the given values of the resultant e.m.f. in each loop regardless of what separate items they consist. Therefore, changing the position of the e.m.f. sources in the circuit for which the resultant e.m.f. appearing in each loop remains constant has no effect on the value of the currents.

Analogously, the equations of Kirchhoff's current law (Chapter IV, Section 3) shows that the voltages between the nodes are determined by the given values of net source currents at the nodes, regardless of the position of the individual current sources. Changing the arrangement of the current sources in the circuit for which the summary current fed into each node remains constant, has no effect on the values of the voltage between the nodes.

The e.m.f. source can be removed from any branch by introducing a compensating e.m.f. into the given branch, this e.m.f. also being introduced into all the other branches connected to one of the nodes in the given branch (the compensating e.m.f. and the supplementary e.m.f.s have the same direction with respect to the node). As a result, the e.m.f. source is removed from the branch and e.m.f. sources appear in the other branches of the circuit.

If there are identical e.m.f. sources all directed towards (or away from) one node in all the branches, except one, connected at the node (Fig. 13*a*),

then they can be replaced by an analogous procedure by one e.m.f. source in the branch in which there was no source (Fig. 13*b*).

Thus, when e.m.f. sources are being transferred, the compensating and the other e.m.f.s are introduced into the network in such a way that the loop e.m.f.s do not change.

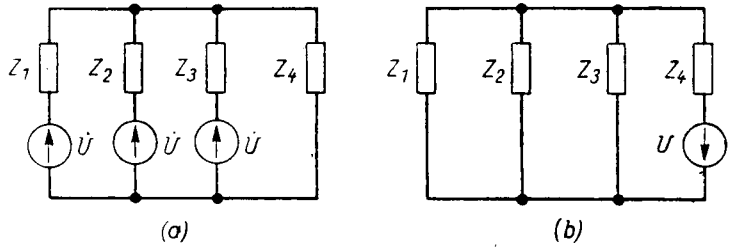



FIG. 13. Illustration of mobility of e.m.f. sources.

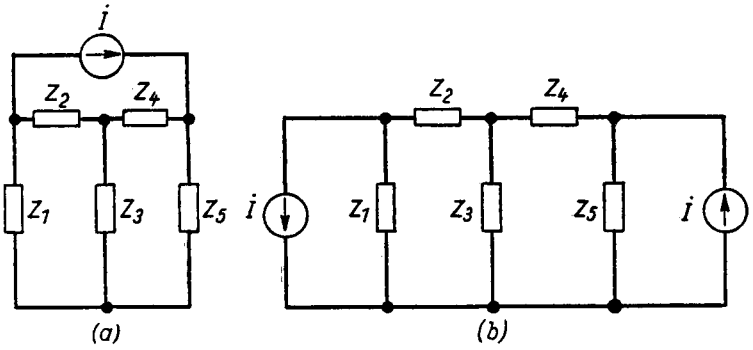



FIG. 14. Illustration of mobility of current sources.

When transferring current sources the compensating source and the other current sources are connected to the circuit nodes in such a way that the sum of the source currents is unchanged at each node.

Thus, for example, the voltages between the nodes in the circuits shown in Fig. 14*a* and *b* are indentical, despite the fact that the arrangements of their current sources differ.

CHAPTER IV

# METHODS OF SOLVING NETWORKS

## 1. Direct Application of Kirchhoff's Laws (Branch-current Method)

There are two types of problem in circuit theory: the problems of analysis and synthesis.

By network *analysis* we understand the calculation of the response of the given network, i.e. the determining of the electrical values or their relationships for the given circuit.

By network *synthesis* we mean the reverse problem; namely, determining an electrical circuit meeting certain prescribed specifications, which are usually given as frequency or time functions.

Unlike analysis problems which, as a rule, have only one solution, problems of synthesis may have a few or many solutions, or none.

This chapter is devoted to general methods of network analysis in steady-state conditions. Some topics of network synthesis will be discussed in subsequent sections of the book, particularly in Chapters VI–VIII.

With a given network and its parameters, it is possible to obtain the desired branch currents and potentials by directly solving network equations which express Kirchhoff's current and voltage laws (so called branch-current method).

Let us suppose that in a network containing $p$ branches and $q$ nodes the voltage sources are given and the currents in the branches are to be found. The number of unknowns, therefore, equals the number of branches.

According to *Kirchhoff's current law* $q-1$ independent equations can be written; the equation for the last $q$-th node is the result of the preceding $q-1$ equations. Actually, in view of the fact that each branch connects two nodes, the current of each branch appears twice in the equations written for $q$ nodes. If, therefore, the $q$ equations are added, the result is an identity in the form $0 = 0$. Consequently, one of the equations is dependent.

Nodes for which independent equations based on Kirchhoff's current law can be written are called *independent nodes*. It follows from the above that any $q-1$ nodes of a general number of nodes $q$ are independent, while the remaining node is dependent.

According to *Kirchhoff's voltage law* $p-q+1$ independent equations can be written. This is explained by the fact that if Ohm's law is applied

to all the branches, then $p$ equations in the form

$$\dot{U}_{ik} = -\dot{U}_n + Z_n\dot{I}_n \tag{4.1}$$

are obtained.

Here $\dot{U}_{ik}$ is the voltage between nodes $i$ and $k$;

$\dot{U}_n, \dot{I}_n$ are the e.m.f. source and the current in the $n$-th branch between nodes $i$ and $k$;

$Z_n$ is the impedance of the same branch.

A set of equations of the form of (1) which must be satisfied simultaneously includes $p$ unknown currents $\dot{I}_n$ and $q - 1$ unknown nodal potentials (the potential of one node being taken as zero). If $q - 1$ potentials are excluded from the system, then $p - q + 1$ equations remain, connecting the e.m.f. sources with the voltage drops in the impedances and expressing Kirchhoff's voltage law.

The calculation of electrical circuits with the aid of Kirchhoff's current and voltage laws is thus reduced to solving $(q-1) + (p-q+1) = p$ simultaneous equations, according to the number of branches.

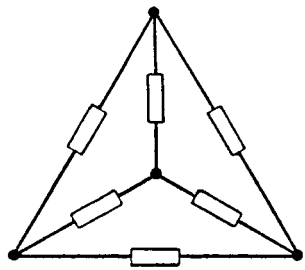


FIG. 1. A network of three independent loops.

Loops for which equations based on Kirchhoff's voltage law are independent are called *independent loops*. For voltage-law equations, and consequently the loops themselves, to be independent, it is sufficient that each loop should differ from the preceding loop if only with respect to one new branch.

As an example, Fig. 1 shows a network with a number of branches $p = 6$ and a number of nodes $q = 4$. Consequently the Kirchhoff's current law yields $4 - 1 = 3$ equations and Kirchhoff's voltage law yields $6 - 4 + 1 = 3$ equations. The network shown in Fig. 1 has three independent loops.

It should be noted that in individual cases an independent loop may not contain a new branch (e.g. loop 4 in the network shown in Fig. 2).

For choosing a suitable set of independent loops the following procedure may be carried out. We find any loop in the network. Then we image that

any one branch of this loop has been removed from the network. In the network that remains, we find any loop, and then remove one branch from this loop. This procedure is to be continued step by step until no loop remains and the process terminates. The loops thus set up are independent and suffice to solve the network.

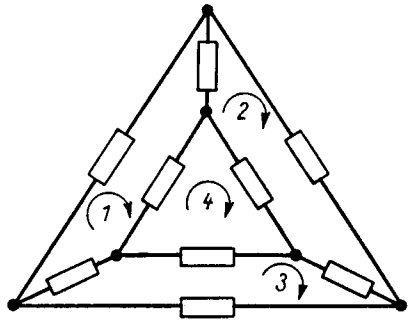



FIG. 2. A network of four independent loops.

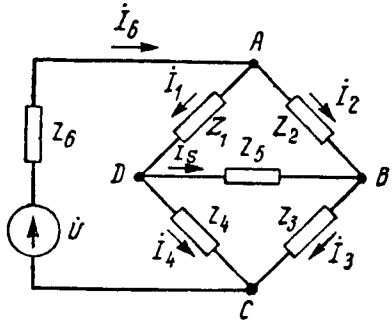



FIG. 3. Example 1 (bridge network).

EXAMPLE 1 (Ref. 24). In the bridge circuit represented in Fig. 3, all the impedance and the e.m.f. $\dot{U}$ are given. We are required to find current $\dot{I}_5$ in branch $Z_5$ (the current in the diagonal of the bridge circuit). The circuit contains four nodes and six branches. Consequently, three equations can be compiled from Kirchhoff's current law and three from Kirchhoff's voltage law:

for node $A$ $\quad -\dot{I}_1 - \dot{I}_2 + \dot{I}_6 = 0;$

for node $B$ $\quad I_2 - \dot{I}_3 + \dot{I}_5 = 0;$

for node $C$ $\quad \dot{I}_3 + \dot{I}_4 - \dot{I}_6 = 0;$

for loop $ABDA$ $\quad -Z_1\dot{I}_1 + Z_2\dot{I}_2 - Z_5\dot{I}_5 = 0;$

for loop $BCDB$ $\quad Z_3\dot{I}_3 - Z_4\dot{I}_4 + Z_5\dot{I}_5 = 0;$

for loop $ABCA$ $\quad Z_2\dot{I}_2 + Z_3\dot{I}_3 + Z_6\dot{I}_6 = \dot{U}.$

In the set of six equations obtained the currents in the branches are unknown. Solving the set of equations with respect to the unknown current we find

$$\dot{I}_5 = \frac{\dot{U}}{M}(Z_2Z_4 - Z_1Z_3),$$

where

$$M = Z_5[(Z_1 + Z_4)(Z_2 + Z_3) + Z_6(Z_1 + Z_2 + Z_3 + Z_4)]$$

$$+ Z_1Z_4(Z_2 + Z_3) + Z_2Z_3(Z_1 + Z_4) + Z_6(Z_1 + Z_2)(Z_3 + Z_4).$$

The expression obtained shows that the current in the diagonal is equal to zero if the condition $Z_1Z_3 = Z_2Z_4$ is fulfilled (fundamental balance relation).

## 2. Loop-current Method*

The *loop-current method* consists in determining the so-called loop or circulating currents (instead of determining the branch currents), using only Kirchhoff's voltage law.

As an example Fig. 4 shows a network of two loops in which the loop currents $\dot{I}_1$ and $\dot{I}_2$ are unknown. The currents in the impedances $Z_1$ and $Z_2$ are equal to the corresponding loop currents; the current in impedance $Z_3$

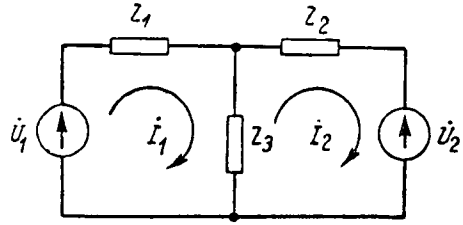



FIG. 4. Loop currents in a two-loop network.

which is common to both loops is equal to the difference between loop currents $\dot{I}_1$ and $\dot{I}_2$ since these currents are opposed in branch $Z_3$.† If the positive direction of the unknown current in branch $Z_3$ is then taken to coincide with the direction of loop current $\dot{I}_1$, the current in the branch will equal $\dot{I}_1 - \dot{I}_2$. And if the positive direction of the unknown current of branch $Z_3$ is selected to coincide with the direction of loop current $\dot{I}_2$, it will equal $\dot{I}_2 - \dot{I}_1$.

* Known as Maxwell's circulating current method.

† If the positive direction of one of the loop currents ($\dot{I}_1$ or $\dot{I}_2$) is reversed, the current in arm $Z_3$ will be equal to the sum of these currents.

The number of equations based on Kirchhoff's voltage law which can be written for the loop currents equals the number of independent loops, i.e. for an electrical circuit with $q$ nodes and $p$ branches, the problem of finding loop currents is reduced to solving a set of simultaneous $p - q + 1$ equations. Thus, in the Fig. 4 diagram $q = 2, p = 3$, and so the number of equations is $3 - 2 + 1 = 2$ (in accordance with the number of independent loops).

Let us agree to call the sum of the complex impedances included in the loop the *self-impedance* of the loop, and the complex impedance common to two loops the *mutual impedance* of these loops.

The positive directions of these loop currents are set arbitrarily; it is assumed that each loop is followed in the current direction. Then the voltage drop caused by the given loop current in the self-impedance of the loop is taken with a plus sign in voltage-law equations. The voltage drop caused by the current of the adjacent loop in the impedance common to both adjacent loops should be taken with a minus sign if the loop currents in this impedance are opposed, as happens, for example, in the circuit shown in Fig. 4 where the direction of both loop currents is taken as being clockwise.

In accordance with the Fig. 4 circuit, the self-impedance of loop *1* is $Z_1 + Z_3$, and impedance $Z_3$, which belongs to both loops at once is a mutual impedance of loops *1* and *2*, a minus sign being ascribed to it in accordance with positive directions taken for the loop currents in Fig. 4 (see Example 2).

If a given electrical circuit contains $n$ independent loops then, on the basis of Kirchhoff's voltage law, we get a set of $n$ equations:

$$\left.\begin{aligned} \dot{U}_1 &= Z_{11}\dot{I}_1 + Z_{12}\dot{I}_2 + \cdots + Z_{1n}\dot{I}_n, \\ \dot{U}_2 &= Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2 + \cdots + Z_{2n}\dot{I}_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{U}_n &= Z_{n1}\dot{I}_1 + Z_{n2}\dot{I}_2 + \cdots + Z_{nn}\dot{I}_n. \end{aligned}\right\} \tag{4.2}$$

Where $\dot{U}_i$ is the total e.m.f. in loop $i$, i.e. the algebraic sum of the e.m.f.s appearing in the given loop, the rise in loop-current direction being taken as positive;

$Z_{ii}$ is the self-impedance of loop $i$;

$Z_{ik}$ is the mutual impedance of loops $i$ and $k$; it requires the negative sign if the loop currents in loops $i$ and $k$ have opposite directions in the common impedance of these loops.

The solution of the set of simultaneous equations (2) with respect to the unknown loop currents can be found in terms of the determinants:

$$\dot{I}_1 = \frac{1}{\Delta}\begin{vmatrix} \dot{U}_1 & Z_{12} & \dots & Z_{1n} \\ \dot{U}_2 & Z_{22} & \dots & Z_{2n} \\ \dots & \dots & \dots & \dots \\ \dot{U}_n & Z_{n2} & \dots & Z_{nn} \end{vmatrix} \tag{4.3}$$

$$\dot{I}_2 = \frac{1}{\Delta}\begin{vmatrix} Z_{11} & \dot{U}_1 & \dots & Z_{1n} \\ Z_{21} & \dot{U}_2 & \dots & Z_{2n} \\ \dots & \dots & \dots & \dots \\ Z_{n1} & \dot{U}_n & \dots & Z_{nn} \end{vmatrix} \tag{4.4}$$

etc., where the determinant of the system

$$\Delta = \begin{vmatrix} Z_{11} & Z_{12} & \dots & Z_{1n} \\ Z_{21} & Z_{22} & \dots & Z_{2n} \\ \dots & \dots & \dots & \dots \\ Z_{n1} & Z_{n2} & \dots & Z_{nn} \end{vmatrix} \tag{4.5}$$

According to the rule for expanding a determinant in terms of the elements of the column and the cofactors, the determinant is equal to the sum of the products of the elements of the column and their cofactors:

$$\left.\begin{aligned} \dot{I}_1 &= \dot{U}_1 \frac{\Delta_{11}}{\Delta} + \dot{U}_2 \frac{\Delta_{21}}{\Delta} + \dots + \dot{U}_n \frac{\Delta_{n1}}{\Delta}, \\ \dot{I}_2 &= \dot{U}_1 \frac{\Delta_{12}}{\Delta} + \dot{U}_2 \frac{\Delta_{22}}{\Delta} + \dots + \dot{U}_n \frac{\Delta_{n2}}{\Delta}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{I}_n &= \dot{U}_1 \frac{\Delta_{1n}}{\Delta} + \dot{U}_2 \frac{\Delta_{2n}}{\Delta} + \dots + \dot{U}_n \frac{\Delta_{nn}}{\Delta}. \end{aligned}\right\} \tag{4.6}$$

Here $\Delta_{ik}$ is the cofactor of element $Z_{ik}$ of determinant (5), i.e. its minor with the sign factor $(-1)^{i+k}$.

In abbreviated form the set of equations (6) is written thus:

$$\dot{I}_k = \frac{1}{\Delta}\sum_{i=1}^{n} \dot{U}_i \Delta_{ik}. \tag{4.7}$$

The first subscript of the cofactor ($i$) denoting the number of the row omitted in the determinant of the system, corresponds to the number of the loop, the total e.m.f. of which is multiplied by the given cofactor. The second subscript ($k$) denoting the number of the column in the determinant

of the system, corresponds to the number of the loop for which the loop current is calculated.

Equations (2) expressing Kirchhoff's voltage law are written on the supposition that the sources of electrical energy are voltage sources. When there are current sources in the electrical circuit they can be replaced by equivalent voltage sources. But in the case of ideal current sources (whose admittances are zero) it is better to take the given source currents as loop

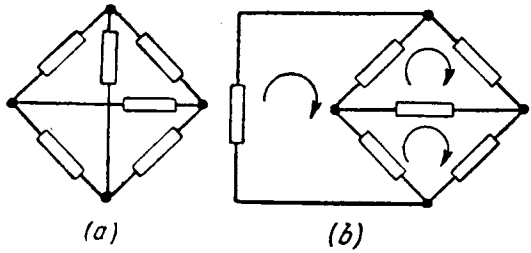



FIG. 5. Planar network.

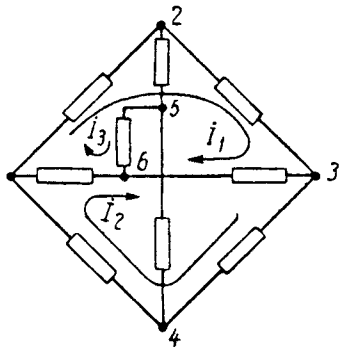



FIG. 6. Non-planar network.

currents; then the number of unknown loop currents and, correspondingly, the number of voltage-law equations, is reduced to the number of given currents.

Electrical networks can be either planar or non-planar.

A *planar* or flat network can be drawn in a plane without a crossover. In certain cases the crossovers resulting from the manner in which the circuit diagram is drawn, can be removed by representating the given planar network in another way, e.g. as shown in Fig. 5.

The network given in Fig. 5*a* is planar since its crossover point is removable, as shown in Fig. 5*b*.

A *non-planar* or non-flat network cannot be shown schematically in one plane with no wires crossing. The network of Fig. 6 is an example of a non-

planar network since it is not possible to draw its diagram in a plane without crossover.

If the positive direction of the loop currents in all the loops of a planar network is the same, e.g. is clockwise*, then the mutual impedances of the adjacent loops contained in the set of equations (2) require a minus sign, as the loop currents of adjacent loops are opposed in the common branches.

In the case of a non-planar electrical circuit it is not possible to have loop current differences only in the common branches as is seen, for example, from the Fig. 6 diagram. Let us suppose that the loop currents $\dot{I}_1$ (in loop *12361*) and $I_2$ (in loop *16341*) are clockwise. If loop current $\dot{I}_3$ in loop *12611* travels clockwise, it is added to current $\dot{I}_1$ in branches *1–2* and *1–6*. And if current $\dot{I}_3$ is sent anticlockwise, it is added to current $\dot{I}_2$ in branch *1–6*. Thus, in the case of a non-planar electrical circuit not all the mutual impedances of adjacent loops have the same sign.

EXAMPLE 2. To determine the currents in the Fig. 4 diagram by the loop method.

The problem is reduced to the joint solution of two equations based on Kirchhoff's voltage law:

$$\dot{U}_1 = (Z_1 + Z_3)\dot{I}_1 - Z_3\dot{I}_2,$$

$$-\dot{U}_2 = (Z_2 + Z_3)\dot{I}_2 - Z_3\dot{I}_1.$$

The determinant of the system is

$$\Delta = \begin{vmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{vmatrix} = (Z_1 + Z_3)(Z_2 + Z_3) - Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1.$$

The current in impedance $Z_1$ is

$$\dot{I}_1 = \frac{\begin{vmatrix} \dot{U}_1 & -Z_3 \\ -U_2 & Z_2 + Z_3 \end{vmatrix}}{\Delta} = \frac{(Z_2 + Z_3)\dot{U}_1 - Z_3\dot{U}_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}.$$

The current in impedance $Z_2$ is

$$\dot{I}_2 = \frac{\begin{vmatrix} Z_1 + Z_3 & \dot{U}_1 \\ -Z_3 & -\dot{U}_2 \end{vmatrix}}{\Delta} = \frac{-(Z_1 + Z_3)\dot{U}_2 + Z_3\dot{U}_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}.$$

The current in impedance $Z_3$ is

$$\dot{I}_1 - \dot{I}_2 = \frac{Z_2\dot{U}_1 + Z_1\dot{U}_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}.$$

EXAMPLE 3 (Ref. 24). To determine the current in the diagonal of the bridge circuit in Fig. 7 by the loop method.

* This holds for all loops with the exception of the exterior loop enclosing the whole network diagram, in which the loop current should be directed anticlockwise (see Example 3).

The positive directions specified for the loop currents $\dot{I}_1$, $\dot{I}_2$ and $\dot{I}_3$ are shown on the diagram by the arrows. There are three equations using Kirchhoff's voltage law (in accordance with the number of independent loops):

$$(Z_1 + Z_2 + Z_5)\dot{I}_1 - Z_5\dot{I}_2 - Z_2\dot{I}_3 = 0,$$

$$-Z_5\dot{I}_1 + (Z_3 + Z_4 + Z_5)\dot{I}_2 - Z_3\dot{I}_3 = 0,$$

$$-Z_2\dot{I}_1 - Z_3\dot{I}_2 + (Z_2 + Z_3 + Z_6)\dot{I}_3 = -\dot{U}.$$

The solution of the set of equations obtained, with respect to loop currents $\dot{I}_1$ and $\dot{I}_2$, gives:

$$\dot{I}_1 = -\frac{\dot{U}}{M}[Z_3Z_5 + Z_2(Z_3 + Z_4 + Z_5)],$$

$$\dot{I}_2 = -\frac{\dot{U}}{M}[Z_2Z_5 + Z_3(Z_1 + Z_2 + Z_5)],$$

where $M$ has the same value as in Example 1.

The unknown current in the diagonal of the bridge circuit is equal to difference in the loop currents

$$\dot{I}_5 = \dot{I}_2 - \dot{I}_1 = \frac{\dot{U}}{M}(Z_2Z_4 - Z_1Z_3),$$

which agrees with the answer obtained in Example 1.

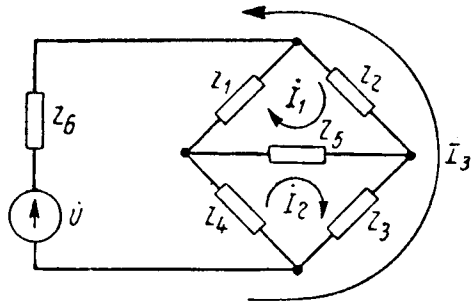



FIG. 7. Example 3 (bridge network).

It should be noted that if the loops in the given diagram are selected in such a way that only one loop current flows through arm $Z_5$, then the unknown current in arm $Z_5$ will be that same current, i.e. the problem is reduced to finding only one loop current, instead of two.

## 3. Node-Voltage Method

The *node-voltage method* consists in determining the voltages in the nodes of an electrical circuit with respect to a certain reference or datum node, using Kirchhoff's current law. These unknown voltages are called *node voltages*, the direction from the nodes to the reference node being taken as their positive direction.

The voltage at the terminals of any branch is obviously equal to the difference between the nodal voltages of the nodes of the given branch, and the product of this voltage with the admittance of the given branch is equal to the current in the branch. Thus, knowing the nodal voltages in the electrical circuit, we can find the currents in the branches.

As an example, Fig. 8 shows an electrical circuit with two current sources and three nodes. We shall select node *3* as the datum node of the

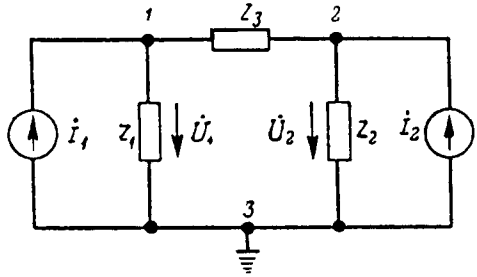



FIG. 8. Node voltages in a network with three nodes.

given circuit and denote the nodal voltages of nodes *1* and *2* by $\dot{U}_1$ and $\dot{U}_2$. In accordance with the notations used in Fig. 8, the branch admittances are respectively,

$$Y_1 = \frac{1}{Z_1}, \quad Y_2 = \frac{1}{Z_2}, \quad Y_3 = \frac{1}{Z_3}.$$

For the given electrical circuit with three nodes, these two equations can be written using Kirchhoff's current law:
for node *1*

$$\dot{I}_1 = Y_1\dot{U}_1 + Y_3(\dot{U}_1 - \dot{U}_2) = (Y_1 + Y_3)\dot{U}_1 - Y_3\dot{U}_2, \qquad (4.8)$$

for node *2*

$$\dot{I}_2 = Y_2\dot{U}_2 + Y_3(\dot{U}_2 - \dot{U}_1) = -Y_3\dot{U}_1 + (Y_2 + Y_3)\dot{U}_2. \qquad (4.9)$$

The quantity $Y_1 + Y_3$, representing the sum of the admittances of the branches connected at node *1*, is called the *self-admittance* of node *1*; the admittance $Y_3$, common to nodes *1* and *2*, is called the *mutual admittance* of nodes *1* and *2* (it requires a negative sign according to the assumed positive direction of the nodal voltages).

In equations (8) and (9) currents $\dot{I}_1$ and $\dot{I}_2$ and the branch admittances are given and the nodal voltages $\dot{U}_1$ and $\dot{U}_2$ which are the joint solution of the equations, are to be found.

In general, if an electrical circuit contains $q$ nodes, then using Kirchhoff's current law a set of $q - 1$ equations is obtained (the node $q$ is taken as datum node):

$$\left.\begin{aligned} \dot{I}_1 &= Y_{11}\dot{U}_1 + Y_{12}\dot{U}_2 + \cdots + Y_{1,q-1}\dot{U}_{q-1}, \\ \dot{I}_2 &= Y_{21}\dot{U}_1 + Y_{22}\dot{U}_2 + \cdots + Y_{2,q-1}\dot{U}_{q-1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{I}_{q-1} &= Y_{q-1,1}\dot{U}_1 + Y_{q-1,2}\dot{U}_2 + \cdots + Y_{q-1,q-1}\dot{U}_{q-1}. \end{aligned}\right\} \quad (4.10)$$

Here the current entering the node is taken to have a plus sign and the current leaving the node a minus sign;

$Y_{ii}$ is the self-admittance of all the branches connected at the given node;

$Y_{ik}$ is the mutual admittance of nodes $i$ and $k$ with a negative sign for the specified direction of all the nodal voltages to the datum node, regardless of whether the given electrical circuit is planar or non-planar.

By solving the set of simultaneous equations (10) in terms of determinants we obtain a formula for the voltage of the $k$-th node with respect to the datum node

$$\dot{U}_k = \frac{1}{\Delta} \sum_{i=1}^{q-1} \dot{I}_i \Delta_{ik}, \quad (4.11)$$

where $\Delta$ is a determinant of the form

$$\Delta = \begin{vmatrix} Y_{11} & Y_{12} \cdots & Y_{1,q-1} \\ Y_{21} & Y_{22} \cdots & Y_{2,q-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ Y_{q-1,1} & Y_{q-1,2} \cdots & Y_{q-1,q-1} \end{vmatrix}, \quad (4.12)$$

and $\Delta_{ik}$ is the cofactor of element $Y_{ik}$ of the given determinant.

The first subscript of the cofactor ($i$), denoting the number of the row omitted in the determinant of the system, corresponds to the number of the node whose total source current is multiplied by the given cofactor. The second subscript ($k$), denoting the number of the column omitted in the determinant of the system, corresponds to the number of the node for which the nodal voltage is calculated.

Equations (10), which express Kirchhoff's current law, suppose that current sources are used as the sources of electrical energy. When there are voltage sources in the electrical circuit they can be replaced by equivalent current sources.

If any branch contains an e.m.f. only, i.e. the impedance of the branch is zero and, consequently, the voltage between the two nodes is given, it is best to select one of the nodes of the given branch as the datum node. In this case the number of unknown nodal voltages and, correspondingly, the number of equations is reduced to one.

The node method has an advantage over the loop method when the number of current-law equations is fewer than the number of voltage-law equations. If the given network has $q$ nodes and $p$ branches then, in accordance with the above, the node method is better when $q-1<p-q+1$ or, the same thing, when

$$2(q-1)<p. \tag{4.13}$$

This takes into account the general case when the number of equations is not reduced on account of the known loop currents or nodal voltages.

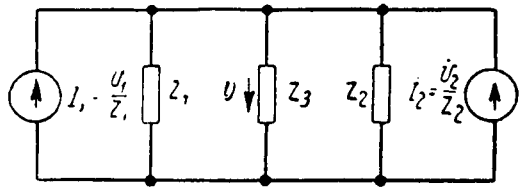



FIG. 9. Example 4.

EXAMPLE 4. To determine the current in branch $Z_3$ of the Fig. 4 diagram, using the node method.

As a result of replacing the voltage sources given in Fig. 4 by equivalent current sources, the Fig. 9 circuit is obtained. This circuit contains only two nodes.

If the voltage between the nodes is $\dot{U}$,

$$Y_1=\frac{1}{Z_1},\quad Y_2=\frac{1}{Z_2},\quad Y_3=\frac{1}{Z_3};\quad \dot{I}_1=Y_1\dot{U}_1,\quad \dot{I}_2=Y_2\dot{U}_2,$$

the problem of finding nodal voltage $\dot{U}$ is reduced to solving an equation based on Kirchhoff's current law,

$$Y_1\dot{U}_1+Y_2\dot{U}_2=(Y_1+Y_2+Y_3)\,\dot{U},$$

from which

$$\dot{U}=\frac{Y_1\dot{U}_1+Y_2\dot{U}_2}{Y_1+Y_2+Y_3}.$$

The current in impedance $Z_3$

$$\dot{I}_3=Y_3\dot{U}=\frac{Z_2\dot{U}_1+Z_1\dot{U}_2}{Z_1Z_2+Z_2Z_3+Z_3Z_1},$$

which agrees with the result of Example 2.

EXAMPLE 5 (Ref. 24). To determine the current in the diagonal of the Fig. 7 bridge circuit, using the node method.

As a result of replacing the given voltage source by an equivalent current source, the Fig. 10 circuit containing 4 nodes is obtained.

For this circuit $4 - 1 = 3$ equations based on Kirchhoff's current law (in accordance with the number of independent nodes) can be written. If node *4* is selected as the datum node in the given circuit and the nodal voltages are directed to the datum node, the equations take the form:

for node *1*

$$\dot{Y}_6\dot{U} = (Y_1 + Y_2 + Y_6)\,\dot{U}_1 - Y_2\dot{U}_2 - Y_6\dot{U}_3,$$

for node *2*

$$0 = -Y_2\dot{U}_1 + (Y_2 + Y_3 + Y_5)\,\dot{U}_2 - Y_3\dot{U}_3,$$

for node 3

$$-Y_6\dot{U} = -Y_6\dot{U}_1 - Y_3\dot{U}_2 + (Y_3 + Y_4 + Y_6)\,\dot{U}_3.$$

The solution of the set of equations obtained with respect to $\dot{U}_2$ gives

$$\dot{U}_2 = \frac{\dot{U}}{N}\,Y_6(Y_2Y_4 - Y_1Y_3),$$

where

$$N = Y_5[(Y_1 + Y_2)(Y_3 + Y_4) + Y_6(Y_1 + Y_2 + Y_3 + Y_4)]$$
$$+ Y_1Y_4(Y_2 + Y_3) + Y_2Y_3(Y_1 + Y_4) + Y_6(Y_1 + Y_4)(Y_2 + Y_3).$$

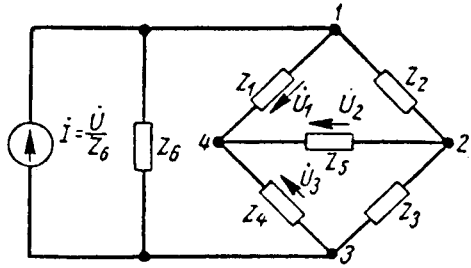



FIG. 10. Example 5 (bridge network).

By multiplying the discovered nodal voltage $\dot{U}_2$ by admittance $Y_5$ of the diagonal arm of the bridge circuit and changing the sign in accordance with the voltage of current $\dot{I}_5$ previously selected (Fig. 3), we find the required current

$$\dot{I}_5 = \frac{\dot{U}}{N}\,Y_5Y_6(Y_1Y_3 - Y_2Y_4).$$

## 4. Superposition Method

In a linear electrical circuit containing voltage sources, the loop currents (and, correspondingly, the branch currents) are *linear functions* of e.m.f.s. Mathematically, they are expressed by formula (7)

$$\dot{I}_k = \frac{1}{\Delta}\sum_{i=1}^{n}\dot{U}_i\Delta_{ik}.$$

The physical significance of this formula is that *the current in any loop of a linear electrical circuit can be obtained as the algebraic sum of the currents produced in that loop by each e.m.f. individually*, with all other e.m.f.s reduced to zero.

The method of calculating the currents, based on determining the partial currents in one loop (or branch) caused by each source acting separately and subsequent summation of these partial currents is known as the *superposition method.* In view of the fact that the voltages across the sections of the circuit are proportional to the currents, this method is also applicable to voltages.

When partial currents are being determined by the superposition method the internal impedances of those voltage sources, which are to be reduced to zero, must remain in the circuit. The e.m.f. sources are reduced to zero by replacing them by short circuits.

In turn, in a linear electrical circuit containing current sources, the nodal voltages (and, correspondingly, the voltages across the branches) are linear functions of the source currents. Mathematically, they are expressed by formula (11)

$$\dot{U}_k = \frac{1}{\Delta} \sum_{i=1}^{q-1} \dot{I}_i \Delta_{ik}.$$

The physical significance of this formula is that *the nodal voltage in any node of a linear electrical circuit can be obtained as an algebraic sum of the voltages produced in this node by each of the source currents individually.* Thus, formula (11), as well as formula (7), is a mathematical expression of the superposition method which is true for linear electrical circuits.

When partial nodal voltages are being determined by the superposition method, the internal admittances of those current sources which are to be reduced to zero, must remain in the circuit. The ideal current sources are reduced to zero by replacing them by open circuits.

In a linear electrical circuit where the voltage sources and current sources are both given, superimposing is also applicable. For example, the current in any loop of the given circuit can be obtained as the result of the algebraic addition of the partial currents owing to the individual voltage- and current-source excitations. The unwanted sources are then replaced by their internal impedances.

EXAMPLE 6. To determine the current in branch $Z_3$ of the Fig. 4 circuit, using the superposition method.

The unknown current $\dot{I}_3$ is determined as the sum of currents $\dot{I}_3'$ and $\dot{I}_3''$ in the branch $Z_3$ owing to e.m.f.s $\dot{U}_1$ (Fig. 11 *a*) and $\dot{U}_2$ (Fig. 11 *b*), acting separately. Currents $\dot{I}_3'$ and $\dot{I}_3''$ are added and not subtracted since their positive directions are assumed to coincide. Then

$$\dot{I}_3' = \frac{Z_2}{Z_2 + Z_3} \cdot \frac{\dot{U}_1}{Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}} = \frac{Z_2 \dot{U}_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

$$\dot{I}_3'' = \frac{Z_1}{Z_1 + Z_3} \cdot \frac{\dot{U}_2}{Z_2 + \dfrac{Z_1 Z_3}{Z_1 + Z_3}} = \frac{Z_1 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

consequently

$$\dot{I}_3 = \dot{I}_3' + \dot{I}_3'' = \frac{Z_2 \dot{U}_1 + Z_1 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

which agrees with the results of Example 2 obtained above.

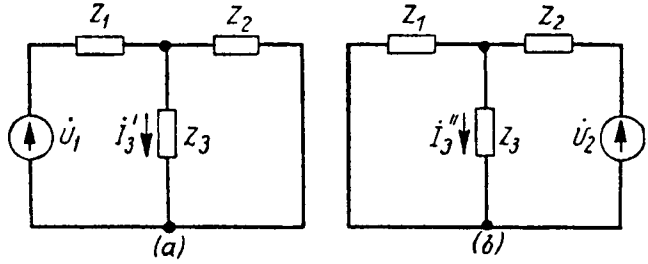



FIG. 11. Example 6 (superposition method).

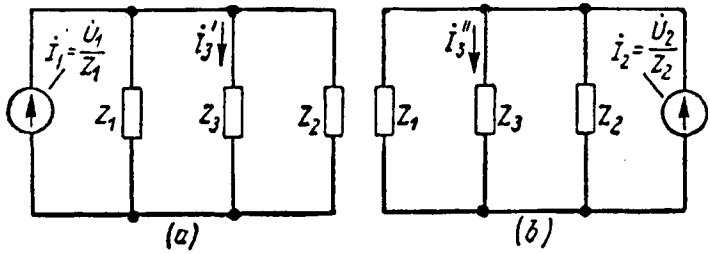



FIG. 12. Example 7 (superposition method).

EXAMPLE 7. To determine the current in branch $Z_3$ of the Fig. 9 circuit, using the superposition method.

The unknown current $\dot{I}_3$ is determined as the sum of currents $\dot{I}_3'$ and $\dot{I}_3''$ in the branch $Z_3$ due to the current sources $\dot{I}_1 = \dot{U}_1/Z_1$ (Fig. 12*a*) and $\dot{I}_2 = \dot{U}_2/Z_2$ (Fig. 12*b*), acting separately.

Using the notations adopted in Example 4, we get:

$$\dot{I}_3' = \frac{Y_3}{Y_1 + Y_2 + Y_3} \dot{I}_1 = \frac{Z_2 \dot{U}_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

$$\dot{I}_3'' = \frac{Y_3}{Y_1 + Y_2 + Y_3} \dot{I}_2 = \frac{Z_1 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

i.e. we obtain the same result as above (Example 4).

## 5. Input and Transfer Impedances and Admittances. Current and Voltage Ratios

Let an e.m.f. source $\dot{U}_i$ be placed into any branch of loop $i$ of an electrical circuit (Fig. 13*a*), the given branch not being common but belonging to loop $i$ only.

If the rest of the electrical circuit is passive, i.e. does not contain any electrical energy sources, then in accordance with (7) the given e.m.f. $\dot{U}_i$ will produce currents

$$\dot{I}_i = \frac{\Delta_{ii}}{\Delta_z}\,\dot{U}_i \tag{4.14}$$

and

$$\dot{I}_k = \frac{\Delta_{ik}}{\Delta_z}\,\dot{U}_i \tag{4.15}$$

in the given loop $i$ and in any other loop $k$.

$$\frac{\dot{U}_i}{\dot{I}_i} = \frac{\Delta_z}{\Delta_{ii}} = Z_i.$$

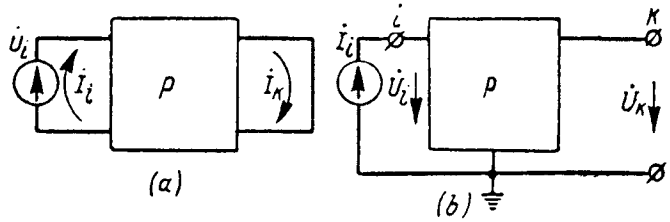



FIG. 13. Input and transfer impedances and admittances on a loop and node basis.

Figure 13 shows loops $i$ and $k$ of the given electrical circuit, letter $P$ denoting a passive electrical circuit. The relationship of the e.m.f. $\dot{U}_i$ to current $\dot{I}_i$ produced in loop $i$ is known as the *input (or driving-point) impedance* of the electrical circuit

$$\frac{\dot{U}_i}{\dot{I}_i} = \frac{\Delta_z}{\Delta_{ii}} = Z_i. \tag{4.16}$$

Correspondingly, the relationship of the e.m.f. $\dot{U}_i$ to the current $\dot{I}_k$ produced in loop $k$ is known as the *transfer impedance* of loops $i$ and $k$ (not to be confused with mutual impedance)

$$\frac{\dot{U}_i}{\dot{I}_k} = \frac{\Delta_z}{\Delta_{ik}} = Z_{ik}. \tag{4.17}$$

The self- and mutual impedances of the given electrical circuit serve as the elements of the determinant $\Delta_z$ contained in equations (16) and (17); $\Delta_z$ has the dimension of an impedance raised of the $n$-th power, where $n$ is the order of the determinant, defined by the number of independent loops in the given circuit. Correspondingly, the cofactors $\Delta_{ii}$ and $\Delta_{ik}$ in (16) and (17) have the dimension of impedances raised to the degree $n-1$. As a result of dividing the determinant of the system by a cofactor, the impedance–dimensional quantity is obtained.

From (16) and (17) we can conclude that *the input impedance is numerically equal to the e.m.f. in loop i which produces a current of one unit in the given loop; the transfer impedance of loops i and k is numerically equal to the e.m.f. in loop i which produces a current of one unit in loop k.*

Analogous conclusions may be drawn about the relationship of nodes $i$ and $k$ of the electrical circuit assuming that current source $\dot{I}_i$ is connected between node $i$ and the datum node, all the rest of the circuit being passive (Fig. 13*b*). In accordance with (11) the given current $\dot{I}_i$ causes nodal voltages in nodes $i$ and $k$

$$\dot{U}_i = \frac{\Delta_{ii}}{\Delta_y} \dot{I}_i \tag{4.18}$$

and

$$\dot{U}_k = \frac{\Delta_{ik}}{\Delta_y} \dot{I}_i. \tag{4.19}$$

The relationship of current $\dot{I}_i$ to nodal voltage $\dot{U}_i$ is called the *input (or driving-point) admittance* of the electrical circuit

$$\frac{\dot{I}_i}{\dot{U}_i} = \frac{\Delta_y}{\Delta_{ii}} = Y_i \tag{4.20}$$

and, correspondingly, the relationship of current $\dot{I}_i$ to nodal voltage $\dot{U}_k$ is called the *transfer admittance* of nodes $i$ and $k$ (not to be confused with mutual admittance)

$$\frac{\dot{I}_i}{\dot{U}_k} = \frac{\Delta_v}{\Delta_{ik}} = Y_{ik}. \tag{4.21}$$

The self- and mutual admittances of the given electrical circuit serve as the elements of the determinant $\Delta_y$ contained in (20) and (21); $\Delta_y$ has an admittance to the degree $m$, where $m = q - 1$ is the order of the determinant ($q$ being the number of nodes in the given circuit). Correspondingly, the cofactors in (20) and (21) have admittance to the degree $m-1$. As a result of dividing the determinant of the system by the cofactor, the quantity of the admittance dimension is obtained.

From (20) and (21) we can conclude that *the input admittance is numerically equal to the source current applied to node i (with a return through the datum node) and producing one unit of voltage in the given node with respect to the datum node; the transfer admittance of nodes i and k is numerically equal to the source current feeding node i and producing one unit of voltage in node k with respect to the datum node.*

In certain cases, with the appropriate reservation, the inverse value of expressions (16) and (17) are meant by input and transfer admittance, and

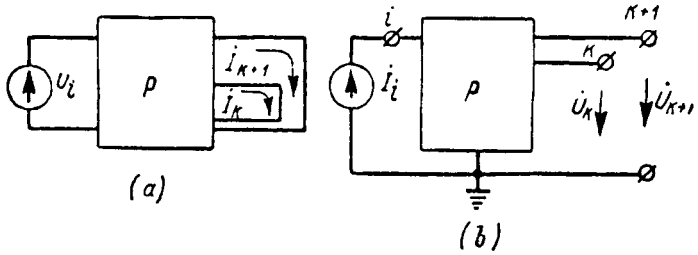



FIG. 14. Current (*a*) and voltage (*b*) ratios.

the inverse values of expressions (20) and (21) are meant by input and transfer impedance.

The relationship between two loop currents $\dot{I}_k$ and $\dot{I}_{k+1}$ produced in loops $k$ and $k+1$ by a e.m.f. source $\dot{U}_i$ connected in loop $i$ (Fig. 14*a*) is determined on the basis of (17) from the relationship of the corresponding cofactors or transfer impedances, namely

$$\dot{I}_k = \frac{\Delta_{ik}}{\Delta_z}\dot{U}_i = \frac{\dot{U}_i}{Z_{ik}},$$

$$\dot{I}_{k+1} = \frac{\Delta_{ik+1}}{\Delta_z}\dot{U}_i = \frac{\dot{U}_i}{Z_{i,\,k+1}}$$

and

$$\frac{\dot{I}_k}{\dot{I}_{k+1}} = \frac{\Delta_{ik}}{\Delta_{i,\,k+1}} = \frac{Z_{i,\,k+1}}{Z_{ik}}. \tag{4.22}$$

The relationship between the two nodal voltages $\dot{U}_k$ and $\dot{U}_{k+1}$ produced in nodes $k$ and $k+1$ by current source $\dot{I}_i$ given in node $i$ (Fig. 14*b*) can be determined analogously on the basis of (21) from the relationship of the corresponding cofactors or transfer admittances:

$$\frac{\dot{U}_k}{\dot{U}_{k+1}} = \frac{\Delta_{ik}}{\Delta_{i,\,k+1}} = \frac{Y_{i,\,k+1}}{Y_{ik}}. \tag{4.23}$$

It should then be borne in mind that the self- and mutual impedances of the given circuit are the elements of the cofactors in (22) and that the self- and mutual admittances are the elements of the cofactors in (23).

## 6. Reciprocity Theorem

The solving of networks may be simplified in a number of cases by using the property of linear electrical circuits known as the *principle* or *theorem of reciprocity*.

This reciprocity theorem can be formulated in two ways: one applicable to e.m.f. sources and the other applicable to current sources.

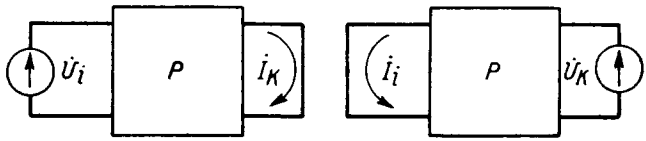



FIG. 15. Illustration of reciprocity on a loop basis.

*The version for an e.m.f. source.* In Fig. 15 a generalized passive network is drawn so as to place two separate loops $i$ and $k$ in evidence.

When there is only one source in the given network, namely an e.m.f. $\dot{U}_i$, inserted in a branch of the $i$-th loop which is not common to both, the current in the $k$-th loop is

$$\dot{I}_k = \frac{\Delta_{ik}}{\Delta_z}\,\dot{U}_i.$$

Correspondingly, when an e.m.f. $\dot{U}_k$ is placed in a branch of the $q$-th loop which is common to both, the current in the $i$-th loop is

$$\dot{I}_i = \frac{\Delta_{ki}}{\Delta_z}\,\dot{U}_k.$$

It follows from this that

$$\frac{\dot{I}_k}{\dot{I}_i} = \frac{\Delta_{ik}}{\Delta_{ki}} \cdot \frac{\dot{U}_i}{\dot{U}_k}.$$

The cofactors $\Delta_{ik}$ and $\Delta_{ki}$ are only distinguished by the fact that their rows are replaced by columns, since the mutual impedances are unchanged if the order of their subscripts is interchanged. It is generally known that when rows are replaced by columns the value of the determinant does not

change. Therefore, generally $\Delta_{ik} = \Delta_{ki}$ and, correspondingly

$$\frac{\dot{I}_k}{\dot{I}_i} = \frac{\dot{U}_i}{\dot{U}_k}.$$

If we take $\dot{U}_i = \dot{U}_k$, then $\dot{I}_i = \dot{I}_k$.

Thus the following proposition is true: if an e.m.f. applied in one loop of a linear, passive network produces a certain current in any other loop of the network, the same e.m.f. applied in the second loop will produce the same current in the first loop.

When the loop currents are selected properly, the current in the branch equals the loop current. The given theorem is therefore also true for the branch currents.

*The version for a current source.* In Fig. 16 a generalized passive network is drawn so as to place three separate nodes $i$, $k$ and $k+1$ in evidence. The earthed node is taken as the reference node, and node $k+1$ may, in certain cases, coincide with this reference node.

A current source $\dot{I}_i$ when applied at node $i$ (Fig. 16*a*), produces the voltage between nodes $k$ and $k+1$ equal to

$$\dot{U}_{k,\,k+1} = \dot{U}_k - \dot{U}_{k+1} = \frac{\Delta_{ik} - \Delta_{i,\,k+1}}{\Delta_y}\,\dot{I}_i.$$

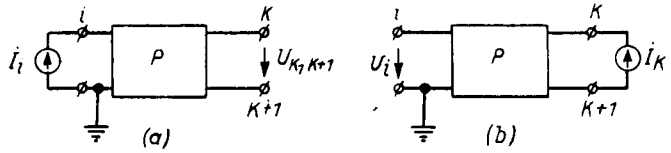



FIG. 16. Illustration of reciprocity on a node basis.

Correspondingly, when there are currents $\dot{I}_k$ and $\dot{I}_{k+1} = -\dot{I}_k$ in nodes $k$ and $k+1$ (Fig. 16*b*), the voltage of node $i$ with respect to the reference node is

$$\dot{U}_i = \frac{\Delta_{ki} - \Delta_{k+1,\,i}}{\Delta_y}\,\dot{I}_k.$$

From this it follows that

$$\frac{\dot{U}_{k,\,k+1}}{\dot{U}_i} = \frac{\Delta_{ik} - \Delta_{i,\,k+1}}{\Delta_{ki} - \Delta_{k+1,\,i}} \cdot \frac{\dot{I}_i}{\dot{I}_k}$$

or, in view of the equality of the cofactors $\Delta_{ik} = \Delta_{ki}$ and $\Delta_{i,k+1} = \Delta_{k+1,i}$

$$\frac{\dot{U}_{k,\,k+1}}{\dot{U}_i} = \frac{\dot{I}_i}{\dot{I}_k}.$$

If we suppose that $\dot{I}_i = \dot{I}_k$, then $\dot{U}_i = \dot{U}_{k,k+1}$ (in the particular case when node $k-1$ coincides with the reference node, $\Delta_{i,k+1} = \Delta_{k+1,i} = 0$ and $\dot{U}_{k,k+1} = \dot{U}_k$)

Thus, *if a current source applied at one node of a network produces a certain voltage between any two nodes of the circuit, the same current source, connected between the said two nodes, will produce the same voltage in the first.*

## 7. Compensation Theorem

The truth of the proposition known as the *compensation theorem* results from the fact that any of the component parts of a voltage drop contained in an equation based on Kirchhoff's voltage law can be transferred to the other side of the equation with the opposite sign, i.e. can be regarded as a fictitious e.m.f. opposed to the current. Consequently, *the currents in an electrical circuit do not change if the impedance in any loop of the circuit is replaced by an e.m.f. equal in value to the voltage drop in the given impedance and opposed to the current flowing through the given impedance.*

Figure 17 illustrates this: the equation for the Fig. 17*a* circuit based on Kirchhoff's voltage law

$$\dot{U} = Z_1\dot{I} + Z_2\dot{I},$$

can be represented in the form

$$\dot{U} - Z_2\dot{I} = Z_1\dot{I}.$$

The circuit shown in Fig. 17*b*, where an e.m.f. $Z_2\dot{I}$ opposed to current $\dot{I}$ is inserted instead of impedance $Z_2$, corresponds to an equation in this form.

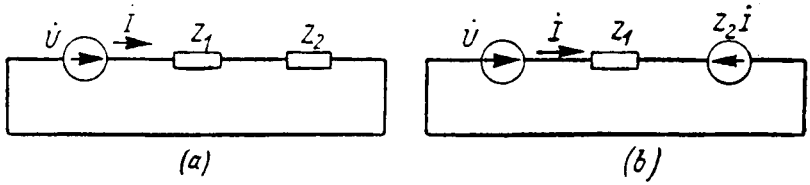



FIG. 17. Illustration of compensation theorem (impedance replaced by e.m.f. source).

The given theorem is also true for any complex network. Figure 18*b* illustrates the possibility of replacing impedance $Z$ by an e.m.f. source $U = Z(\dot{I}_1 - \dot{I}_2)$ acting against current $\dot{I}_1 - \dot{I}_2$ flowing through the removed impedance $Z$.

A current source $\dot{I}_1 - \dot{I}_2$, causing the same current as in Fig. 18*a* to flow between the nodes, may be connected in place of the removed im-

pedance (Fig. 18*c*), and the distribution of currents and voltages in the rest of the network will not change as a result.

It should be noted that the e.m.f. or the source current replacing the impedance is determined according to the current passing through the given impedance.

If anything is changed in the remainder of the network, the current in the impedance generally changes, and therefore the above-mentioned equivalent sources are not independent but are so-called non-autonomous or *dependent* sources.

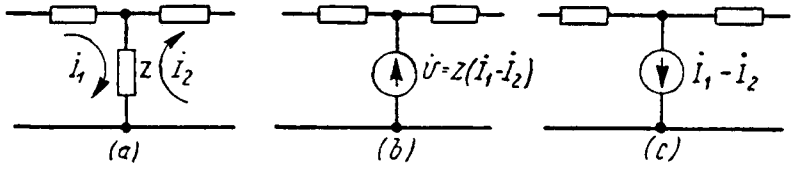



FIG. 18. Illustration of compensation theorem (impedance replaced by e.m.f. or current source).

By replacing any impedance by a non-autonomous e.m.f. or current source and using the superposition method, it is easy to verify that the voltages and currents in the remainder or the network are related linearly to the voltage across the variable impedance of the current flowing through it. If, for example, when impedance $Z$ in any branch changes, the current $\dot{I}_1$ in this branch changes within the limits $\dot{I}_{1(z=\infty)}$ to $\dot{I}_{1(z=0)}$, then current $\dot{I}_2$ in any branch is determined from the equation

$$\dot{I}_2 = \dot{I}_{2(z=\infty)} + k\dot{I}_1.$$

Coefficient $k$ is found from the condition

$$\dot{I}_{2(z=0)} = \dot{I}_{2(z=\infty)} + k\dot{I}_{1(z=0)},$$

from which

$$\dot{I}_2 = \dot{I}_{2(z=\infty)} + \frac{\dot{I}_{2(z=0)} - \dot{I}_{2(z=\infty)}}{\dot{I}_{1(z=0)}}\dot{I}_1.$$

## 8. Alteration Theorem

From the superposition methods and the compensation theorem this theorem follows on the change in the currents produced by changing the impedance of one of the branches of the given network:

*If, in a network, the impedance of any branch that carries the current* $\dot{I}$ *be altered by an amount* $\pm Z'$, *the current increments will be the same as*

*those which would result from an e.m.f.* $\pm Z'\dot{I}$ *acting alone in the modified branch, in the direction opposite to the original current.*

This is explained by Fig. 19. Let us suppose that a current $\dot{I}$ is flowing in a branch with an impedance $Z$ (Fig. 19*a*). The question is, how will the current change if the impedance of the branch is increased by an amount $Z'$, i.e. becomes equal to $Z+Z'$.

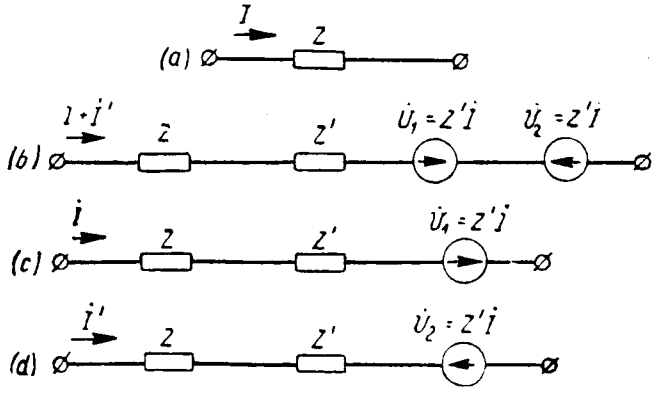



FIG. 19. Illustration of alteration theorem.

Figure 19*b* shows the branch with the impedance changed. It is supposed that as a result of changing impedance $Z$ by a value $Z'$ the original current $\dot{I}$ will change by $\dot{I}'$ and, consequently, the current in the branch will equal $\dot{I}+\dot{I}'$ ($\dot{I}'$ can have any sign).

Let us insert into the modified branch two e.m.f.s $\dot{U}_1$ and $\dot{U}_2$ without affecting the network, for they are equal and opposite, and, using the superposition method, examine the effect of these e.m.f.s in turn. When there is no $\dot{U}_2$, conditions identical to the original conditions obtain, since the circuits in Fig. 19*a* and *c* are equivalent; the voltage drop produced by current $\dot{I}$ in an incremental impedance $Z'$ is cancelled on the basis of the compensation theorem by a complementary e.m.f. $\dot{U}_1 = Z'\dot{I}$. Consequently, a change in the current of the circuit is produced by the action of the e.m.f. $\dot{U}_2 = Z'\dot{I}$ directed against $\dot{I}$ (Fig. 19*d*), which was to be proved.

The given theorem simplifies the calculation of currents in cases where the original currents are known before the impedance of the branch changes.

It should be noted that opening any branch of the given network corresponds to the extreme case $z' = \infty$, when the said solution becomes indefinite. In this case we apply the following artificial procedure: we connect two current sources $\dot{I}$ coupled in parallel between the opened

points. When the direction of the currents of these sources is as in Fig. 20*a*, the current in the given branch is zero in accordance with Kirchhoff's current law. Using the superposition method and taking into account that the current $\dot{I}$ is selected equal to the current originally in that branch, we arrive at the conclusion that opening the branch will have the same effect on the currents in the network as the insertion in the passive network of a source current $\dot{I}$, opposed to the primary branch current.

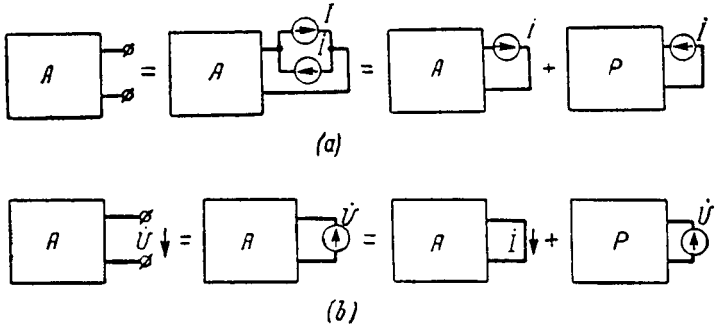



FIG. 20. Opening a branch.

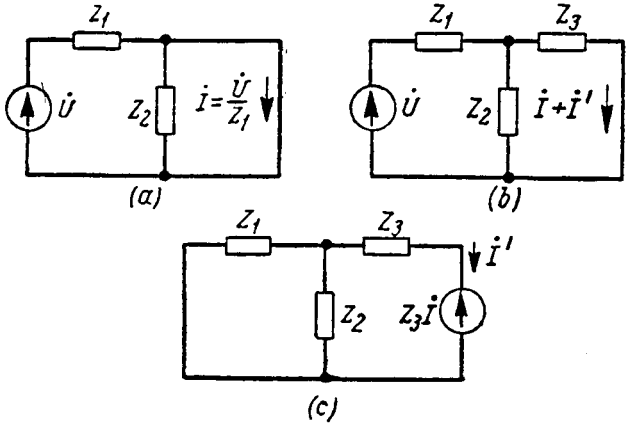



FIG. 21. Example 8.

Let us denote the voltage between the terminals of the electrical circuit by $\dot{U}$ and connect an e.m.f. source $\dot{U}$ to them. The currents and voltages of the circuit do not change as a result. By using the superposition method we find that the distribution of currents and voltages in the given network is composed of the corresponding electrical quantities in two networks (Fig. 20*b*): an active network with closed terminals, and a passive network with an e.m.f. source $\dot{U}$ applied to the said terminals.

In Fig. 20 letter $A$ denotes the given network containing electrical energy sources (active network) and letter $P$ denotes the same network, supposing that only impedances are left in the place of these sources (passive network). The addition and equality signs in Fig. 20 relate to the current (or voltage) distribution.

EXAMPLE 8. To determine the current in branch $Z_3$ of Fig. 21*b* using the alteration theorem, if it is known that when $Z_3 = 0$ the current in the given branch is $\dot{I} = \dot{U}/Z_1$ (Fig. 21*a*).

On the basis of the said theorem the change in the currents of the given circuit is produced by the effect of the e.m.f. $Z_3\dot{I}$ in branch $Z_3$, there being no sources in the rest of the circuit (Fig. 21*c*). Taking the direction of this e.m.f. into account we get:

$$\dot{I}' = -\frac{Z_3\dot{I}}{Z_3 + \dfrac{Z_1Z_2}{Z_1 + Z_2}}.$$

Consequently, the required current in branch $Z_3$ is

$$\dot{I} + \dot{I}' = \left(1 - \frac{Z_3}{Z_3 + \dfrac{Z_1Z_2}{Z_1 + Z_2}}\right)\dot{I} = \frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\dot{I}.$$

## 9. Equivalent Source Theorem

Using the *equivalent source* theorem a complicated network with an arbitrary number of electrical energy sources can be reduced to a circuit with one source, which simplifies analysis of the network.

There are two versions of the equivalent source theorem: the *voltage source* version and the *current source* version.

*Equivalent Voltage Source Theorem**

*The current in any branch mn of a linear network does not change if the network to which the given branch is connected is replaced by an equivalent voltage source;* the e.m.f. of this source should equal the voltage across the terminals *mn* when these terminals are open-circuited, and the internal impedance of the source should equal the input impedance to the passive network measured from these same terminals.

The given theorem is proved as follows: two e.m.f.s $\dot{U}_{mn}$, equal in value and opposed to each other, are introduced into branch *mn* on the condition that $\dot{U}_{mn}$ is equal to the voltage across terminals *m* and *n* when branch *mn* is open (Fig. 22).

* Known as Helmholtz–Thevenin's theorem.

The application of the superposition method, in accordance with Fig. 22, leads to the conclusion that the current in branch $Z$ is equal to

$$\dot{I} = \frac{\dot{U}_{mn}}{Z_0 + Z}, \tag{4.24}$$

where $\dot{U}_{mn}$ is the voltage measured across terminals $m$ and $n$, when the circuit is open (the so-called *open-circuit voltage*), and $Z_0 + Z$ is the impedance of the circuit consisting of branch $Z$ connected in series with passive network $P$.

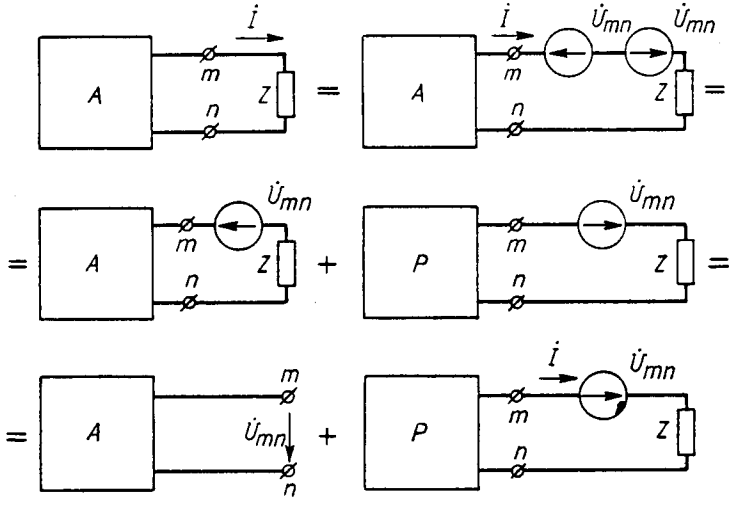



FIG. 22. Illustration of the theorem on an equivalent voltage source.

Thus the current in branch $Z$ is obtained on the assumption that the given branch is connected to a voltage source whose e.m.f. equals open-circuit voltage $\dot{U}_{mn}$, and the internal impedance is $Z_0$ (the input impedance to the network measured between terminals $m$ and $n$ branch $Z$ is open and all active sources within the network are reduced to zero*.

It should be noted that in accordance with Fig. 22 the current in any other branch of the given network can be obtained as a result of the algebraic addition of the current flowing through this branch when terminals $mn$ are open and the current which is produced in it by the effect of the e.m.f. $\dot{U}_{mn}$ in branch $Z$ (when the rest of the circuit is passive). Therefore, if we know the distribution of the currents in the network when branch $Z$ is open, then the subsequent distribution of currents when branch $Z$ is connected can easily be found by superimposing the initial currents on

* It should be remembered that the internal impedance of the sources still remain in the circuit when the sources are reduced to zero.

the currents resulting from the e.m.f. $\dot{U}_{mn}$ acting independently in series with branch $Z$.

In particular, by using this procedure it is possible to show that when there is an impedance $Z$ in one of the branches of the network the current in the second branch equals

$$\dot{I} = \dot{I}_{z=\infty} + \frac{k}{Z + Z_0}.$$

Here $\dot{I}_{(z=\infty)}$ is the current in the second branch when the first is open; $Z_0$ is the impedance of the circuit to which impedance $Z$ is connected, when there are no sources; $k$ is the proportionality factor which can be found from the condition that, when $Z = 0$

$$\dot{I} = \dot{I}_{(z=0)}.$$

From this

$$k = Z_0\left[\dot{I}_{(z=0)} - \dot{I}_{(z=\infty)}\right]$$

and, correspondingly

$$\dot{I} = \dot{I}_{(z=\infty)} + \frac{\dot{I}_{(z=0)} - \dot{I}_{(z=\infty)}}{Z + Z_0} Z_0.$$

The same follows from the last equation in Section 7, since $\dot{I}_1/\dot{I}_{1(Z=0)} = Z_0/(Z + Z_0)$.

As was mentioned above, the superposition method is applicable not only to currents but also to voltages. Therefore, using the procedure described, we can also find the relationships for voltages.

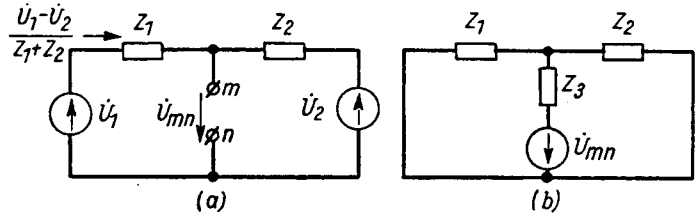



FIG. 23. Example 9.

EXAMPLE 9. To determine the currents in the branch of the circuit in Fig. 4 (Example 2), using the equivalent voltage source theorem.

Any of the three branches of the given electrical circuit can be opened and, as a result, the open-circuit voltage found. Figure 23 shows the case when branch $Z_3$ is open.

The open circuit voltage $\dot{U}_{mn}$ is then found as the difference between the e.m.f. $\dot{U}_1$ and the voltage drop which is due to the current $(\dot{U}_1 - \dot{U}_2)/(Z_1 + Z_2)$ through impedance $Z_1$ (Fig. 23*a*)

$$\dot{U}_{mn} = \dot{U}_1 - Z_1 \frac{\dot{U}_1 - \dot{U}_2}{Z_1 + Z_2} = \frac{Z_2\dot{U}_1 + Z_1\dot{U}_2}{Z_1 + Z_2}.$$

Under the effect of the e.m.f. $\dot{U}_{mn}$, through the impedance $Z_3$ in the Fig. 23*b* circuit flows a current

$$\dot{I}_3 = \frac{\dot{U}_{mn}}{Z_3 + \dfrac{Z_1 Z_2}{Z_1 + Z_2}} = \frac{Z_2 \dot{U}_1 + Z_1 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

which is divided between parallel branches $Z_1$ and $Z_2$: a current

$$\frac{Z_2}{Z_1 + Z_2} \dot{I}_3$$

flows through $Z_1$ and a current

$$\frac{Z_1}{Z_1 + Z_2} \dot{I}_3$$

flows through $Z_2$.

The unknown currents in arms $Z_1$ and $Z_2$ are determined as a result of superimposing the currents flowing in the circuits shown in Fig. 23*a* and *b*, i.e.

$$\dot{I}_1 = \frac{\dot{U}_1 - \dot{U}_2}{Z_1 + Z_2} + \frac{Z_2}{Z_1 + Z_2} \times \frac{Z_2 \dot{U}_1 + Z_1 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} = \frac{(Z_2 + Z_3)\dot{U}_1 - Z_3 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1},$$

$$\dot{I}_2 = \frac{\dot{U}_1 - \dot{U}_2}{Z_1 + Z_2} - \frac{Z_1}{Z_1 + Z_2} \times \frac{Z_2 \dot{U}_1 + Z_1 \dot{U}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} = \frac{-(Z_1 + Z_3)\dot{U}_2 + Z_3 \dot{U}_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}.$$

When there are several e.m.f. and current sources of the same frequency in the circuit, the voltage $\dot{U}_{mn}$ is a linear function of the given e.m.f.s and currents of the sources.

The linear relationship between voltage $\dot{U}_k$ at the terminals *mn* and the e.m.f. $\dot{U}_l$ of one source producing this voltage can be established on the basis of Fig. 20*b*.

According to (15) the e.m.f. $\dot{U}_l$ given in the *i*-th loop of the circuit produces a current $I_k = \Delta_{lk}/\Delta \dot{U}_l$, in the *k*-th loop of the circuit. In accordance with Fig. 20*b* this current is equal to the current produced in the *k*-th loop by the e.m.f. $\dot{U}_k$. And in accordance with (14) $\dot{I}_k = \Delta_{kk}/\Delta \dot{U}_k$. Comparing the expressions obtained for $\dot{I}_k$ we get

$$\dot{U}_k = \frac{\Delta_{lk}}{\Delta_{kk}} \dot{U}_l.$$

In turn, a current source applied at any node of the circuit produces a voltage in another node proportional to the current of the source.

Consequently, in general, under the influence of $n_1$ e.m.f. sources and $n_2$ current sources, the voltage across the open terminals *mn* is

$$\dot{U}_{mn} = \sum_{i=1}^{n_1} \frac{\Delta_{lk}}{\Delta_{kk}} \dot{U}_l + \sum_{i=1}^{n_2} Z_{kl} \dot{I}_l.$$

The factors in the form $\Delta_{ik}/\Delta_{kk}$ and $Z_{kl}$ contained in the expression for $\dot{U}_{mn}$ are transfer functions (Chapter VII, Section 5).

EXAMPLE 10. To determine the current in the diagonal of the arm of the bridge circuit in Fig. 24, using the equivalent voltage source theorem.

By opening arm $Z_5$ we find the open-circuit voltage $\dot{U}_{mn}$ as the difference between the voltage drops in impedances $Z_4$ and $Z_3$ (Fig. 24*b*).

$$\dot{U}_{mn} = Z_4 \frac{\dot{U}}{Z_1 + Z_4} - Z_3 \frac{\dot{U}}{Z_2 + Z_3} = \frac{Z_2 Z_4 - Z_1 Z_3}{(Z_1 + Z_4)(Z_2 + Z_3)} \dot{U}.$$

The input impedance measured between terminals $m$ and $n$ when $\dot{U} = 0$ and arm $Z_5$ is open is

$$Z_0 = \frac{Z_1 Z_4}{Z_1 + Z_4} + \frac{Z_2 Z_3}{Z_2 + Z_3}.$$

On the basis of (24)

$$\dot{I}_5 = \frac{\dot{U}_{mn}}{Z_0 + Z_5}.$$

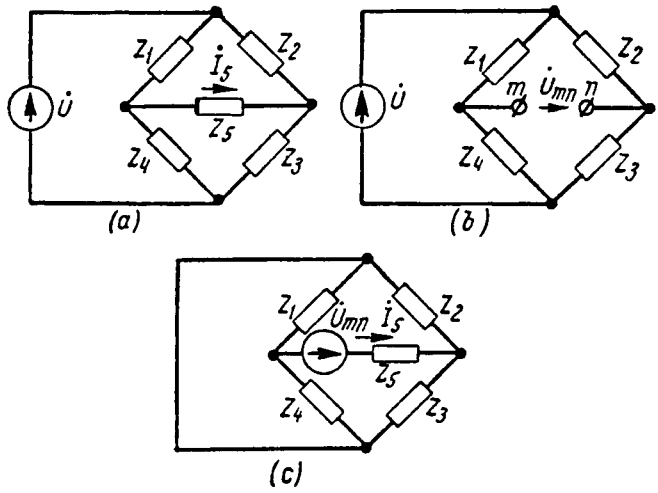



FIG. 24. Example 10.

*The Equivalent Current Source Theorem**

*The current in any branch mn of a linear network does not change if the network to which the given branch is connected is replaced by an equivalent current source;* the current of this source should equal the current flowing through the terminals *mn* when these terminals are short-circuited, and the internal admittance of the source should be equal to the input admittance to the passive network measured from these same terminals.

* Known as Norton's theorem.

This proposition follows from the condition of equivalence of the voltage and current sources, namely: a voltage source whose e.m.f. equals the open-circuit voltage $\dot{U}_{mn}$ and whose internal impedance is $Z_0$ can be replaced by a current source (Fig. 25)

$$\dot{I}_{mn} = \frac{\dot{U}_{mn}}{Z_0} = Y_0 \dot{U}_{mn}.$$

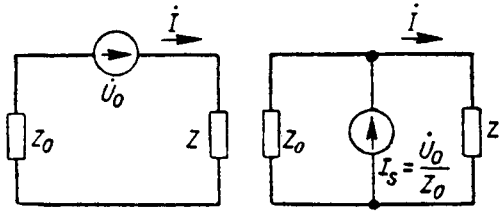



FIG. 25. Illustration of the theorem on an equivalent current source.

The last expression is the same as the current flowing through the short-circuited terminals $m$ and $n$ (the so-called *short-circuit current*). The unknown current in the branch is equal to

$$\dot{I} = \frac{Z_0}{Z_0 + Z} \dot{I}_{mn} = \frac{Y}{Y_0 + Y} \dot{I}_{mn}, \tag{4.25}$$

where

$$Y = \frac{1}{Z}.$$

If we know the distribution of the currents in the network when terminals $mn$ are short-circuited, then the distribution of the currents in the circuit when branch $Z$ is connected up can easily be found by superimposing the initial currents on the currents resulting from the current source $\dot{I}_{mn}$ acting independently in parallel with branch $Z$ when all active sources within the network are reduced to zero.

It can be shown that when there are several e.m.f. and current sources of the same frequency in the circuit, the current $\dot{I}_{mn}$ is expressed by the linear relationship

$$\dot{I}_{mn} = \sum_{i=1}^{n_1} Y_{ki} \dot{U}_i + \sum_{i=1}^{n_2} \frac{\Delta_{ik}}{\Delta_{ii}} \dot{I}_i.$$

The self- and mutual impedances of the circuit serve as the elements of the cofactors $\Delta_{ik}$ and $\Delta_{ii}$. The factors in the form $\Delta_{ik}/\Delta_{ii}$ and $Y_{ki}$ contained in the expression for $\dot{I}_{mn}$ are transfer functions (Chapter VII, Section 5).

The equivalent source theorem is widely used in present-day electrical and radio engineering for the practical solving of complex networks.

EXAMPLE 11. To determine the current $\dot{I}_3$ in branch $Z_3$ of the circuit in Fig. 4 (Example 2), using the equivalent current source theorem.

The current of the equivalent current source is equal to the short-circuit current ($Z_3 = 0$, Fig. 26*a*).

$$\dot{I}_s = \frac{\dot{U}_1}{Z_1} + \frac{\dot{U}_2}{Z_2} = Y_1\dot{U}_1 + Y_2\dot{U}_2.$$

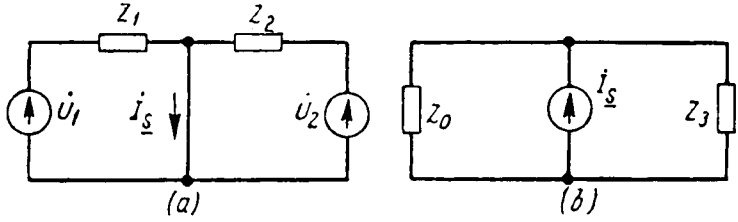



FIG. 26. Example 11.

The admittance of the equivalent current source $Y_0 = Y_1 + Y_2$. Consequently, on the basis of (25) the required current is

$$\dot{I}_3 = \frac{Y_3}{Y_0 + Y_3}\dot{I}_s = \frac{Y_3}{Y_1 + Y_2 + Y_3}\dot{I}_s = \frac{Z_2\dot{U}_1 + Z_1\dot{U}_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}.$$

## 10. Circuits with Magnetic Coupling

When a magnetic field connected with any turn changes, an e.m.f. is induced in the latter and is determined, in accordance with Faraday's and Maxwell's law of electromagnetic induction, by the time rate of the change in the magnetic flux, regardless of which produces the changing flux.

As we mentioned above (Chapter I, Section 2), a change in the magnetic flux of an induction coil produced by a change in the current in the same coil causes the phenomenon of self inductance, and a change in the magnetic flux of the induction coil produced by a change in the current in any other induction coil placed in its immediate proximity causes the phenomenon of mutual induction.

Circuits in which e.m.f.s of mutual inductance are induced are called *circuits with magnetic-field coupling*.

Formulae (8) and (9) in Chapter I for the e.m.f. of mutual inductance

$$e_{1M} = -M\frac{di_2}{dt}, \quad e_{2M} = -M\frac{di_1}{dt}$$

correspond to the so-called *additive polarity* of the magnetically coupled coils for which the positive directions of the magnetic fluxes of the self

inductance and the mutual inductance in each element coincide. The positive direction of the current and the magnetic flux created by it then agree each time, according to the right-hand screw rule.

The additive polarity of two magnetically coupled elements is shown in Fig. 27; the crosses mark the so-called *like* or *corresponding* terminals* (the other two terminals are the second pair of like terminals).

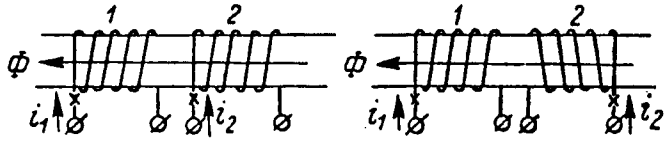



FIG. 27. Magnetically coupled elements (additive polarity).

The like terminals of the magnetically coupled elements are characterized by the property that when the currents $i_1$ and $i_2$ are in the same direction with respect to these terminals, the magnetic fluxes of the self inductance and mutual inductance in each element are added.

Current $i_1$ passing through coil *1* induces an e.m.f. $e_2$ in coil *2*, the positive direction of which coincides with the positive direction of current $i_2$.

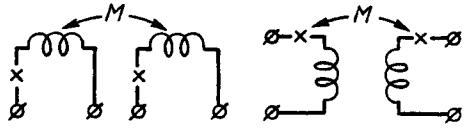



FIG. 28. Schematic representation of magnetically coupled coils with winding labelling.

Then, if $di_1/dt > 0$, $e_2 = -M\,di_1/dt < 0$, i.e. the potential of the terminal of coil *2* marked with a cross, is positive with respect to the second terminal of the same coil.

It may be concluded from this that like terminals of the two magnetically coupled coils possess this same property: that if a current increasing in magnitude is supplied to one of them, an increase in potential is produced on the like terminal of the other element.

Figure 28 gives the diagramatic representation of magnetically coupled coils, with the like terminals indicated. The purpose of the marking on the terminals is to clarify the meaning of the circuit diagram, without making it necessary to show the actual winding directions.

The experimental determining of the like terminals of magnetically coupled coils is based on the property of the potential increasing on the terminals of the second coil when a current $i_1$ is supplied to the like po-

* The general British practice is to denote the corresponding terminals of two coils with dots.

tential of the first coil, on the condition that $di_1/dt > 0$. One of the coils is connected into the circuit of a constant voltage source and a d.c. voltmeter is connected to the other (Fig. 29).

If, at the instant the circuit is closed, the voltmeter swings upscale, the terminals of the magnetically coupled coils connected to the positive pole of the electrical energy source and to the positive terminal of the voltmeter are corresponding.

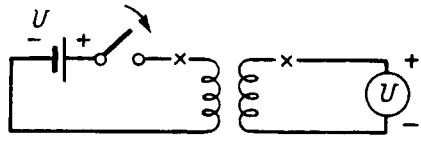



FIG. 29. Determining corresponding terminals experimentally.

Figure 30*a* shows the so-called *series aiding:* currents $i_1$ and $i_2$ (being equal) flow in the same direction with respect to the like poles and thus the magnetic fluxes of the self inductance and mutual inductance linked with each coil are added in this case (additive polarity).

When current directions are the same relative to the like terminals, the e.m.f. of mutual inductance $e_{1M} = -M\,di_2/dt$ induced in a positive direction (coinciding with the direction of current $i$), can be replaced by the

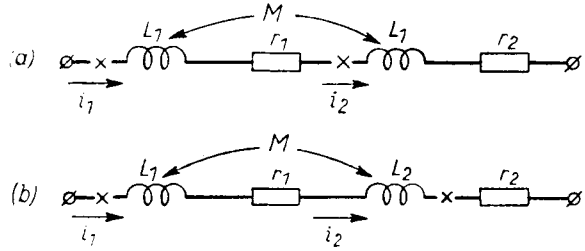



FIG. 30. Series aiding (*a*) and series opposing (*b*).

voltage drop of the mutual inductance $u_{1M} = -e_{1M} = M\,di_2/dt$, having the same reference direction. With series aiding, since the coils carry the same current ($i_1 = i_2 = i$), the voltage across the terminals of the circuit is expressed thus:

$$u' = r_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + r_2 i_2 + L_2 \frac{di_2}{di} + M \frac{di_1}{dt} = (r_1 + r_2)\, i + (L_1 + L_2 + 2M) \frac{di}{dt}. \quad (4.26)$$

The expression obtained shows that a series-aiding circuit is equivalent to a coil with a resistance $r' = r_1 + r_2$ and an effective inductance $L' = L_1 + L_2 + 2M$.

Thus, when the currents in two series-connected and magnetically coupled coils are in the same direction with respect to like terminals, the total inductance of both of them together is the sum of the individual self-inductances plus twice the mutual inductance.

Figure 30*b* shows the so-called *series opposing:* the currents in the coils are opposite with respect to like terminals, and therefore the magnetic fluxes of self inductance and mutual inductance linked to each coil are subtracted *(subtractive polarity).*

With series opposing, the voltage drop which is due to the mutual induction, assumed in the positive current direction, is obtained with a minus sign:

$$u'' = r_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} - M \frac{\mathrm{d}i_2}{\mathrm{d}t} + r_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} - M \frac{\mathrm{d}i}{\mathrm{d}t}$$

$$= (r_1 + r_2)\, i + (L_1 + L_2 - 2M) \frac{\mathrm{d}i}{\mathrm{d}t}. \qquad (4.27)$$

The given expression shows that a series-opposing circuit is equivalent to a circuit with a resistance $r'' = r_1 + r_2$ and an effective inductance $L'' = L_1 + L_2 - 2M$.

Consequently, when two series-connected coils are magnetically coupled in opposition the total inductance of both of them together is the sum of the individual self-inductances minus twice the mutual inductance.

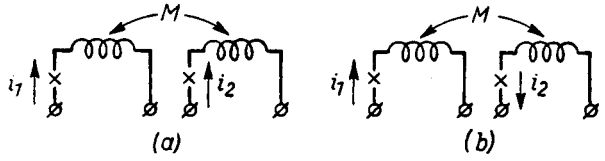



FIG. 31. Magnetic coupling: (*a*) additive polarity; (*b*) subtractive polarity.

The aiding and opposing coupling of two magnetically coupled coils connected arbitrarily are represented conventionally in Fig. 31.

For additive polarity the expression for the voltage drop caused by the mutual induction is taken with a plus sign and for subtractive polarity, a minus sign.

By representing the current in complex form it is possible to obtain an expression for the complex e.m.f. of the mutual inductance for the case of additive polarity

$$-M \frac{\mathrm{d}\dot{I}_m e^{j\omega t}}{\mathrm{d}t} = -j\omega M \dot{I}_m e^{j\omega t},$$

from which the sinor representation of the r.m.s.-valued e.m.f. of the mutual inductance is

$$\dot{E}_M = -j\omega M \dot{I}, \qquad (4.28)$$

and, correspondingly, the voltage drop of the mutual inductance in terms of sinors is

$$\dot{U}_M = j\omega M\dot{I}, \tag{4.29}$$

where $j\omega M$ is the so-called *complex impedance of the mutual inductance.*

For subtractive polarity the signs in (28) and (29) should be reversed. This is illustrated by the sinor diagrams in Fig. 32.

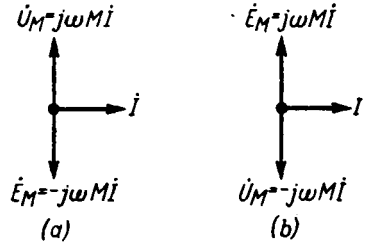



FIG. 32. Sinor diagrams for additive (*a*) and subtractive (*b*) polarities.

In terms of sinors the equations (26) and (27) may be written

$$\left.\begin{aligned} \dot{U}' &= [r_1 + r_2 + j\omega(L_1 + L_2 + 2M)]\,\dot{I}, \\ \dot{U}'' &= [r_1 + r_2 + j\omega(L_1 + L_2 - 2M)]\,\dot{I}. \end{aligned}\right\} \tag{4.30}$$

The two series connections of magnetically coupled coils furnish a means of measuring the mutual inductance between the coils. If $Z'$ denotes the complex impedance of a series-aiding circuit and $Z''$ denotes the same of the series-opposing circuit, i.e. we suppose that

$$\left.\begin{aligned} Z' &= r_1 + r_2 + j\omega(L_1 + L_2 + 2M), \\ Z'' &= r_1 + r_2 + j\omega(L_1 + L_2 - 2M), \end{aligned}\right\} \tag{4.31}$$

then, the subtraction of equations (31) yields

$$M = \frac{Z' - Z''}{j4\omega}. \tag{4.32}$$

The way in which complicated circuits with magnetic coupling are calculated is illustrated below by the example given in Fig. 33. It is supposed that the coils $L_1$, $L_2$ and $L_3$ contained in the circuit are coupled magnetically.

The e.m.f.s $\dot{U}_1$ and $\dot{U}_2$ and all the circuit parameters are iven. The loop currents $\dot{I}_1$ and $\dot{I}_2$ are to be found.

Let us use the loop current method: following the loops in the direction of currents $\dot{I}_1$ and $\dot{I}_2$ we can write voltage equations, taking into account

the voltage drops in the impedances of mutual inductance. In the case of additive polarity a plus sign will be ascribed to the voltage drop in the impedance of mutual inductance ($\dot{U}_M = j\omega M\dot{I}$), and in the case of subtractive polarity, a minus sign ($\dot{U}_M = -j\omega M\dot{I}$). In the case where the loop current $\dot{I}$ passes through two magnetically coupled elements connected in series, the voltage drop resulting from the given current in the impedance of mutual inductance is taken into account by the expression $\dot{U}_M = j2\omega M\dot{I}$ for additive polarity (series aiding) and $\dot{U}_M = -j2\omega M\dot{I}$ for subtractive polarity (series opposing).

Using the above-mentioned rules, let us compose loop equations for the Fig. 33 circuit:

$$\dot{U}_1 = [Z_1 + Z_3 + j\omega(L_1 + L_3 - 2M_{13})]\dot{I}_1 + [-Z_3 - j\omega L_3 + j\omega(M_{13} + M_{23} - M_{12})]\dot{I}_2, \tag{4.33}$$

$$-\dot{U}_2 = [Z_2 + Z_3 + j\omega(L_2 + L_3 - 2M_{23})]\dot{I}_2 + [-Z_3 - j\omega L_3 + j\omega(M_{13} + M_{23} - M_{12})]\dot{I}_1. \tag{4.34}$$

The signs ascribed to the mutual inductances $M_{12}$, $M_{13}$ and $M_{23}$ in equations (33) and (34) take into account the given polarity of the terminals of the magnetically coupled coils. Thus, for example, a minus sign is taken for $M_{12}$ since currents $\dot{I}_1$ and $\dot{I}_2$ enter at the unlike terminals of the coupled

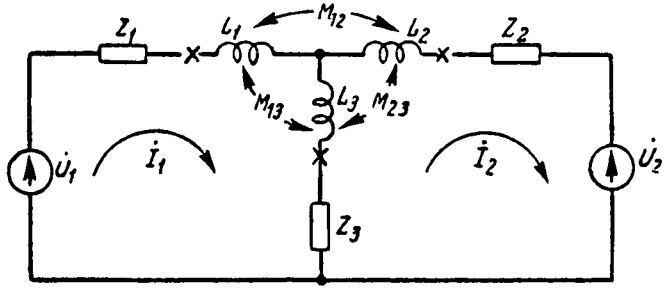



FIG. 33. Typical network with three magnetically coupled elements.

coils $L_1$ and $L_2$ (subtractive polarity); but a plus sign is provided for $M_{12}$ in the equations since currents $\dot{I}_1$ and $\dot{I}_2$ enter at the like terminals of the coupled coils $L_1$ and $L_3$ (additive polarity). The unknown loop currents $\dot{I}_1$ and $\dot{I}_2$ are found by the joint solution of equations (33) and (34).

It is evident from the above that any complicated network with mutual inductance between elements can be solved by one of the methods discussed previously, taking into account the voltage drops in the impedances of the

mutual inductance. But when there are several magnetic couplings, as in the diagram shown in Fig. 33, the branch-current method, i.e. the direct application of Kirchhoff's current and voltage laws, leads to less cumbersome equations than the loop-current method.

In this case the node method in its pure form as given above (Section 3 of this chapter) is not applicable since the currents in the branches not only depend on the voltages between the nodes to which these branches are attached but also on the currents in other branches coupled with them

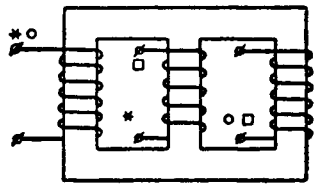


FIG. 34. System of winding labelling.

magnetically. (For further discussion see References (14) and (105), and Section 12 of this chapter.)

In the polarity markings shown above the like terminals of all three magnetically connected elements were denoted in the same way (by a cross) since it was supposed that these elements had a common single-path core.

When three or more coils are arranged on a three-bar core (Fig. 34), the like terminals of each pair of inductively coupled coils must be denoted by different conventional symbols (Example 13).

EXAMPLE 12. To determine the complex impedance at the terminals of two magnetically coupled coils connected in parallel (Fig. 35).

Taking the positive directions of the loop currents in accordance with Fig. 35, we can compile the loop equations

$$\dot{U} = (r_1 + j\omega L_1)\,\dot{I}_1 - (r_1 + j\omega L_1)\,\dot{I}_2 + j\omega M\dot{I}_2,$$

$$0 = [r_1 + r_2 + j\omega(L_1 + L_2 - 2M)]\,I_2 - (r_1 + j\omega L_1)\,I_1 + j\omega MI_1.$$

Let us denote

$$Z_1 = r_1 + j\omega L_1, \quad Z_2 = r_2 + j\omega L_2, \quad Z_M = j\omega M.$$

Thus

$$\dot{U} = Z_1\dot{I}_1 - (Z_1 - Z_M)\,\dot{I}_2,$$

$$0 = (Z_1 + Z_2 - 2Z_M)\,\dot{I}_2 - (Z_1 - Z_M)\,\dot{I}_1.$$

The solution of the equations with respect to $\dot{I}_1$ yields

$$\dot{I}_1 = \frac{\dot{U}}{Z_1 - \dfrac{(Z_1 - Z_M)^2}{Z_1 + Z_2 - 2Z_M}}.$$

The required input impedance is

$$Z = \frac{\dot{U}}{\dot{I}_1} = Z_1 - \frac{(Z_1 - Z_M)^2}{Z_1 + Z_2 - 2Z_M} = \frac{Z_1 Z_2 - Z_M}{Z_1 + Z_2 - 2Z_M}.$$

If the terminals of one coil were interchanged, the minus sign in the denominator in the last equation would be changed into a plus sign. When there is no mutual induction between coils ($M = 0$), the well known formula for parallel impedances is obtained.

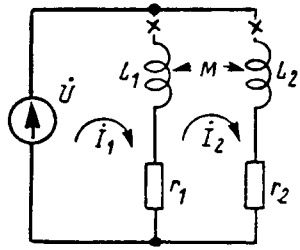



FIG. 35. Example 12.

EXAMPLE 13. To write loop equations for the network given in Fig. 36*a* consisting of an e.m.f. source $\dot{U}$, a capacitance $C$, and three magnetically coupled coils mounted on a three-bar core. The mutual inductance of the coils are $M_{12}$, $M_{13}$ and $M_{23}$. The circuit is considered to be linear.

Figure 36*b* represents a circuit diagram of the given network. The asterisk, circle and square denote the like terminals corresponding to the given coil winding.

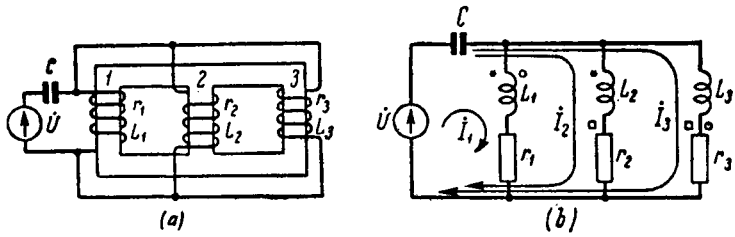



FIG. 36. Example 13.

By selecting the directions of the loop currents shown on the Fig. 36*b* diagram, we can compile the loop equations:

$$\dot{U} = \frac{1}{j\omega C}(\dot{I}_1 + \dot{I}_2 + \dot{I}_3) + (r_1 + j\omega L_1)\dot{I}_1 + j\omega M_{12}\dot{I}_2 - j\omega M_{13}\dot{I}_3,$$

$$\dot{U} = \frac{1}{j\omega C}(\dot{I}_1 + \dot{I}_2 + \dot{I}_3) + (r_2 + j\omega L_2)\dot{I}_2 + j\omega M_{12}\dot{I}_1 + j\omega M_{23}\dot{I}_3,$$

$$\dot{U} = \frac{1}{j\omega C}(\dot{I}_1 + \dot{I}_2 + \dot{I}_3) + (r_3 + j\omega L_3)\dot{I}_3 - j\omega M_{13}\dot{I}_1 + j\omega M_{23}\dot{I}_2.$$

## 11. Coefficient of Coupling. Leakage Inductance

The degree of the magnetic coupling of two elements in a circuit is characterized by the *coupling coefficient*

$$k = \frac{M}{\sqrt{(L_1 L_2)}}, \tag{4.35}$$

where $L_1$ and $L_2$ are the inductances of the circuit elements; $M$ is the mutual inductance.

The magnitude of coupling coefficient is always less than unity because the magnetic flux created by the current in one of the elements is not com-

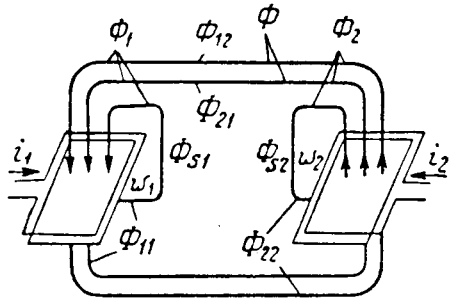



FIG. 37. Magnetic fluxes of two magnetically coupled elements.

pletely linked with the turns of the second element, and the phenomenon of *leakage* takes place. This is demonstrated pictorially in Fig. 37 which schematically represents the components of the magnetic flux of coupled elements with additive polarity.

Let us suppose that the first element consists of $w_1$ turns and the second of $w_2$ turns arranged in each element as close as possible to each other, and that the magnetic flux entering one of the turns is linked with all the other turns of the given element as well.

Generally, when currents $i_1$ and $i_2$ flow through both elements, the magnetic fluxes can be represented as a result of superimposing the fluxes produced by each of the currents individually, assuming a linear system.

The following notations for the magnetic fluxes have been adopted in Fig. 37.

$\Phi_1$ the total flux linking the first element;
$\Phi_2$ the total flux linking the second element;
$\Phi_{11}$ the flux produced by current $i_1$;
$\Phi_{22}$ the flux produced by current $i_2$;
$\Phi_{12}$ the flux produced by current $i_2$ and linking the turns of the first element;

$\Phi_{21}$ the flux produced by current $i_1$ and linking the turns of the second element;

$\Phi_{s1}$ the flux produced by current $i_1$ and linking the turns of the first element only (leakage flux of first element);

$\Phi_{s2}$ the flux produced by current $i_2$ and linking the turns of the second element only (leakage flux of second element);

$\Phi_{s3}$ the flux common to both elements (the mutual flux).

From the above it follows that

$$\left.\begin{aligned} \Phi_{11} &= \Phi_{s1} + \Phi_{21}, \quad \Phi_{22} = \Phi_{s2} + \Phi_{12}, \\ \Phi_1 &= \Phi_{11} + \Phi_{12}, \quad \Phi_2 = \Phi_{22} + \Phi_{21}, \\ \Phi &= \Phi_{12} + \Phi_{21}. \end{aligned}\right\} \tag{4.36}$$

Obviously $\Phi_{11} > \Phi_{21}$ and $\Phi_{22} > \Phi_{12}$. The smaller the leakage fluxes $\Phi_{s1}$ and $\Phi_{s2}$, the more $\Phi_{11}$ approximates to $\Phi_{21}$ and, correspondingly, $\Phi_{22}$ to $\Phi_{12}$.

When currents $i_1$ and $i_2$ change in time, the fluxes produced by these currents also change. In accordance with the law of electromagnetic induction a change on flux $\Phi_1$ induces an e.m.f. in the first element

$$e_1 = -w_1 \frac{d\Phi_1}{dt} = -\frac{d\Psi_1}{dt}, \tag{4.37}$$

and a change in flux $\Phi_2$ induces an e.m.f. in the second element

$$e_2 = -w_2 \frac{d\Phi_2}{dt} = -\frac{d\Psi_2}{dt}. \tag{4.38}$$

Here $\Psi_1 = w_1\Phi_1$ and $\Psi_2 = w_2\Phi_2$ are the flux linkages of each of the elements. Taking (36) into account, equations (37) and (38) can be written thus:

$$\left.\begin{aligned} e_1 &= -\left(\frac{d\Psi_{11}}{dt} + \frac{d\Psi_{12}}{dt}\right), \\ e_2 &= -\left(\frac{d\Psi_{22}}{dt} + \frac{d\Psi_{21}}{dt}\right), \end{aligned}\right\} \tag{4.39}$$

where $\Psi_{11} = w_1\Phi_{11}$ and $\Psi_{22} = w_2\Phi_{22}$ are the *flux linkages of the self-inductance* of each of the elements; and $\Psi_{12} = w_1\Phi_{12}$ and $\Psi_{21} = w_2\Phi_{21}$ are the *flux linkages of the mutual inductance* of both elements.

In the case of a linear circuit the relationships (2), (6) and (7) of Chapter I obtain:

$$\left.\begin{aligned} L_1 &= \frac{\Psi_{11}}{i_1}, \quad L_2 = \frac{\Psi_{22}}{i_2}, \\ M &= \frac{\Psi_{12}}{i_2} = \frac{\Psi_{21}}{i_1} = \sqrt{\left(\frac{\Psi_{12}\Psi_{21}}{i_1 i_2}\right)}. \end{aligned}\right\} \tag{4.40}$$

Substituting (40) in (35) yields

$$k = \sqrt{\left(\frac{\Psi_{12}\Psi_{21}}{\Psi_{22}\Psi_{11}}\right)}. \tag{4.41}$$

It is evident from equation (41) that the magnitude of the coefficient of coupling is less than unity since $\Psi_{12}\Psi_{21} < \Psi_{22}\Psi_{11}$.

Introducing the concept of the *leakage flux linkages* $\Psi_{s1} = w_1\Phi_{s1}$ and $\Psi_{s2} = w_2\Phi_{s2}$ for each of the elements, on the basis of (36) we can write:

$$\left.\begin{aligned} \Psi_{s1} &= w_1(\Phi_{11} - \Phi_{21}) = \Psi_{11} - \frac{w_1}{w_2}\Psi_{21}, \\ \Psi_{s2} &= w_2(\Phi_{22} - \Phi_{12}) = \Psi_{22} - \frac{w_2}{w_1}\Psi_{12}. \end{aligned}\right\} \tag{4.42}$$

By analogy with the previous case the leakage flux linkages of each element relate to the *leakage inductance* of the given element thus:

$$\left.\begin{aligned} L_{s1} &= \frac{\Psi_{s1}}{i_1}, \\ L_{s2} &= \frac{\Psi_{s2}}{i_2}. \end{aligned}\right\} \tag{4.43}$$

Substituting (42) in (43) results in the following relationships:

$$\left.\begin{aligned} L_{s1} &= \frac{\Psi_{11}}{i_1} - \frac{w_1}{w_2}\frac{\Psi_{21}}{i_1} = L_1 - \frac{w_1}{w_2}M, \\ L_{s2} &= \frac{\Psi_{22}}{i_2} - \frac{w_2}{w_1}\frac{\Psi_{12}}{i_2} = L_2 - \frac{w_2}{w_1}M. \end{aligned}\right\} \tag{4.44}$$

We shall go on to examine these expressions in the next paragraph when studying an equivalent circuit of an air-core transformer.

As was mentioned above, the schematic diagram of a magnetic field given in Fig. 37 corresponds to additive polarity. If the positive direction

of the current $i_2$ is changed in it, then the direction of the fluxes $\Phi_{s2}$ and $\Phi_{12}$ changes, which corresponds to subtractive polarity.

In this case the sign should be changed in front of $i_2$, $\Psi_{s2}$, $\Psi_{12}$ and $\Psi_{22}$ in the equations given above. Then, as (40) and (44) show, the values $L_1$, $L_2$, $M$, $k$, $L_{s1}$ and $L_{s2}$ remain unaltered.

The mutual inductance, leakage inductance and the magnitude of the coefficient of coupling depend on the orientation of the coils and on the core material. Figure 38*a* gives a diagrammatic representation of a bifilar

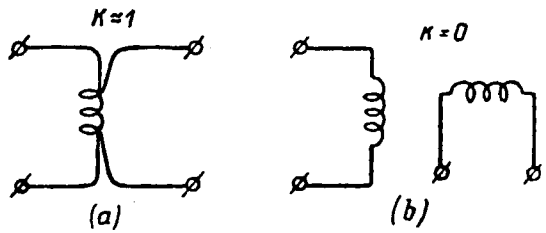



FIG. 38. Examples of coils with a coefficient of coupling: (*a*) $k \approx 1$; (*b*) $k \approx 0$.

winding, which makes it possible to obtain a coupling coefficient close to unity.

When the axes of the coils are arranged perpendicularly (Fig. 38*b*), the coupling coefficient is zero.

By shifting one coil with respect to the other, it is possible to change the coupling coefficient within required limits *(variometer)*.

The use of a ferromagnetic coil raises the coupling coefficient, bringing it nearer to unity. But the circuit then loses its linear property. In those cases when, in accordance with working conditions, the magnetic flux density varies within the limits of the linear part of the magnetization curve, and permeability of the core material can be taken as constant, the given circuit is considered to be linear and the techniques for its solution remain the same as for an air-core system.

## 12. Equations and Equivalent Circuits of an Air-Core Transformer

A *transformer* is a device which transfers energy from one circuit to another by electromagnetic induction. It consists of two or more magnetically coupled coils or windings mounted on a common core. In this section the two-winding transformer without a ferromagnetic core (so called air-core transformer) is considered. This type of transformer can be a component part of a linear electrical circuit in automation, technical measuring and communication equipment. If we disregard the capacitance between the turns in the winding of the transformer, the latter may be

represented by Fig. 39 in which the resistances of the windings have been extracted and shown separately.

The transformer winding connected to the feed source is called the *primary winding* and that to which the load is connected is known as the *secondary*. Correspondingly the voltages and currents at the terminals of these windings are called primary and secondary. It should be noted that in certain cases this appellation is conventional if, depending on the energy conditions, transmission can take place in either direction.

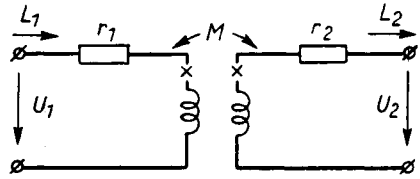



FIG. 39. Air-core transformer.

The positive direction of the currents and the winding markings adopted in Fig. 39 correspond to subtractive polarity.

In this case, the transformer equations in differential form are:

$$\left.\begin{aligned} u_1 &= r_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} - M \frac{\mathrm{d}i_2}{\mathrm{d}t}, \\ -u_2 &= r_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} - M \frac{\mathrm{d}i_1}{\mathrm{d}t}. \end{aligned}\right\} \tag{4.45}$$

If the voltages and currents are sinusoidal, the transformer equations in terms of sinors are written thus:

$$\left.\begin{aligned} \dot{U}_1 &= (r_1 + j\omega L_1)\dot{I}_1 - j\omega M\dot{I}_2, \\ -\dot{U}_2 &= (r_2 + j\omega L_2)\dot{I}_2 - j\omega M\dot{I}_1 \end{aligned}\right\} \tag{4.46a}$$

or

$$\left.\begin{aligned} \dot{U}_1 &= [r_1 + j\omega(L_1 - M) + j\omega M]\dot{I}_1 - j\omega M\dot{I}_2, \\ -\dot{U}_2 &= [r_2 + j\omega(L_2 - M) + j\omega M]\dot{I}_2 - j\omega M\dot{I}_1. \end{aligned}\right\} \tag{4.46b}$$

The last equations represent the loop analysis of the circuit of Fig. 40.

Consequently this circuit can be regarded as an *equivalent circuit* of an air-core transformer. Unlike Fig. 39, in this equivalent circuit the primary and secondary sides of the transformer are not magnetically but conductively coupled*.

* It should be noted that if a circuit using magnetic coupling is to be solved by the node method, an equivalent circuit as shown in Fig. 40 can first be introduced. The node method is then applied as in any network.

If $L_1 = L_2$, then $L_1 - M > 0$ and $L_2 - M > 0$, since the coupling coefficient $k = M/\sqrt{L_1 L_2} < 1$. When the values $L_1$ and $L_2$ are not equal, one of the differences $L_1 - M$ or $L_2 - M$ may be negative. In this case the Fig. 40 equivalent circuit can only be realized physically at a fixed frequency when the negative inductance can be replaced by a capacitive element. In general, a circuit with a linear element which has a negative inductance is unrealizable.

The differences $L_1 - M$ and $L_2 - M$ contained in the Fig. 40 circuit are of physical significance only when there is the same number of turns in the primary and secondary windings ($w_1 = w_2$). In this case, as (44) shows, they represent the leakage inductances of the primary and secondary windings of the transformer.

When there is a different number of turns in the primary and secondary windings ($w_1 \neq w_2$) an equivalent circuit *referred* to primary or secondary winding is sometimes used in practice.

With scaling of voltage and current on one side (say, when referring to primary), voltage $\dot{U}_2$ is multiplied by $n$ and current $\dot{I}_2$ is divided by $n = w_1/w_2$ ($n$ is the ratio of the numbers of turns).

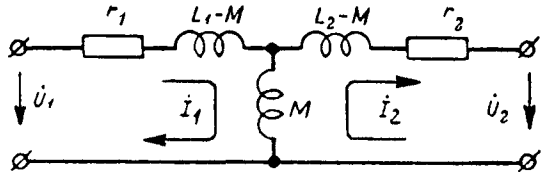



FIG. 40. Equivalent circuit of an air-core transformer.

By giving equations (46a) the form

$$\dot{U}_1 = (r_1 + j\omega L_1)\dot{I}_1 - j\omega nM\frac{\dot{I}_2}{n},$$

$$-n\dot{U}_2 = n^2(r_2 + j\omega L_2)\frac{\dot{I}_2}{n} - j\omega nM\dot{I}_1,$$

it is possible to rearrange them thus:

$$\dot{U}_1 = [r_1 + j\omega(L_1 - nM)\dot{I}_1 + j\omega nM\dot{I}_1 - j\omega nM\frac{\dot{I}_2}{n},$$

$$-n\dot{U}_2 = n^2\left[r_2 + j\omega\left(L_2 - \frac{M}{n}\right)\right]\frac{\dot{I}_2}{n} + j\omega nM\frac{\dot{I}_2}{n} - j\omega nM\dot{I}_1.$$

The resulting equations are recognized as the loop equations of the

equivalent circuit of the transformer referred to the primary winding, shown in Fig. 41.

An equivalent circuit of a transformer referred to the primary winding contains: a resistance $r_1$ and a leakage inductance $L_{s1}$ of the primary transformer winding; an inductance $(w_1/w_2)M$ in the central branch (magnetizing branch); a resistance $r_2$ and a leakage inductance $L_{s2}$ of the secondary winding, referred to the primary, i.e. multiplied by $n^2 = (w_1/w_2)^2$ (the square of the ratio of the number of turns).

The inductive reactances $\omega L_{s1}$ and $\omega L_{s2}$ are the leakage reactances of the primary and secondary windings, and the inductive reactance $n\omega M$ is the reactance of the magnetizing branch.

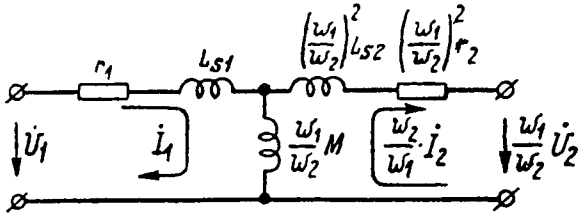



FIG. 41. Equivalent circuit of a two-winding transformer referred to primary.

The current $\dot{I}_1 - \dot{I}_2/n$ flowing through the magnetizing branch is called the transformer's *magnetizing current.*

The sinor diagram given in Fig. 42 corresponds to the Fig. 41 circuit. The scaled secondary voltage and current can be used to constuct the sinor diagram.

The voltage drop of the scaled secondary current in the resistance and inductive leakage reactance of the secondary winding referred to primary are added vectorially to the scaled secondary voltage. The resulting voltage equals the voltage drop of the magnetizing current in the inductive reactance of the magnetizing branch, the magnetizing current lagging behind the voltage obtained by 90°. The primary current is found as the vectorial sum of the magnetizing current and the scaled secondary current

$$\left(\dot{I}_1 - \frac{\dot{I}_2}{n}\right) + \frac{\dot{I}_2}{n} = \dot{I}_1.$$

The voltage drop of current $\dot{I}_1$ in the resistance and the inductive leakage reactance of the primary winding are added vectorially to the voltage of the magnetizing branch, forming a primary voltage.

In view of the fact that the secondary electrical values in Fig. 41, voltage $\dot{U}_2$ and current $\dot{I}_2$, are referred to the primary winding, i.e. are scaled by the ratio of the numbers of turns, the given circuit is not equivalent to the transformer's initial circuit.

To make the circuit diagram equivalent to the given transformer circuit we can use a so-called *ideal transformer* to which we shall ascribe the following property: in any conditions the relationship of the primary to the secondary voltage at the terminals of the ideal transformer, which is equal

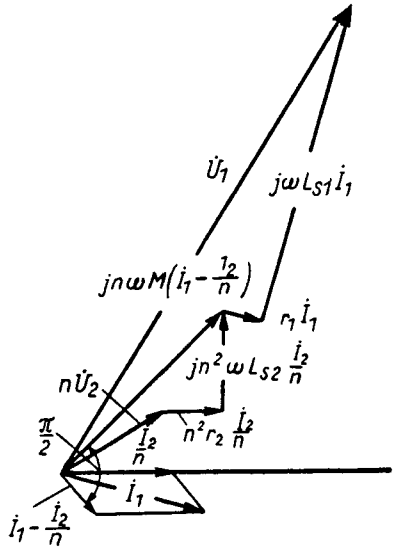



FIG. 42. Sinor diagram for the circuit of Fig. 41.

to the relationship between the secondary and primary currents, is determined by the winding turns ratio (transformer ratio). An ideal transformer has no energy losses, and the current does not flow through its primary winding when its secondary winding is open.

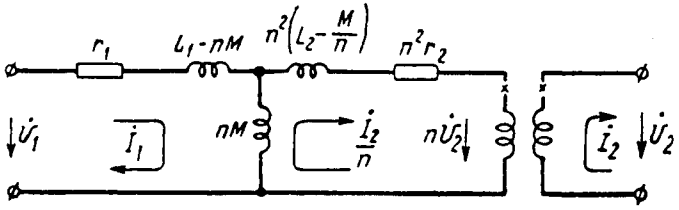



FIG. 43. Equivalent circuit, with $n = w_1/w_2$.

In fact, this transformer does not exist, but a transformer with a coupling coefficient close to unity and with such a large number of turns that the impedance of its magnetizing branch can be regarded as infinite, approximates to it by virtue of its qualities.

By supplementing the Fig. 41 circuit with an ideal transformer with a transformation ratio $n$, we get the equivalent transformer circuit given in

Fig. 43. It should be noted that this equivalent circuit is true for an arbitrary value of factor $n$; the latter need not be selected equal to the ratio $w_1/w_2$.

When $n = 1$ the Fig. 43 circuit becomes the Fig. 40 circuit. The case $n = \sqrt{(L_1/L_2)}$ is also of interest.

Taking into account formula (35)

$$L_1 - nM = L_1 - \sqrt{\left(\frac{L_1}{L_2}\right)} k \sqrt{(L_1 L_2)} = (1 - k)L_1,$$

$$L_2 - \frac{M}{n} = L_2 - \sqrt{\left(\frac{L_2}{L_1}\right)} k \sqrt{(L_1 L_2)} = (1 - k)L_2,$$

$$nM = \sqrt{\left(\frac{L_1}{L_2}\right)} k \sqrt{(L_1 L_2)} = kL_1$$

and the equivalent circuit in this case is shown in Fig. 44.

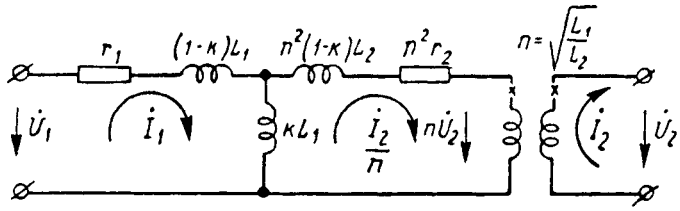



FIG. 44. Equivalent circuit, with $n = \sqrt{(L_1/L_2)}$.

The Figs. 40–44 correspond to the subtractive polarity shown in Fig. 39.

If the positive direction of the secondary current were changed, the equivalent circuits would correspond to additive polarity. Equivalent circuits for additive polarity can also be obtained by changing the sign of the mutual inductance $M$ and voltage $\dot{U}_2$.

Let us examine the question of the energy of magnetically coupled windings.

For additive polarity the transformer's differential equations have the form

$$u_1 = r_1 i_1 + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t},$$

$$u_2 = r_2 i_2 + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + M \frac{\mathrm{d}i_1}{\mathrm{d}t}. \qquad (4.47)$$

Multiplying the first equation by $i_1$ and the second by $i_2$ and adding the results we get

$$p = u_1 i_1 + u_2 i_2 = r_1 i_1^2 + r_2 i_2^2 + L_1 i_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 i_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + M\left(i_1 \frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2 \frac{\mathrm{d}i_1}{\mathrm{d}t}\right)$$

or

$$p = r_1 i_1^2 + r_2 i_2^2 + \frac{\mathrm{d}W}{\mathrm{d}t}, \tag{4.48}$$

where $W$ is the energy of the magnetic field, equal to

$$W = \frac{L_1 i_1^2}{2} + \frac{L_2 i_2^2}{2} + M i_1 i_2. \tag{4.49}$$

As we should expect, the instantaneous power supplied to the coupled windings is equal to the sum of the instantaneous values of the power converted into heat and the time rate of the change in the energy $W$ stored in the magnetic field.

The first item of the energy of the magnetic field equals the energy of the field of the first winding when $i_2 = 0$; the second item equals the energy of the field of the second winding when $i_1 = 0$; the third item is the energy resulting from the interaction of the two currents.

For subtractive polarity the third item in the energy expression has a minus sign.

The first and second items are essentially positive, while the third, depending on the signs of the instantaneous currents $i_1$ and $i_2$, can have a positive or a negative sign. Therefore the energy of a system consisting of magnetically coupled windings can be more or less than the sum of the energy of each winding taken separately.

EXAMPLE 14. To solve Example 12 with the aid of a transformer equivalent circuit.

Regarding the magnetically coupled elements in Fig. 45*a* as a transformer with its primary and secondary terminals connected in pairs and using the Fig. 50 equivalent circuit (changing the direction of current $\dot{I}_2$) we get the equivalent circuit of Fig. 45*b*, without magnetic coupling.

The complex impedance at the terminals of the circuit is

$$Z = \frac{\dot{U}}{\dot{I}} = j\omega M + \frac{[r_1 + j\omega(L_1 - M)]\,[r_2 + j\omega(L_2 - M)]}{r_1 + r_2 + j\omega(L_1 + L_2 - 2M)}.$$

Using the shortened form

$$Z_1 = r_1 + j\omega L_1, \quad Z_2 = r_2 + j\omega L_2, \quad Z_M = j\omega M,$$

we find

$$Z = Z_M + \frac{(Z_1 - Z_M)(Z_2 - Z_M)}{Z_1 + Z_2 - 2Z_M} = \frac{Z_1 Z_2 - Z_M}{Z_1 + Z_2 - 2Z_M}.$$

EXAMPLE 15. Two magnetically coupled elements have inductances $L_1 = 4$ H and $L_2 = 25$ H; the coupling coefficient $k = 0{\cdot}5$. To determine the energy of the magnetic field produced by these elements when currents $i_1 = 10$ A and $i_2 = 20$ A.

The mutual inductance is

$$M = k\sqrt{(L_1 L_2)} = 0{\cdot}5\sqrt{(4\,.\,25)} = 5 \text{ H}.$$

The energy of each element, taken separately, is

$$\frac{L_1 i_1^2}{2} = \frac{4\,.\,10^2}{2} = 200 \text{ J}, \quad \frac{L_2 i_2^2}{2} = \frac{25\,.\,20^2}{2} = 5000 \text{ J}.$$

The energy of the mutual interaction

$$M i_1 i_2 = 5\,.\,10\,.\,20 = 1000 \text{ J}.$$

For additive polarity the energy of the field of the entire system

$$W = 200 + 5000 + 1000 = 6200 \text{ J}$$

and for subtractive polarity

$$W = 200 + 5000 - 1000 = 4200 \text{ J}.$$

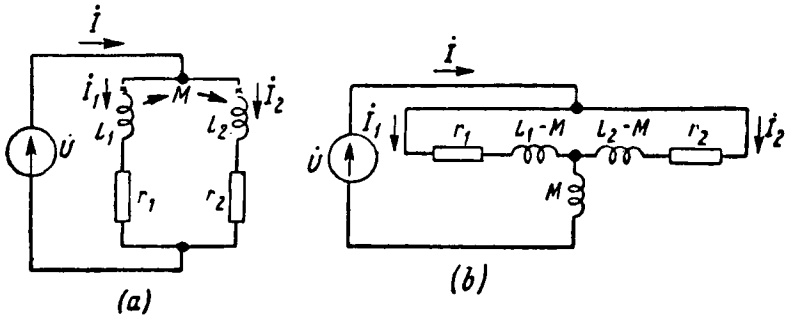



FIG. 45. Example 14.

## 13. Input Impedance of Coupled Circuits

If the load $Z_l$ is coupled indirectly to a source through a transformer (Fig. 46), then in accordance with (46)

$$\dot{U}_1 = (r_1 + j\omega L_1)\dot{I}_1 \pm j\omega M\dot{I}_2,$$

$$0 = \pm j\omega M\dot{I}_1 + (r_2 + j\omega L_2 + Z_1)\dot{I}_2.$$

Consequently

$$\dot{I}_2 = \frac{\pm j\omega M\dot{I}_1}{r_2 + j\omega L_2 + Z_1},$$

from which the input impedance of the transformer is

$$Z = \frac{\dot{U}_1}{\dot{I}_1} = r_1 + j\omega L_1 + \frac{(\omega M)^2}{r_2 + j\omega L_2 + Z_1}. \tag{4.50}$$

The third item in the right-hand part of equation (50) is a *complex impedance introduced into the primary circuit from the secondary circuit* (insertion impedance). The Fig. 46 circuit is equivalent to the Fig. 47 circuit.

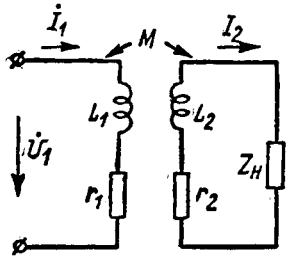



FIG. 46. Coupling of an impedance to a source through a transformer.

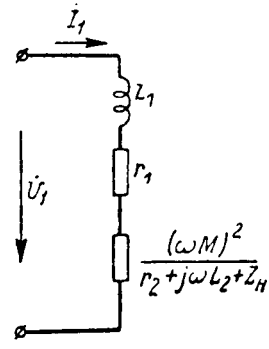



FIG. 47. Equivalent circuit for determining the input impedance of a transformer.

As we mentioned above the ideal transformer is characterized by the conditions $M^2 = L_1L_2$ (perfect coupling, i.e. $k = 1$) and $r_1 = r_2 = 0$, where $L_1$ and $L_2$ are infinitely great. Consequently, in the case of an ideal transformer

$$Z = \left(j\omega L_1 + \frac{\omega^2 L_1 L_2}{j\omega L_2 + Z_1}\right)_{\substack{L_1\to\infty \\ L_2\to\infty}}$$

$$= \left(\frac{j\omega L_1 Z_1}{j\omega L_2 + Z_1}\right)_{\substack{L_1\to\infty \\ L_2\to\infty}} = \left(\frac{L_1}{L_2}\right)_{\substack{L_1\to\infty \\ L_2\to\infty}} \cdot Z_1. \tag{4.51}$$

When $L_1$ and $L_2$ are tending to infinity their ratio remains finite. The inductance of a winding is proportional to the square of the number of turns, therefore

$$Z = \left(\frac{w_1}{w_2}\right)^2_{\substack{w_1\to\infty \\ w_2\to\infty}} \cdot Z_1 = n^2 Z_1. \tag{4.52}$$

Consequently the ideal transformer, placed between the load and the electrical energy source, changes the load impedance by the ratio of the square of the numbers of turns.

In practice this phenomenon is used in the various fields of electrical engineering, communications, radio, instrument making and automation, for impedance-matching purposes, i.e. to match the impedances of the source and the load (in order to achieve the condition for maximum power transfer).

Although the ideal transformer is also a theoretical concept, the properties of an actual transformer can approximate to it to a considerable extent. Therefore, when it is necessary to change the magnitude of the

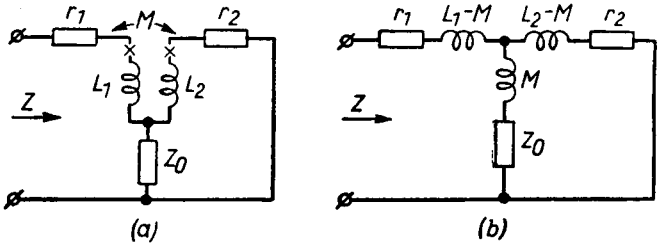



FIG. 48. Example 16: (*a*) initial circuit; (*b*) equivalent circuit.

impedance of any element without changing the element itself, an intermediate transformer with a transformer ratio determined by (52) is connected:

$$n = \sqrt{\frac{Z}{Z_1}}, \tag{4.53}$$

where $Z$ is the required value of the impedance.

Formulae (51) and (52) follow directly from the conditions characterizing the ideal transformer, i.e.

$$\frac{\dot{U}_1}{\dot{U}_2} = \frac{\dot{I}_2}{\dot{I}_1} = n,$$

from which

$$Z = \frac{\dot{U}_1}{\dot{I}_1} = \frac{n\dot{U}_2}{\frac{1}{n}\dot{I}_2} = n^2 Z_1.$$

When the secondary winding is short circuited the input impedance of the ideal transformer reverts to zero, and when the secondary winding is open it reverts to infinity.

EXAMPLE 16. To determine the input impedance of the circuit shown in Fig. 48*a*. Using the substitution circuit given in Fig. 48*b* we find:

$$Z = r_1 + j\omega(L_1 - M) + \frac{[r_2 + j\omega(L_2 - M)]\,[j\omega M + Z_0]}{r_2 + j\omega L_2 + Z_0}.$$

EXAMPLE 17 (Ref. 90). To determine the input impedance of a circuit consisting of two transformers connected in cascade with a load $r_l$ at the output (Fig. 49*a*). The resistance of the windings of the transformers are not taken into account.

In accordance with (50) the input impedance of the second transformer

$$Z_2 = j\omega L_2 + \frac{(\omega M)^2}{j\omega L_2 + r_l}.$$

Regarding $Z_2$ as the load of the first transformer we can determine the input impedance of the circuit

$$Z = j\omega L_1 + \frac{(\omega M_1)^2}{j\omega L_1 + Z_2}.$$

After a number of conversions we find

$$Z = \frac{\omega r_l(L_1^2 + L_1L_2 - M_1^2) + j\omega^2(L_1^2L_2 + L_1L_2^2 - L_1M_2^2 - L_2M_1^2)}{\omega(L_1L_2 + L_2^2 - M_2^2) - jr_l - (L_1 + L_2)}.$$

The same result can be arrived at by using the equivalent circuit shown in Fig. 49*b*.

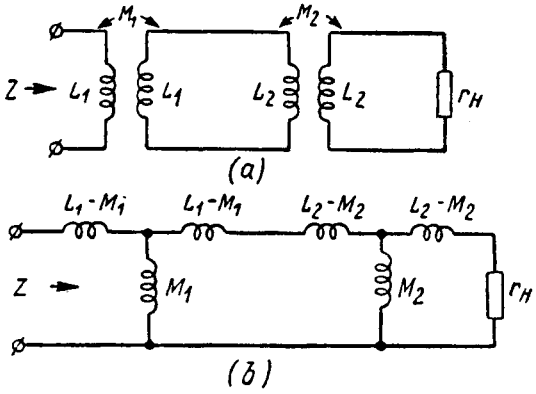



FIG. 49. Example 17: (*a*) initial circuit; (*b*) equivalent circuit.

## 14. Duality

By the condition of *duality* we understand the correspondence of two networks for which the loop equations of one have the same form as the node equations of the other.

For the circuit elements shown in Fig. 50 (left and right) the voltage–current relationships have the form:

$$\left.\begin{aligned} u_r &= ri_r, \\ u_L &= L\frac{di_L}{dt}, \\ u_C &= \frac{1}{C}\int_0^t i_C\,dt + u_C(0), \end{aligned}\right\} \qquad (4.54)$$

$$\left.\begin{aligned} i_r &= g' u_r, \\ i_C &= C' \frac{\mathrm{d}u_C}{\mathrm{d}t}, \\ i_L &= \frac{1}{L'} \int_0^t u_L \, \mathrm{d}t + i_L(0). \end{aligned}\right\} \tag{4.55}$$

Here $u_C(0)$ and $i_L(0)$ are the initial values of the voltage on the capacitance and the current in the inductance, i.e. the values of the said electrical quantities when $t = 0$.

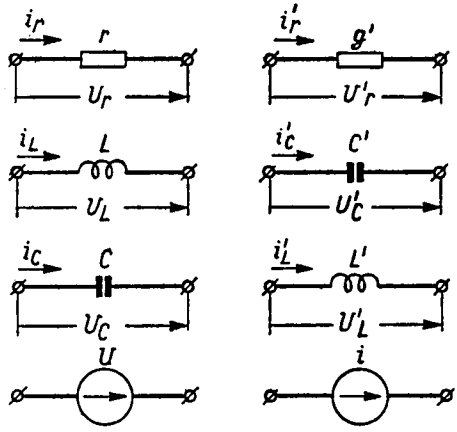



FIG. 50. Analogues or dual elements.

It should be borne in mind that the voltages and currents contained in equations (54) and (55) with identical conventional letters, are not equal to each other.

It is evident from comparing equation (54) with equation (55) that the condition of duality is satisfied by the following elements:

resistance and conductance;
inductance and capacitance;
e.m.f. source and current source.

The circuit elements satisfying the condition of duality are called *analogues* or *dual elements.*

When the circuit elements are connected in series, the voltages are added; when they are in parallel, the currents are added. Therefore a parallel connexion of analogues corresponds to a series connexion of elements, and a series connexion of analogues corresponds to a parallel connexion of elements.

For example, when the series connected elements $r$, $L$ and $C$ (Fig. 51*a*) are replaced by their analogues $g'$, $C'$ and $L'$ connected in parallel (Fig. 51*b*) we get dual circuits.

The loop current equation for the original circuit

$$u(t) = ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{4.56}$$

is analogous to the node voltage equation for the second circuit

$$i(t) = g'u + C'\frac{du}{dt} + \frac{1}{L'}\int u\,dt. \tag{4.57}$$

Node voltage $u$ in equation (57) corresponds to loop current $i$ in equation (56).

If the e.m.f. and the current of the sources are subject to the same law, e.g. the sinusoidal law, and have the same initial phase, the laws of the

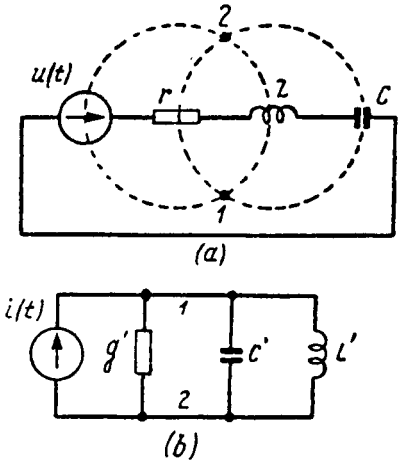



FIG. 51. Dual networks.

change in the loop current in Fig. 51*a* and the nodal voltage in Fig. 51*b* agree with each other when the proportion

$$\frac{r}{g'} = \frac{L'}{C} = \frac{L}{C'} \tag{4.58}$$

is observed.

In the case of any complex network diagram a fixed node of the dual network corresponds to each region bounded by an independent loop of the given network, also taking into account the region which is external

with respect to the whole network diagram and, correspondingly, the number of regions of the given network equals the number of nodes in the dual network.

When constructing a dual network for a given planar network* it is convenient to use the following graphic procedure (Ref. 14).

In each of the regions enclosed by an independent loop of the given network we mark a point, which is regarded as the future node of the dual network (Fig. 51).

The nodes corresponding to each pair of adjacent regions are connected together by parallel branches, the number of which is equal to the number

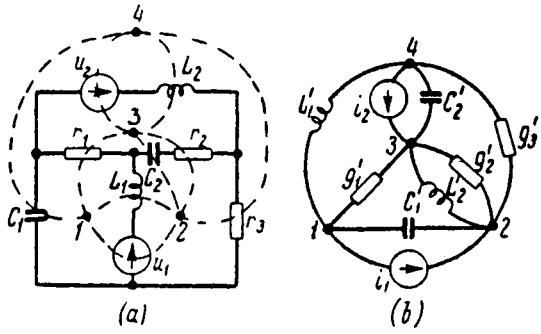



FIG. 52. Example of the construction of a dual network.

of elements connected in series in the circuit adjacent to the said regions. The elements of the parallel branches are analogues of the elements of the given network.

The graphic method of constructing a dual network is illustrated by Fig. 52.

The initial network (Fig. 52*a*) contains three independent loops. Within these loops we fix three points (*1*, *2* and *3*) corresponding to the nodes of the unknown dual network. We fix a fourth point, corresponding to node *4*, in the region which is external with respect to the given network. Let us draw dotted lines between these points, traversing the circuit elements and representing the branches of the dual network: the traversed elements are replaced by their analogues connected between the corresponding nodes of the dual network.

When the directions of the e.m.f. and the currents of the dual sources agree, the following rule is used; if the e.m.f. of the source acts in a positive direction in the loop (clockwise), the current of the current source in the dual network is directed to the node corresponding to the given loop of the initial network.

* Networks which are not planar do not have duals.

It should be noted that if the graphic method of constructing a dual network is applied again to the network of Fig. 52*b*, the original Fig. 52*a* network is obtained.

There are dual networks with an identical lay-out, e. g. bridge type networks (the so-called autodual networks).

When proportion (58) is observed, the complex impedance of the circuit common to two adjacent loops in the diagram is proportional to the complex admittance of the circuit connecting two corresponding nodes of the dual network. For example in the Fig. 51*a* diagram the impedance of the circuit is

$$Z = r + j\left(\omega L - \frac{1}{\omega C}\right),$$

and in Fig. 51*b* the admittance of the circuit is

$$Y = g' - j\left(\frac{1}{\omega L'} - \omega C'\right).$$

Supposing

$$g' = \frac{r}{k^2}, \quad L' = k^2 C, \quad C' = \frac{L}{k^2},$$

where $k$ is the proportionality coefficient with the magnitude of the resistance, we get:

$$Z = k^2 Y.$$

An analogous proportionality can also be obtained between the input impedance and the admittance of more complex dual networks. This fact is used particularly in filter theory.

CHAPTER V

# LOCI METHOD

## 1. Loci in a Complex Plane

The complex notation of electrical quantities (currents, voltages, power) and their relationships (impedances, admittances, transfer functions, distribution factors) makes it possible to estimate the performance of electrical networks extensively and in many aspects by the *loci method.* The main point of this method is that the network functions being examined are represented in a complex plane by sinors or phasors, the loci of whose "tips" are shown by different curves when the parameters contained in these functions are varied. These curves, also called *locus diagrams*, show visually changes in the moduli and phases of the electrical quantities of their relationships, depending on the assumed variable.

Any one of the corresponding values which characterize the given electrical energy sources or passive elements involved can act as the variable: the magnitude or phase of the e.m.f. (or the current) of the source, frequency $\omega$, resistance $r$, inductance $L$, mutual inductance $M$, capacitance $C$.

The reactance and susceptance of any element can also serve as the variable, regardless of whether this component produces a change: by changing the frequency $\omega$ or changing the parameter of the element ($L$, $M$ or $C$) itself.

The magnitude of the impedance $z$ for a fixed value of the argument $\varphi$ or, conversely, the impedance angle $\varphi$ for a constant magnitude of the impedance $z$ can be regarded as the variable.

A change in any of these parameters can be represented graphically in a complex plane. A change in the resistance or the reactance is represented by a straight line parallel, accordingly, to the real or imaginary axis. A change in the magnitude of the impedance for a constant angle $\varphi$ is represented by a straight line at an angle $\varphi$ with the real axis. In turn, a change in the impedance angle with $z$ fixed is shown by a circle of radius $z$. If frequency $\omega$ is taken as the variable, then a change in this parameter is represented by a straight line coinciding with the imaginary axis.

A variable parameter of an electrical circuit can generally be regarded as a certain complex variable $Z$ given in a certain region of a complex plane or, for brevity, the $Z$ plane. To each value of $Z$ there corresponds a fixed value of the function being examined $W = F(Z)$, which expresses the

law of changes in the current, voltage, power, impedance, admittance or any other quantity. The total values of $W = F(Z)$ corresponding to the given $Z$ cover a certain region of the new plane of the complex variable $W$. Thus, the function $F(Z)$ completes the transformation of the $Z$ plane into the $W$ plane or, the same thing, maps points of the $Z$ plane on to the $W$ plane.

If, in a linear electrical network of any complexity, any one of the quantities $r, L, C, x, z$ or $\varphi$ is the variable, then the function being examined can usually express a *linear fractional form* (which, in the language of function theory, is a bilinear transformation)

$$W = \frac{A + BZ}{C + DZ}. \tag{5.1}$$

According to the *circle-preserving mapping property* of this function, when the tip of phasor $Z$ describes a circle in the complex plane, the tip of phasor $W$ also describes a circle or a straight line; and if $Z$ describes a straight line, then $W$ describes a straight line or a circle.

By solving (1) with respect to $Z$, in its turn, we get a linear fractional function

$$Z = \frac{A - CW}{-B + DW}, \tag{5.2}$$

which transforms a circular locus in the $W$ plane into a circular or a straight-line locus in the $Z$ plane.

With frequency $\omega$ as the variable, the character of the function under examination and of the curve which it expresses depends on the degree of $\omega$. If there is an equation of the first order with respect to $\omega$ (or, the same thing, a function such as (1) is linear in $\omega$ in both the numerator and denominator), then the circular property of the function is preserved. For higher degrees curves of a higher order are obtained.

Thus, an examination of the complex equations of electrical circuits when any parameter of the circuit changes results in loci in the form of straight lines, circles or more complex curves.

## 2. Transformation of $Y = 1/Z$

Let us examine a case where a transformation of the form $Y = 1/Z$ is applied to complex equations expressing a circle or a straight line in a complex plane and deduce the corresponding calculation equations for constructing the loci.

**Circle Equation**

Let a circle of radius $R$ with its centre at point $Z_0$ be placed in a $Z$ plane. The complex form of the equation for such a circle is

$$(Z - Z_0)(\overset{*}{Z} - \overset{*}{Z}_0) = R^2$$

or

$$Z\overset{*}{Z} - Z\overset{*}{Z}_0 - \overset{*}{Z}Z_0 + Z_0\overset{*}{Z}_0 = R^2. \tag{5.3}$$

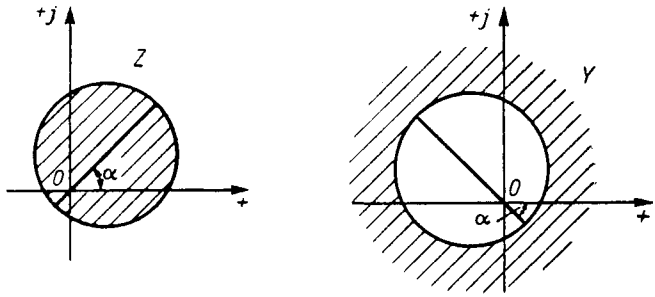



FIG. 1. Mapping the interior of a $Z$-circle into the exterior of a $Y$-circle.

Substituting $Z = 1/Y$ yields

$$Y\overset{*}{Y} - Y\frac{Z_0}{Z_0\overset{*}{Z}_0 - R^2} - \overset{*}{Y}\frac{\overset{*}{Z}_0}{Z_0\overset{*}{Z}_0 - R^2} + \frac{1}{Z_0\overset{*}{Z}_0 - R^2} = 0$$

or, equivalently,

$$Y\overset{*}{Y} - Y\frac{Z_0}{Z_0\overset{*}{Z}_0 - R^2} - \overset{*}{Y}\frac{\overset{*}{Z}_0}{Z_0\overset{*}{Z}_0 - R^2} + \frac{Z_0\overset{*}{Z}_0}{(Z_0\overset{*}{Z}_0 - R^2)^2} = \frac{R^2}{(Z_0\overset{*}{Z}_0 - R^2)^2}. \tag{5.4}$$

It follows from a comparison of (4) and (3) that the transformation $Y = 1/Z$ maps the circle of radius $R$ with its centre at point $Z_0$ in the $Z$ plane on to a circle of radius $R/(Z_0\overset{*}{Z}_0 - R^2)$ with its centre at point $\overset{*}{Z}/(Z_0\overset{*}{Z}_0 - R^2)$ of the $Y$ plane.

Analogously, starting from the equation of the circle in the $Y$ plane

$$Y\overset{*}{Y} - Y\overset{*}{Y}_0 - \overset{*}{Y}Y_0 + Y_0\overset{*}{Y}_0 = \varrho^2, \tag{5.5}$$

we get, as a result of substituting $Y = 1/Z$,

$$Z\overset{*}{Z} - Z\frac{Y_0}{Y_0\overset{*}{Y}_0 - \varrho^2} - \overset{*}{Z}\frac{\overset{*}{Y}_0}{Y_0\overset{*}{Y}_0 - \varrho^2} + \frac{Y_0\overset{*}{Y}_0}{(Y_0\overset{*}{Y}_0 - \varrho^2)^2}$$

$$= \frac{\varrho^2}{(Y_0\overset{*}{Y}_0 - \varrho^2)^2}. \tag{5.6}$$

The circle acts as the boundary of two different regions: the interior and the exterior of the circle. Depending on whether the origin is inside or outside the circle, the function $Y = 1/Z$ accordingly transforms the inside of the circle into either the outside (Fig. 1) or the inside of the mapped circle (Fig. 2).

The point on the circumference $Y$ nearest to the origin then corresponds to the point on the circumference $Z$ farthest away from the origin, and vice

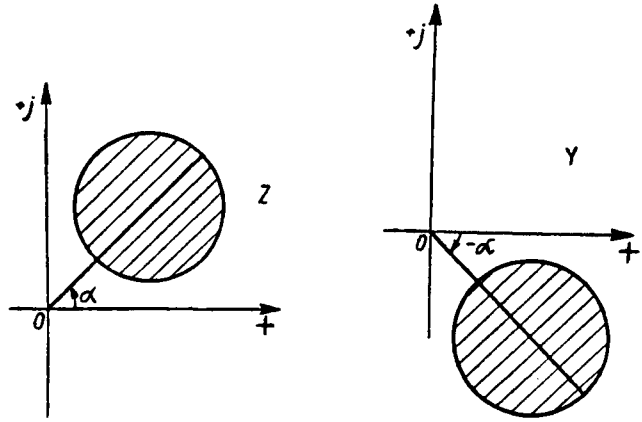



FIG. 2. Mapping the interior of a $Z$-circle into the exterior of a $Y$-circle.

versa. The diameters of the corresponding circles, passing through the origin, form identical angles, reckoned from opposite directions, with the real axis.

It should be noted that the coordinates of the centres of the circles in the $Z$ and $Y$ planes do not observe the condition $Y = 1/Z$.

In a case where the circle passes through the origin, $Z_0\overset{*}{Z}_0 = R^2$, and on the basis of (3)

$$Z\overset{*}{Z} - Z\overset{*}{Z}_0 - \overset{*}{Z}Z_0 = 0. \tag{5.3a}$$

Function $Y = 1/Z$ transforms this circle into a straight line at a distance $1/(2R)$ from the origin (Fig. 3). The diameter of circle $Z$, passing

through the origin, and the perpendicular dropped from the origin to the straight line $Y$, form identical angles with the real axis.

The equation for the straight line $Y$ can be obtained if we multiply both parts of (4) by $Z_0\overset{*}{Z}_0 - R^2$ and then proceed to the limit $Z_0\overset{*}{Z}_0 - R^2 \to 0$. The same result can be obtained directly from (3$a$) by substituting $Y = 1/Z$:

$$Y\overset{*}{Z}_0 + \overset{*}{Y}Z_0 - 1 = 0.$$

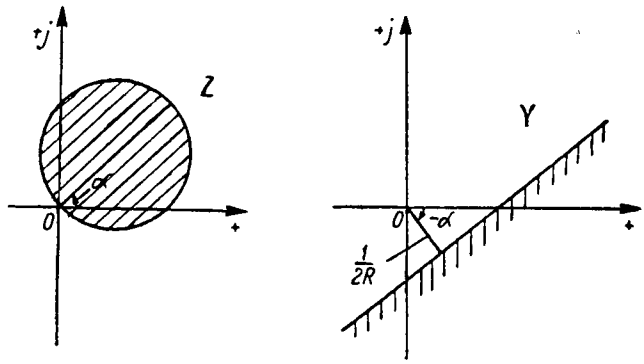



FIG. 3. Mapping a circle into a straight line, using the function $Y = 1/Z$.

**Straight-line Equation**

By multiplying both parts of (6) by $Y_0\overset{*}{Y}_0 - \varrho^2$ and directing $Y_0\overset{*}{Y}_0 - \varrho^2$ to zero we get an equation for a straight line in the Z plane, not passing through the origin:

$$Z\overset{*}{Y}_0 + \overset{*}{Z}Y_0 - 1 = 0.$$

This straight line is mapped on to the $Y$ plane in the form of a circle

$$Y\overset{*}{Y} - Y\overset{*}{Y}_0 - \overset{*}{Y}Y_0 = 0.$$

If the straight line in the $Z$ plane passes through the origin then its complex equation has the form (Ref. 76)

$$\frac{Z + \overset{*}{Z}}{Z - \overset{*}{Z}} = -j \cot \varphi, \tag{5.7}$$

when $\varphi$ is the straight line's angle of inclination (Fig. 4)

The given equation follows from the fact that the phasors $Z - \overset{*}{Z}$ and $Z + \overset{*}{Z}$ are sides of a right-angled triangle, angle $\varphi$ adjoining side $Z + \overset{*}{Z}$.

Equation (7) can be rewritten thus

$$Z(1 + j \cot \varphi) + \overset{*}{Z}(1 - j \cot \varphi) = 0.$$

The straight line also passing through the origin (Fig. 5) will be a transformation of this straight line on to the $Y$ plane.

Its equation is

$$\frac{Y + \overset{*}{Y}}{Y - \overset{*}{Y}} = j \cot \varphi \tag{5.8}$$

or

$$Y(1 - j \cot \varphi) + \overset{*}{Y}(1 + j \cot \varphi) = 0.$$

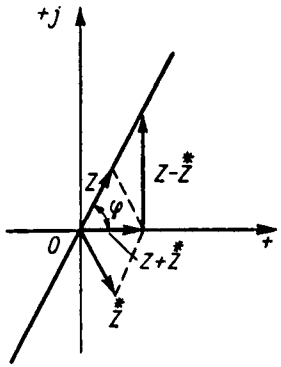



FIG. 4. Straight line passing through the origin.

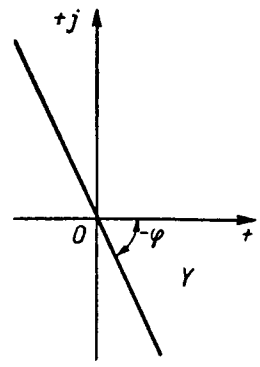



FIG. 5. Transformation of the straight line in Fig. 4 on to the $Y$ plane.

### 3. Transformation of $W = (A + B \cdot k)/(C + D \cdot k)$

Let us suppose that an electrical quantity under examination or the ratio of quantities is expressed by a linear-fractional function of the form

$$W = \frac{A + B \cdot k}{C + D \cdot k}, \tag{5.9}$$

where $A = |A| e^{j\alpha}$, $B = |B| e^{j\beta}$, $C = |C| e^{j\gamma}$, $D = |D| e^{j\delta}$ are complex numbers and $k$ is the real variable.

In accordance with the circular property of the linear-fractional function, when parameter $k$ changes the end of phasor $W$ describes a circle. The centre and radius of this circle can be found analytically or graphically.

**Analytic Method (Ref. 76)**

According to (2)

$$k = \frac{A - CW}{-B + DW}. \tag{5.10}$$

Coefficient $k$, being real, equals its conjugate

$$k = \overset{*}{k} = \frac{\overset{*}{A} - \overset{*}{C}\overset{*}{W}}{-\overset{*}{B} + \overset{*}{D}\overset{*}{W}}. \tag{5.11}$$

Equating the right-hand parts of (10) and (11) we get

$$-\overset{*}{A}\overset{*}{B} + A\overset{*}{D}\overset{*}{W} + B\overset{*}{C}W - C\overset{*}{D}W\overset{*}{W} = -\overset{*}{A}\overset{*}{B} + A\overset{*}{D}W + B\overset{*}{C}\overset{*}{W} - C\overset{*}{D}W\overset{*}{W}$$

or

$$W\overset{*}{W} + \frac{A\overset{*}{D} - B\overset{*}{C}}{C\overset{*}{D} - \overset{*}{C}D} W + \frac{B\overset{*}{C} - A\overset{*}{D}}{C\overset{*}{D} - \overset{*}{C}D} \overset{*}{W} + \frac{A\overset{*}{B} - \overset{*}{A}B}{C\overset{*}{D} - \overset{*}{C}D} = 0. \tag{5.12}$$

A comparison of (12) with (3) shows that the centre of the circle is at point

$$W_0 = \frac{A\overset{*}{D} - B\overset{*}{C}}{C\overset{*}{D} - \overset{*}{C}D} \tag{5.13}$$

and the radius of the circle is

$$\varrho = \sqrt{\left( \frac{\left(A\overset{*}{D} - B\overset{*}{C}\right)\left(B\overset{*}{C} - A\overset{*}{D}\right)}{\left(C\overset{*}{D} - \overset{*}{C}D\right)} - \frac{A\overset{*}{B} - \overset{*}{A}B}{C\overset{*}{D} - \overset{*}{C}D} \right)}. \tag{5.14}$$

**Graphic Method**

Equation (9) can be converted by dividing the numerator by the denominator

$$W = \frac{B}{D} + \frac{\dfrac{A}{C} - \dfrac{B}{D}}{1 + \dfrac{D}{C}k}. \tag{5.15}$$

The first term, which is not dependent on parameter $k$, is a phasor through the tip of which the required circle passes.

On the basis of (9) the first term $B/D$ equals the value of $W$ when $k = \infty$: $BD = W_{(k=\infty)}$.

The second term of (15) is the circle equation

$$V = \frac{\frac{A}{C} - \frac{B}{D}}{1 + \frac{D}{C}k},$$

from which

$$V + V\frac{D}{C}k = \frac{A}{C} - \frac{B}{D}. \tag{5.16}$$

The given equation shows that phasor $A/C - B/D$ is obtained as a result of adding vectorially phasor $V$ with phasor $V|D/C|ke^{-j(\gamma-\delta)}$, which

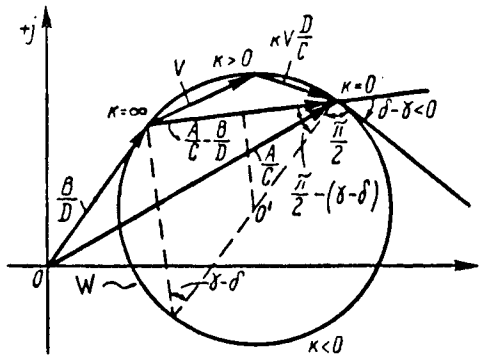



FIG. 6. Constructing a circle diagram from equation (9).

lags behind it by an angle $\gamma - \delta$; the constancy of this angle testifies to the fact that the tip of phasor $V$ describes the arc of the circle which rests on the chord $A/C - B/D$.

On the basis of (9)

$$\frac{A}{C} = W_{(k=0)}.$$

Accordingly

$$\frac{A}{C} - \frac{B}{D} = W_{(k=0)} - W_{(k=\infty)}.$$

From Fig. 6 we derive the following simple method of constructing circle $W$ for the given values of $A/C$, $B/D$ and angle $\gamma - \delta$.

On the continuation of the end of phasor $A/C - B/D$, acting as a chord of the unknown circle, an angle $-(\gamma - \delta) = \delta - \gamma$ is plotted: positive

in a clockwise direction and negative anticlockwise. The straight line obtained is a tangent to the unknown circle. The centre of this circle is at point 0′ at the intersection of the perpendiculars erected on the tangent (at the tip of phasor $A/C$) and the chord in the middle of the phasor $A/C - B/D$.

Attention should be paid to the fact that the arc of the circle on which angle $\gamma - \delta$ rests, corresponds to the positive values of parameter $k$.

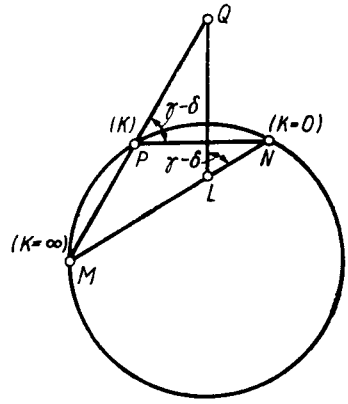



FIG. 7. Finding the point corresponding to a given parameter $k$.

Each point of the resulting circle corresponds to a definite numerical value of parameter $k$. The following graphic procedure is used for finding a point on the circle diagram for a given parameter $k$ (Fig. 7).

From an arbitrary point $L$ on chord $MN = |A/C - B/D|$ a straight line $LQ$ is produced at an angle $\gamma - \delta$. The length of $LQ$ is selected so that the ratio $(LQ)/(LM)$ is equal to $k\,|D/C|$. The straight line $MQ$ intersects the circumference at point $P$. It follows from the similarity of triangles $MPN$ and $MLQ$ that $(PN)/(MP) = (LQ)/(ML) = k\,|D/C|$. Consequently, the point $P$ obtained on the circle diagram corresponds to a value of parameter $k$. As point $Q$ approaches infinity, point $P$ moves towards point $M$ which corresponds to $k = \infty$. The auxiliary line $LQ$ is called the *variable parameter line.*

## 4. Locus Diagrams for the Impedances and Admittances of the Simplest Circuits

Let us examine locus diagrams of the impedance and admittance of series, parallel or series-parallel combinations of $r$ and $x$ elements when one of the parameters is varied.

**Series Connexion**

When the reactance $x$ is maintained constant while the amount of resistance $r$ is varied over all positive values (Fig. 8*a*), a straight line parallel to the axis of positive reals serves as the locus of the impedance phasor $Z = r + jx$ (Fig. 8*b*). The locus of the admittance phasor $Y = 1/(r + jx)$ is a semicircle passing through the origin of the $Y$ plane, the length of diameter being $1/x$ (Fig. 8*c*).

The second half of the circle corresponding to the negative values of $r$ is not taken into consideration. The farthest point on the circumference $Y$ corresponds to the value $r = 0$, and point 0 in the $Y$ plane corresponds to the value $r = \infty$.

When $x > 0$ the impedance locus lies above the real axis, and when $x < 0$ it lies below the real axis.

Accordingly, the semicircle $Y$ lies below the real axis when $x > 0$ and above it when $x < 0$.

When the resistance $r$ is held constant while the reactance $x$ is varied over all values, both positive and negative (Fig. 9*a*), a straight line parallel

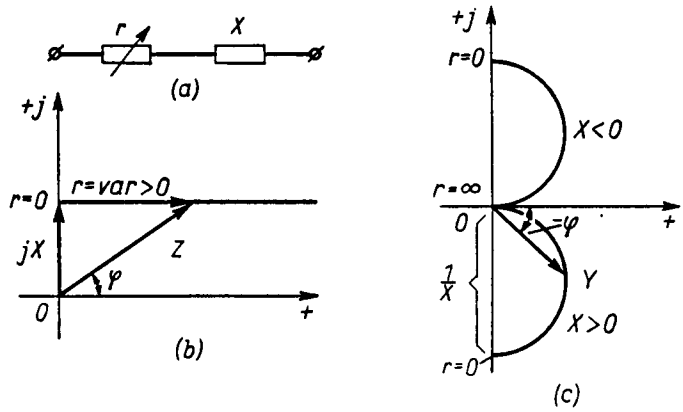



FIG. 8. A series circuit and its loci for variable $r$.

to the imaginary axis is the locus of the impedance $Z$ (Fig. 9*b*). The circle passing through the origin of the complex $Y$ plane with a diameter $1/r$ is the locus of the admittance $Y = 1/Z$ (Fig. 9*c*).

The lower semicircle corresponds to the values $x > 0$, and the upper to $x < 0$. When $x = 0$ the total impedance is a pure resistance and, accordingly, the total admittance is a pure conductance. When $x = \pm\infty$ the conductance of the circuit is zero.

If the reactance $x$ is conditioned by elements $L$ and $C$ connected in series (Fig. 10*a*), then the equation $x = 0$ characterizes the series resonance.

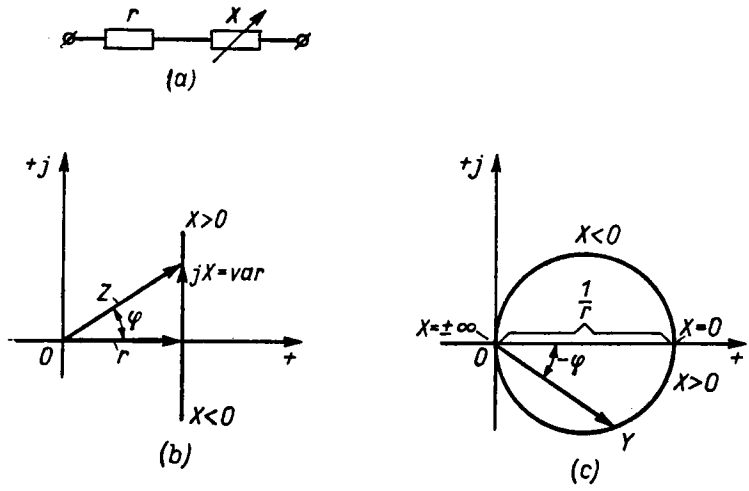



FIG. 9. A series circuit and its loci for variable $x$.

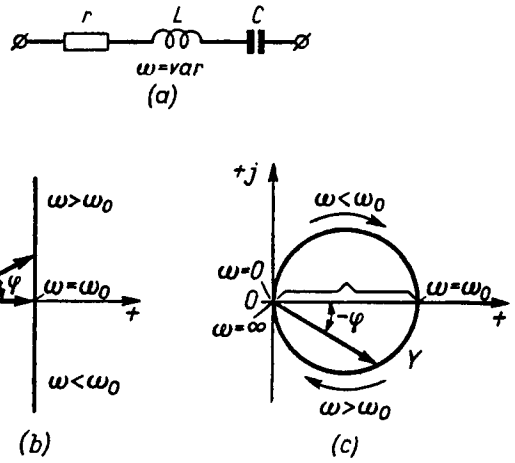



FIG. 10. Impedance and admittance loci of a series $rLC$ combination for variable $\omega$.

The series resonance can be obtained as a result of tuning the $L$ and $C$ elements or selecting the frequency $\omega = \omega_0$ for which $\omega_0 L = 1/\omega_0 C$, as a result of which the resonance frequency is

$$\omega_0 = \frac{1}{\sqrt{(LC)}}. \tag{5.17}$$

Figure 10*b* and *c* shows locus diagrams of $Z$ and $Y$ when the frequency changes from 0 to $\infty$. The reactance $x$ then changes from $-\infty$ to $+\infty$. The Fig. 10 diagrams are analogous to the Fig. 9 diagrams.

When $\omega < \omega_0$ the tip of the admittance phasor describes the upper semicircle and when $\omega > \omega_0$ it describes the lower (Fig. 10*c*). At the resonance frequency $\omega = \omega_0$, the impedance of the circuit is minimum ($r$) and the admittance maximum ($1/r$).

It should be borne in mind that the scales of frequencies in the diagrams in Fig. 10*b* and *c* are not identical.

**Parallel Connexion**

When the reactance $x$ is held constant while the resistance $r$ is varied (Fig. 11*a*), a straight line parallel to the axis of positive reals is the locus of the admittance phasor $Y = 1/r - 1/(jx)$ (Fig. 11*b*). A semicircle is the

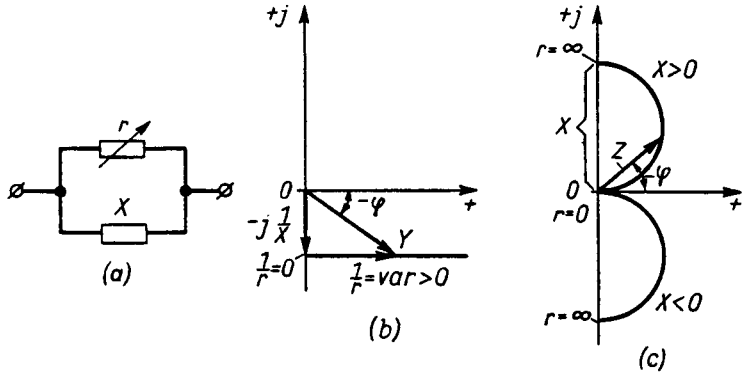



FIG. 11. A parallel circuit and its loci for variable $r$.

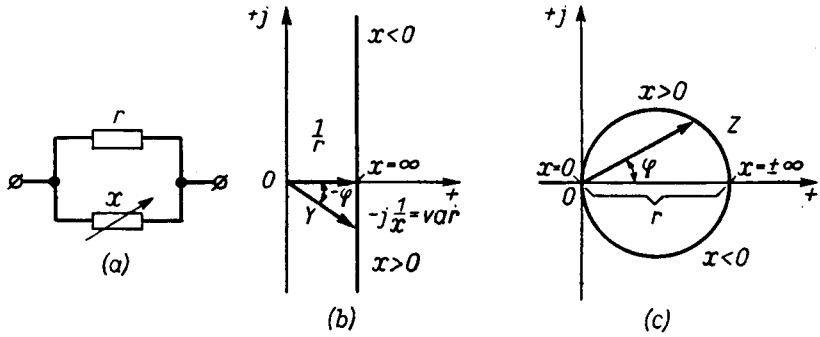



FIG. 12. A parallel circuit and its loci for variable $x$.

locus of the impedance phasor $Z = 1/y$ (Fig. 11*c*)—the upper semicircle when $x > 0$ and the lower when $x < 0$, with a diameter $x$. The origin corresponds to the value $r = 0$, and the diametrically opposed point on the circle corresponds to the value $r = \infty$.

When the reactance $x$ is varied over all values, both positive and negative (Fig. 12*a*), a straight line parallel to the imaginary axis is the admittance locus (Fig. 12*b*) and a circle of diameter $r$ passing through the origin is the impedance locus (Fig. 12*c*).

The upper semicircle corresponds to the values $x > 0$ and the lower to $x < 0$. When $x = 0$ the total impedance of the circuit is zero and the total admittance is infinitely large. When $x = \pm \infty$ the circuit is purely resistive.

If the reactance $x$ is determined by elements $L$ and $C$ connected in parallel (Fig. 13*a*), then condition $x = \infty$ characterizes parallel resonance.

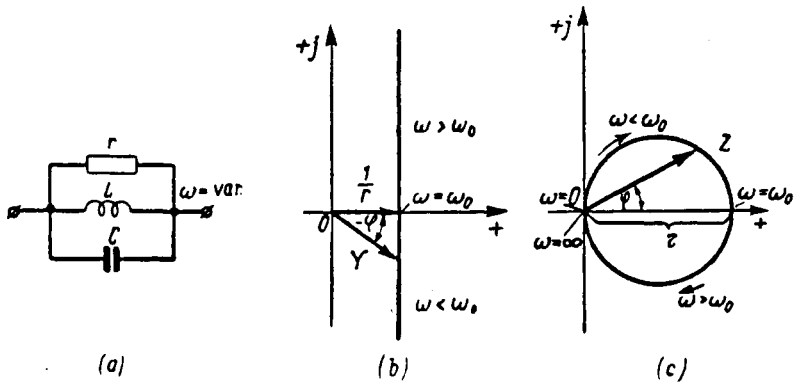



FIG. 13. Impedance and admittance loci of a parallel $rLC$ combination for variable $\omega$.

This can be obtained by tuning $L$ and $C$ or selecting frequency $\omega = \omega_0$ for which condition (17) is satisfied.

Figure 13*b* and *c* gives locus diagrams of $Y$ and $Z$, analogous to the Fig. 12 diagrams, when frequency changes from 0 to $\infty$.

When $\omega < \omega_0$ the end of the impedance phasor $Z$ describes the upper semicircle and when $\omega > \omega_0$ it describes the lower (Fig. 13*c*). At the resonance frequency $\omega = \omega_0$ the impedance of the circuit is maximum ($r$) and the admittance minimum ($1/r$).

As in Fig. 10 the scales of the frequencies in the diagrams in Fig. 13*b* and *c* are not identical.

### Series–Parallel Connexion

As an example let us examine two circuit diagrams with a series–parallel connexion of elements.

Figure 14*a* shows a circuit in which reactance $x$ is the variable. The impedance diagram of such a circuit (Fig. 14*b*) differs from the Fig. 12*c* diagram in that the circle is displaced with respect to the origin by the value

of resistance $r_1$. A change in $x$ can be produced by a change in frequency $\omega$ or a change in the parameter of $L$ or $C$.

Figure 15*a* shows a circuit in which capacitance $C'$ is the variable. The admittance of branch $rL$ is $1/r + j\omega L = r - j\omega L/r^2 + (\omega L)^2$, and the admittance of branch $C'$ is $j\omega C'$. Parallel resonance occurs when

$$\frac{\omega L}{r^2 + (\omega L)^2} = \omega C',$$

i.e. when

$$C' = \frac{L}{r^2 + (\omega L)^2}.$$

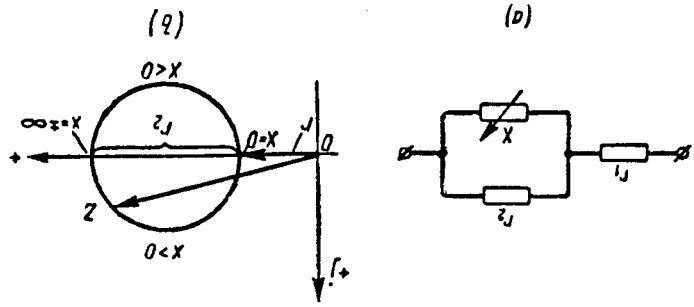



FIG. 14. Impedance locus of a series–parallel circuit for variable $x$.

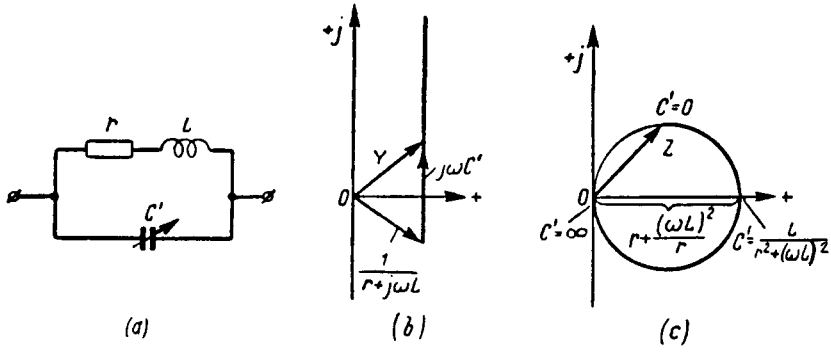



FIG. 15. Impedance and admittance loci for variable $C'$.

The admittance of the circuit then equals

$$Y = \frac{r - j\omega L}{r^2 + (\omega L)^2} + j\omega C' = \frac{r}{r^2 + (\omega L)^2},$$

and the impedance $Z = r + (\omega L)^2/r$.

The admittance locus is a straight line parallel to the imaginary axis (Fig. 15*b*), and the impedance locus is a circle passing the origin (Fig. 15*c*), the length of the diameter being equal to $r + (\omega L)^2/r$.

The part of the circle drawn in the thin line is not taken into consideration since it corresponds to the negative values of $C'$.

The use of formulae (13) and (14) is illustrated below, taking Fig. 15*a* circuit as an example.

The impedance of the given circuit

$$Z = \frac{(r + j\omega L)\frac{1}{j\omega C'}}{r + j\omega L + \frac{1}{j\omega C'}} = \frac{r + j\omega L}{(jr\omega - \omega^2 L)C' + 1}.$$

By comparing this function with the linear-fractional form

$$Z = \frac{A + Bk}{C + Dk}$$

it follows that

$$A = r + j\omega L, \quad B = 0, \quad C = 1, \quad D = jr\omega - \omega^2 L, \quad k = C'.$$

On the basis of (13) the centre of the circle is determined by the phasor

$$Z_0 = \frac{\overset{*}{A}\overset{*}{D} - \overset{*}{B}\overset{*}{C}}{\overset{*}{C}\overset{*}{D} - \overset{*}{C}\overset{*}{D}} = \frac{(r + j\omega L)(-jr\omega - \omega^2 L)}{(-jr\omega - \omega^2 L) - (j\omega r - \omega^2 L)}$$

$$= \frac{1}{2}\left[r + \frac{(\omega L)^2}{r}\right].$$

In accordance with (14) the radius of the circle is

$$\varrho = \sqrt{\left[\frac{\left(\overset{*}{A}\overset{*}{D} - \overset{*}{B}\overset{*}{C}\right)\left(\overset{*}{B}\overset{*}{C} - \overset{*}{A}\overset{*}{D}\right)}{\left(\overset{*}{C}\overset{*}{D} - \overset{*}{C}\overset{*}{D}\right)^2} - \frac{\overset{*}{A}\overset{*}{B} - \overset{*}{A}\overset{*}{B}}{\overset{*}{C}\overset{*}{D} - \overset{*}{C}\overset{*}{D}}\right]}$$

$$= \sqrt{\left[\frac{(r - j\omega L)(j\omega r - \omega^2 L)(-r - j\omega L)(-j\omega r - \omega^2 L)}{[(-jr\omega - \omega^2 L) - (jr\omega - \omega^2 L)]^2}\right]}$$

$$= \frac{1}{2}\left[r + \frac{(\omega L)^2}{r}\right].$$

The locus diagrams of impedances and admittances examined here also represent, on a fixed scale, the following diagrams: current $\dot{I} = \dot{Y}\dot{U}$, with a given voltage $\dot{U} = U$ at the terminals of the circuit; voltage $\dot{U} = \dot{Z}\dot{I}$, with a given current $\dot{I} = I$ passing through the circuit; and power $\dot{S} = U^2\overset{*}{\dot{Y}} = I^2\dot{Z}$, with $U$ or $I$ given.

If the initial phase of the applied voltage (or initial phase of the given current) differs from zero, then the locus diagram of the current (or voltage) turns through the corresponding angle.

A diagram in a complex plane represents a change in the modulus and the angle of the quantity under examination, depending on the variable parameter of the circuit. It replaces two diagrams constructed on the rectangular system of coordination for modulus and angle separately.

Depending on actual requirements and the conditions of the problem, diagrams constructed both in a complex plane and on a rectangular system of coordinates are used in practice.

Both types of diagram are given in the following section for comparison.

## 5. Locus Diagrams for Voltage Ratios in the Simplest Circuits

It was stated at the beginning of the chapter that the loci method is applicable both to impedances and admittances and to the ratios of analogous electrical quantities—voltages or currents.

The construction of locus diagrams for voltage sinor ratios for which frequency is the variable is illustrated below using an $rC$ circuit as an example (Fig. 16*a*). The diagrams are represented in a complex plane (Fig. 16*b*) and with a rectangular system of coordinates (Fig. 16*c* and *d*).

The construction of the diagrams is based on the following equations:

$$\frac{\dot{U}_r}{\dot{U}} = \frac{U_r}{U} e^{j\varphi_r} = \frac{r}{r + \dfrac{1}{j\omega C}} = \frac{j\omega}{\dfrac{1}{rC} + j\omega} \tag{5.18}$$

and

$$\frac{\dot{U}_C}{\dot{U}} = \frac{U_C}{U} e^{j\varphi_C} = \frac{\dfrac{1}{j\omega C}}{r + \dfrac{1}{j\omega C}} = \frac{1}{1 + j\omega rC}. \tag{5.19}$$

Here $\dot{U}_r$, $\dot{U}_C$ and $\dot{U}$ are the voltages across resistance $r$, capacitance $C$ and on the whole $rC$ circuit;

$\varphi_r$, $\varphi_C$ are the phase shifts of $\dot{U}_r$ and $\dot{U}_C$ with respect to $\dot{U}$.

As the frequency approaches zero, ratio (18) tends to zero and ratio (19) to unity. Correspondingly, angles $\varphi_r$ and $\varphi_C$ tend to 90° and 0°.

At a frequency tending to infinity, ratio (18) tends to unity and ratio (19) to zero. Correspondingly angles $\varphi_r$ and $\varphi_C$ tend to zero and $-90°$.

For the value $\omega = 1/rC$ the moduli of both the voltage ratios are 0·707 and the angles $\varphi_r$ and $\varphi_C$ are equal, accordingly, to 45° and −45°.

Thus, as the frequency changes from 0 to ∞, the phasor $\dot{U}_r/\dot{U}$ describes a semicircle in the first quadrant and the phasor $\dot{U}_C/\dot{U}$ a semicircle in the fourth quadrant of the complex plane (Fig. 16*b*). The direction of increasing frequency is clockwise in both cases.

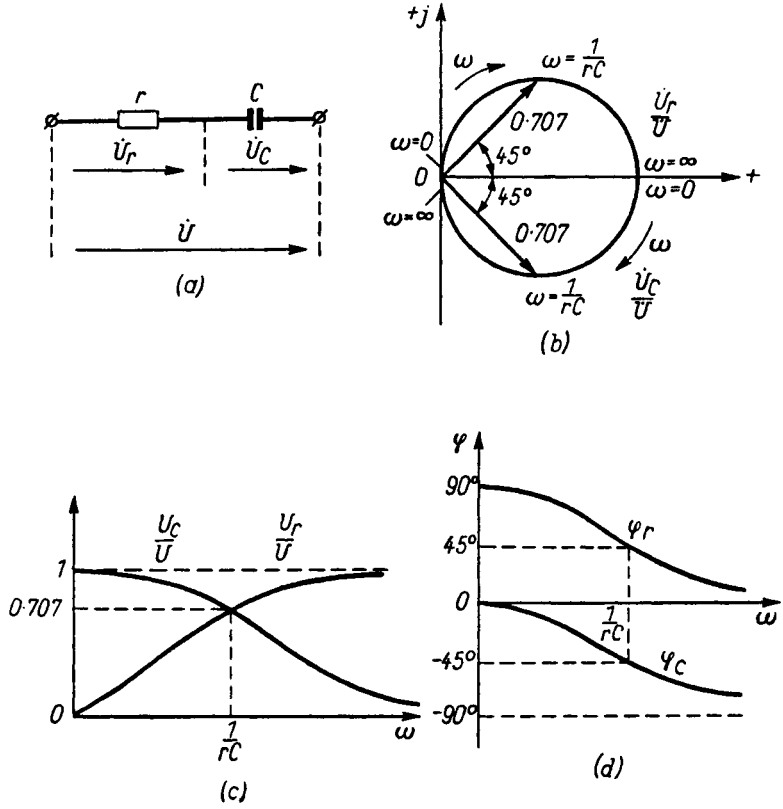
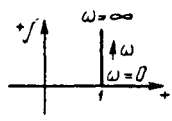
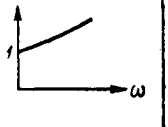
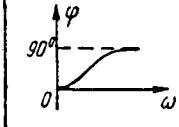
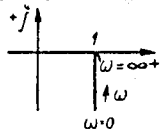
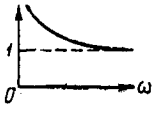
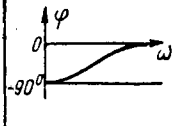
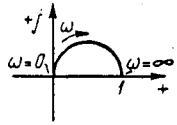
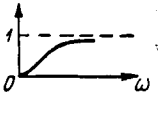
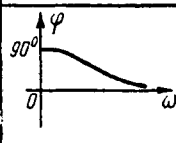
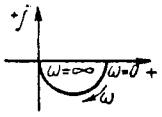
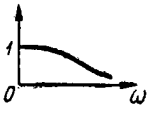
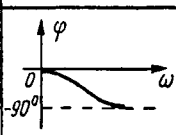
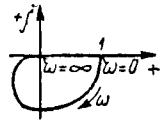
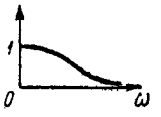
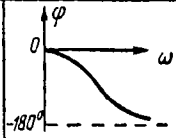
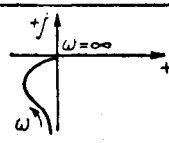
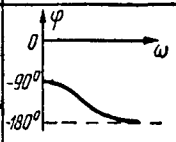
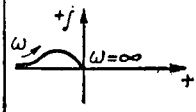
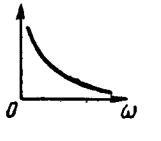
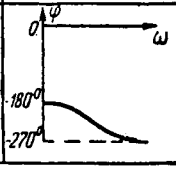



FIG. 16. Network characteristics: (*a*) *rC* circuits; (*b*) locus diagrams in a complex plane; (*c*) and (*d*) diagrams on a rectangular system of coordinates.

A locus diagram in a complex plane replaces the frequency relationships of the moduli of the voltage ratios and the frequency relationships of the angles constructed on a rectangular system of coordinates (Fig. 16*c* and *d*).

Table 1 gives various complex network functions and shows their corresponding loci plots in a complex plane (for variable frequency) and the frequency characteristics of the moduli and angles in a rectangular system of coordinates (Ref. 114).

TABLE 1. CHARACTERISTICS FOR NETWORKS

| Given function $K(j\omega)$ | Locus | Modulus $\lvert K(j\omega)\rvert$ | Angle |
|---|---|---|---|
| *a* | *b* | *c* | *d* |
| $1+j\omega a$ |  |  |  |
| $1-\frac{1}{j\omega a}$ |  |  |  |
| $\frac{j\omega a}{1+j\omega a}$ |  |  |  |
| $\frac{1}{1+j\omega a}$ |  |  |  |
| $\frac{1}{1+j\omega a+(j\omega)^2}$ |  |  |  |
| $\frac{1}{(1+j\omega a)j\omega}$ |  | |  |
| $\frac{1}{(1+j\omega a)(j\omega)^2}$ |  |  |  |

EXAMPLE 1. To determine the character of the change in voltage $\dot{U}_L$ across inductance $L$ connected in series with resistance $r$ (Fig. 17*a*) with the aid of a circle diagram, as the resistance is varied from zero to infinity. To find voltage $\dot{U}_L$ and the angle of the phase difference between voltage $\dot{U}$, applied to the circuit, and current $\dot{I}$. Given: $\dot{U} = U = 100$ V, $f = 400$ c/s, $L = 3{\cdot}18 \times 10^{-2}$ H, $r = 60\,\Omega$.

Voltage $\dot{U}_L$ is expressed by the circuit parameters thus:

$$\dot{U}_L = \frac{j\omega L}{r + j\omega L} U.$$

From a comparison of the given function with (9) it follows that in this case

$$\dot{A} = j\omega LU = U\omega L \angle 90° = 8000 \angle 90°;$$

$$\dot{B} = 0; \quad \dot{C} = j\omega L = \omega L \angle 90° = 80 \angle 90°; \quad \gamma = 90°;$$

$$\dot{D}k = r, \quad \delta = 0, \quad \delta - \gamma = -90°.$$

The circle diagram is constructed on Fig. 17*b* in accordance with Fig. 7, using the conventional notations adopted for the latter. In Fig. 17*b* the line of the variable parameter is

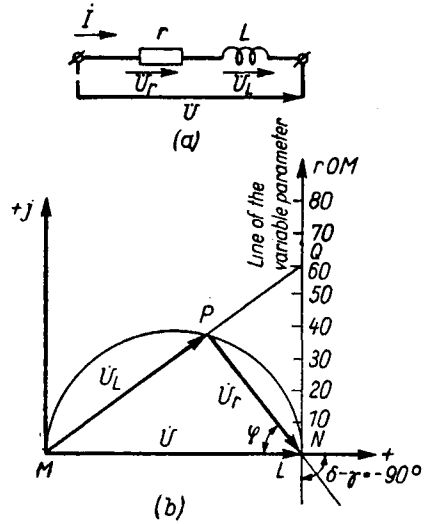



FIG. 17. Example 1.

produced through point $N$ which, in this case, coincides with point $L$. The section $ML$ corresponds to the inductive reactance $\omega L = 80\Omega$. The value of resistance $r$ is plotted on line $LQ$ on the same scale.

For the given value of $r = 60\Omega$, the modulus of the voltage across the inductance equals $U_L \approx 80$ V and the phase difference angle $\varphi \approx 53°$ (the current sinor $\dot{I}$ is in phase with resistive voltage drop $\dot{U}_r$).

CHAPTER VI

# TWO-TERMINAL NETWORKS

## 1. Definition and Classification of Two-terminal Networks

Any electrical circuit regarded with respect to any pair of its terminals may be referred to as a *two-terminal network.** The configuration of a two-terminal network can be as complex as is desired.

When the structure of a two-terminal network is unknown or it is not necessary to illustrate it, it is generally represented in the form of a rectangle with two terminals (Fig. 1).

These networks can be classified according to their various features. Depending on the linearity of the elements contained in the network we

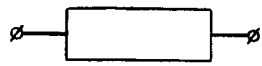


FIG. 1. Conventional symbol for a two-terminal network.

distinguish between *linear* and *non-linear* two-terminal networks. And according to the number of elements involved we distinguish single-element, twin-element and multi-element two-terminal networks.

According to the nature of the elements two-terminal networks are divided into *lossless or non-dissipative (dissipationless) networks*, i.e. consisting of pure inductances and capacitances, and *dissipative networks*, containing resistances. Dissipationless two-terminal networks are idealized electrical systems which approximate in their properties to physically practicable circuits with negligible losses.

If there is a voltage caused by the presence of independent energy sources across the open terminals with respect to which a given two-terminal network is regarded, then this network is called *active*. On the other hand when there is no voltage across the open terminals of the given linear network and there are no dependent sources, it is called *passive*.

It should be noted that not every linear two-terminal network comprising electrical energy sources is active.

If, for example, the presence of independent electrical energy sources does not manifest itself at the terminals of the network, i.e. despite the fact

* Alternative names 'one-port' and 'one-terminal pair' are also used.

that there are sources in the network the voltage across its open terminals is zero, then this type of linear two-terminal network belongs to the passive category. Its input impedance does not change if the sources mentioned above are replaced by passive elements—internal impedances of voltage sources or, correspondingly, internal admittances of current sources. An example of this type of passive two-terminal network containing compensated e.m.f. sources is given in Fig. 2.

Two two-terminal networks of different design are *equivalent* in the electrical sense if their impedances or, correspondingly, their admittances

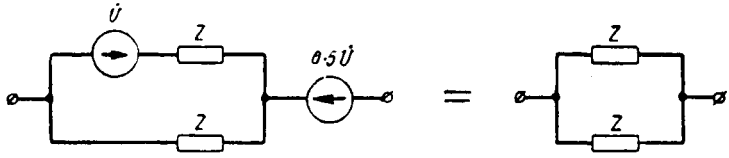



FIG. 2. Example of a passive two-terminal network with voltage sources.

are equal in the whole spectrum of frequencies. Replacing any such network by its equivalent in an electrical circuit has no effect on the currents or voltages in the remainder of the circuit. This is true in both steady-state and transient conditions.

The dependence of the values of the impedances or admittances of two-terminal networks on frequency is called the *frequency function or characteristic.*

The frequency functions of two-terminal networks forming an electrical circuit predetermine the frequency properties of the given circuit, i.e. the dependence of the amplitudes and phases of the currents or voltages on frequency.

The following sections are devoted to a study of the frequency functions of linear two-terminal networks.

## 2. Single-element Reactive Two-terminal Networks

An inductance and a capacitance are single-element reactive two-terminal networks (Fig. 3). The essential difference between these and more complex networks is that the character (sign) of the reactance and susceptance of these two networks is that the character (sign) of the reactance and susceptance of these two networks does not depend on frequency.

An inductive element has positive impedance and negative admittance over the whole scale of frequencies

$$\left.\begin{aligned} Z_L &= jx_L = j\omega L, \\ Y_L &= -jb_L = -j\frac{1}{\omega L}. \end{aligned}\right\} \tag{6.1}$$

A capacitive element has negative impedance and positive admittance in the whole spectrum of frequencies

$$Z_C = -jx_C = -j\frac{1}{\omega C}, \quad Y_C = jb_C = j\omega C. \tag{6.2}$$

The frequency characteristics $Z_L$ and $Y_C$ constructed with a rectangular system of coordinates are straight lines and the frequency characteristics $Z_C$

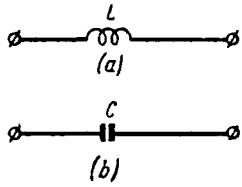



FIG. 3. Single-element reactive two-terminal networks.

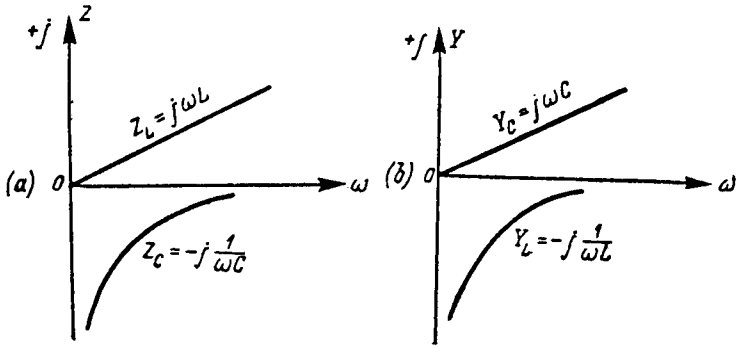



FIG. 4. Frequency functions of single-element two-terminal networks: (*a*) impedance; (*b*) admittance.

and $Y_L$ are hyperbolae (Fig. 4). Thus curves $Z_L$ and $Z_C$ are analogous to curves $Y_C$ and $Y_L$ respectively.

It should be noted that both the impedance and the admittance curves of the single-element reactive two-terminal networks under examination plotted against frequency, have positive slope, i.e.

$$\frac{dZ}{j\,d\omega} > 0, \quad \frac{dY}{jd\omega} > 0.$$

As will be shown below this is the common property of not only single-element but all reactive two-terminal networks.

A two-terminal network consisting of one type of element connected in series or in parallel (inductances or capacitances) belongs to the single-element category of network, since similar elements connected in series or

in parallel can be replaced by one equivalent reactive element of the same character.

As a result of reducing the number of elements in the network to a minimum by substituting their equivalents, a so-called fundamental network representation is obtained.

### 3. Twin-element Reactive Two-terminal Networks

Twin-element reactive two-terminal networks are obtained by connecting an inductance and a capacitance in series or in parallel.

When an inductance and a capacitance are connected in series their reactances are added algebraically. The thick line in Fig. 5*b* shows the frequency characteristic of a two-terminal network obtained as a result of adding curves $Z_L$ and $Z_C$ graphically.

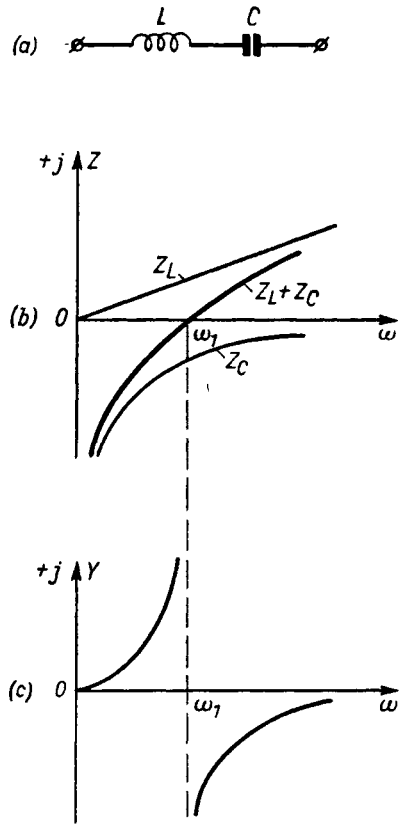



FIG. 5. Frequency functions of a twin-element *LC* reactive two-terminal network: (*a*) circuit diagram; (*b*) impedance; (*c*) admittance.

At a certain frequency $\omega_1$ this characteristic intersects the abscissa axis, i.e. the phenomenon of *series resonance* occurs: the impedance of the network equals zero. Frequency $\omega_1$, called the *resonance frequency* (Chapter V, Section 4), is equal to

$$\omega_1 = \frac{1}{\sqrt{(LC)}}. \tag{6.3}$$

Here and below the frequencies of series resonance are denoted by an odd subscript.

The frequency characteristic of the admittance of a twin-element reactive two-terminal network is obtained by inverting the characteristic $Z = Z_L + Z_C$ since $Y = 1/Z$.

Curve $Y$ is shown in Fig. 5*c*.

The frequency characteristics $Z$ and $Y$ constructed in Fig. 5 correspond to the equations

$$Z = jx = j(x_L - x_C) = j\left(\omega L - \frac{1}{\omega C}\right) = jL\left(\omega - \frac{1}{\omega LC}\right)$$

and

$$Y = -jb = -j\frac{1}{L\left(\omega - \frac{1}{\omega LC}\right)}$$

or, taking (3) into account,

$$\left.\begin{aligned} x &= L\frac{\omega^2 - \omega_1^2}{\omega}, \\ b &= \frac{\omega}{L(\omega^2 - \omega_1^2)}. \end{aligned}\right\} \tag{6.4}$$

It follows from Fig. 5 and equations (4) that in the range of frequencies below the resonance frequency ($\omega < \omega_1$), the impedance of the inductive elements is over-compensated by the impedance of the capacitive element (the impedance of the capacitive element exceeds the impedance of the inductive element in absolute value), the impedance of the resulting network having a capacitive character. In the range of frequencies above the resonance frequency ($\omega > \omega_1$) the impedance of the capacitive element is over-compensated by the impedance of the inductive element (the absolute value of the capacitive reactance is lower than the value of the inductive reactance), the resistance of the resulting network having an inductive character.

When an inductance and a capacitance are connected in parallel their susceptances are added algebraically. The thick line in Fig. 6*b* shows the frequency characteristic of a two-terminal network obtained by adding $Y_L$ and $Y_C$ graphically.

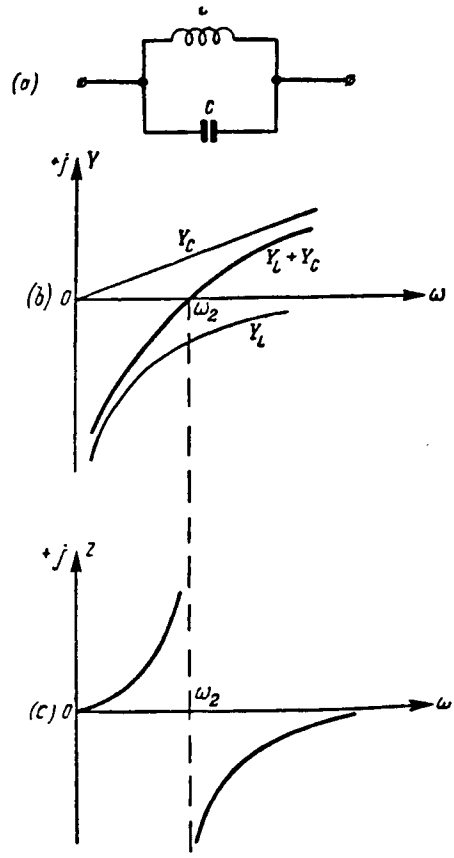



FIG. 6. Frequency functions of a twin-element $LC$ reactive two-terminal network: (*a*) circuit diagram; (*b*) admittance; (*c*) impedance.

At a certain frequency $\omega_2$ this curve intersects the abscissa axis, i.e. the phenomenon of *parallel resonance* occurs: the admittance of the two-terminal network is zero.

The frequency of parallel resonance is determined from the formula

$$\omega_2 = \frac{1}{\sqrt{(LC)}}. \tag{6.5}$$

Here and below the frequencies of parallel resonance are denoted by an even subscript.

The frequency characteristic of the impedance of the same network is obtained by inverting the curve $Y = Y_L + Y_C$ since $Z = 1/Y$.

Curve $Z$ is shown in Fig. 6*c*.

The frequency characteristics $Z$ and $Y$ constructed in Fig. 6 correspond to the equations

$$Y = -jb = -j(b_L - b_C) = -j\left(\frac{1}{\omega L} - \omega C\right) = -jC\left(\frac{1}{\omega LC} - \omega\right)$$

and

$$Z = jx = j\frac{1}{C\left(\frac{1}{\omega LC} - \omega\right)}$$

or, taking (5) into account,

$$\left.\begin{aligned} b &= C\frac{\omega_2^2 - \omega^2}{\omega}, \\ x &= \frac{\omega}{C(\omega_2^2 - \omega^2)}. \end{aligned}\right\} \qquad (6.6)$$

It follows from Fig. 6 and (6) that in the range of frequencies below the resonance frequency the admittance of the capacitive element is over-compensated by the admittance of the inductive element and the impedance of the resulting network has an inductive character. In the range of frequencies above the resonance frequency, the reverse phenomenon is observed: the impedance of the network has a capacitive character.

Thus, depending on frequency, a twin-element reactive two-terminal network may have either an inductive or a capacitive reactance.

Then, as in the case of the single-element reactive two-terminal network, the slope of curves $Z$ and $Y$, plotted against frequency, is everywhere positive, i.e. the derivatives of $Z/j$ and $Y/j$ are positive with respect to $\omega$ at all frequencies.

As distinct from the impedances of single-element two-terminal networks which are expressed by variable frequency only, the impedances of twin-element reactive two-terminal networks contain the difference of the squares of the resonance and variable frequencies in the numerator of formula (4) or in the denominator of formula (6).

## 4. Multi-element Reactive Two-terminal Networks

A *three-element* reactive two-terminal network can be composed of two inductances and one capacitance (Figs. 7*a* and 9*a*) or of two capacitances and one inductance (Figs. 8*a* and 10*a*).

The over-all number of possible circuits for three-element reactive two-terminal networks is four.

By using the frequency characteristics of the single-element and twin-element reactive networks given above, it is easily possible to construct frequency characteristics for three-element two-terminal networks.

When a single-element and a two-element reactive two-terminal network, together forming a three-element network, are connected in series (Figs. 7*a* and 8*a*), the impedances of the component two-terminal networks are added (Figs. 7*b* and 8*b*). The admittances of three-element networks are obtained by inverting the corresponding impedance curves (Figs. 7*c* and 8*c*).

When a single-element and a twin-element reactive two-terminal network together forming a three-element network are connected in parallel (Figs. 9*a* and 10*a*) the admittances of the component networks are added (Figs. 9*b* and 10*b*). The impedances of three-element two-terminal networks

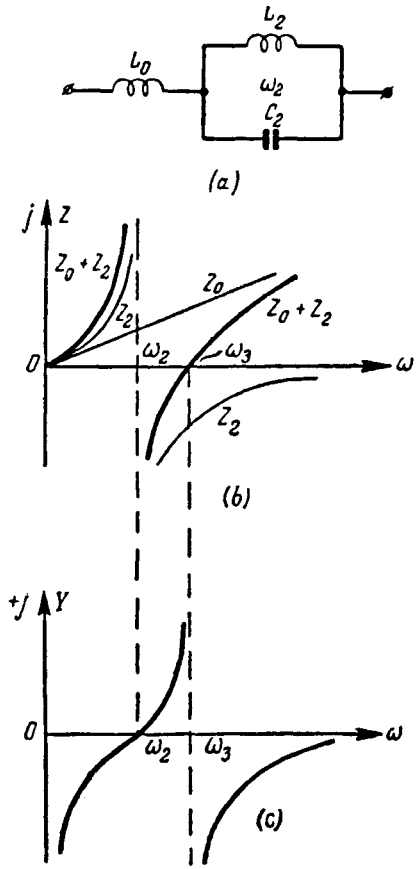



FIG. 7. Frequency functions of a three-element reactive two-terminal network: (*a*) circuit diagram; (*b*) impedance; (*c*) admittance.

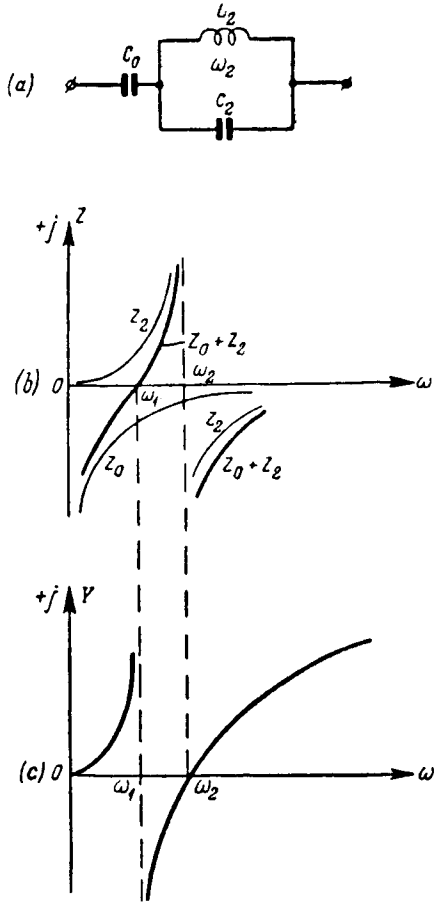



FIG. 8. Frequency functions of a three-element reactive two-terminal network: (*a*) circuit diagram; (*b*) impedance; (*c*) admittance.

are obtained by inverting the corresponding admittance curves (Figs. 9*c* and 10*c*).

It follows from a comparison of the frequency characteristics of three-element two-terminal networks that networks comprising two inductances and one capacitance (Figs. 7*a* and 9*a*) have frequency characteristics of the same kind. The parameters of these two-terminal networks can be selected so that the impedance curves and, correspondingly, the admittance curves of both networks coincide, i. e. the networks will be equivalent in the electrical sense.

The same should be said of three-element networks comprising two capacitances and one inductance (Figs. 8*a* and 10*a*). When the appropriate parameters are selected they may be equivalent.

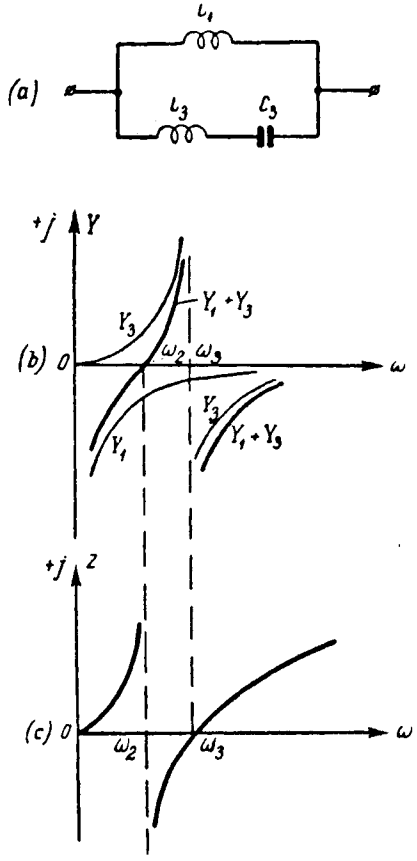



FIG. 9. Frequency functions of a three-element reactive two-terminal network: (*a*) circuit diagram; (*b*) admittance; (*c*) impedance.

The frequency function $Z$ constructed on Fig. 7*b* corresponds to the expression

$$Z = j\omega L_0 + j\frac{\omega}{C_2(\omega_2^2 - \omega^2)}, \tag{6.7}$$

where

$$\omega_2 = \frac{1}{\sqrt{(L_2C_2)}} \tag{6.8}$$

is the resonance frequency of the parallel $L_2C_2$ circuit.

At frequencies $\omega > \omega_2$, the reactance of the parallel $L_2C_2$ circuit has a capacitive character, and at a certain frequency $\omega_3 > \omega_2$ series resonance occurs.

The frequency of the series resonance is determined from the equation

$$\omega_3 L_0 + \frac{\omega_3}{C_2(\omega_2^2 - \omega_3^2)} = 0.$$

Taking (8) into account, we obtain

$$\omega_3 = \sqrt{\left(\frac{L_0 + L_2}{L_0 L_2 C_2}\right)}. \tag{6.9}$$

With the aid of (9), expression (7) can be converted into the following:

$$Z = j\omega L_0 \frac{\omega_3^2 - \omega^2}{\omega_2^2 - \omega^2}. \tag{6.10}$$

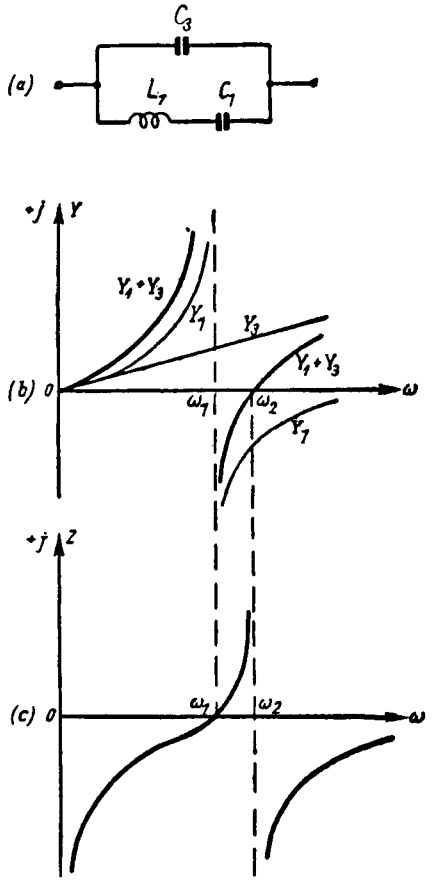



FIG. 10. Frequency functions of a three-element reactive two-terminal network: (*a*) circuit diagram; (*b*) admittance; (*c*) impedance.

Expressions for the impedances and resonance frequencies of other types of three-element network can be obtained analogously.

According to the Fig. 8*c* circuit

$$\left.\begin{aligned} Z &= \frac{C_0 + C_2}{j\omega C_0 C_2} \cdot \frac{\omega_1^2 - \omega^2}{\omega_2^2 - \omega^2}, \\ \omega_1 &= \frac{1}{\sqrt{[L_2(C_0 + C_2)]}}, \quad \omega_2 = \frac{1}{\sqrt{(L_2 C_2)}}. \end{aligned}\right\} \tag{6.11}$$

According to the Fig. 9*a* circuit

$$\left.\begin{aligned} Z &= j\omega \frac{L_1 L_3}{L_1 + L_3} \cdot \frac{\omega_3^2 - \omega^2}{\omega_2^2 - \omega^2}, \\ \omega_2 &= \frac{1}{\sqrt{[C_3(L_1 + L_3)]}}, \quad \omega_3 = \frac{1}{\sqrt{(L_3 C_3)}}. \end{aligned}\right\} \tag{6.12}$$

According to the Fig. 10*a* circuit

$$\left.\begin{aligned} Z &= \frac{1}{j\omega C_3} \frac{\omega_1^2 - \omega^2}{\omega_2^2 - \omega^2}, \\ \omega_2 &= \sqrt{\left(\frac{C_1 + C_3}{L_1 C_1 C_3}\right)}, \quad \omega_1 = \frac{1}{\sqrt{(L_1 C_1)}}. \end{aligned}\right\} \tag{6.13}$$

Thus the impedances of three-element reactive two-terminal networks contain the differences between the squares of the resonance frequency and the present frequency in the numerator and denominator, the series-resonance frequency being in the numerator and parallel-resonance frequency in the denominator of expression for $Z$.

The conditions for the equivalence of three-element two-terminal networks in accordance with Figs. 7*a* and 9*a* and, correspondingly, Figs. 8*a* and 10*a*, are obtained by equating the expressions for the impedances and, correspondingly, for the resonance frequencies.

On the basis of (8)–(10) and (12) the networks shown in Figs. 7*a* and 9*a* are equivalent if

$$L_0 = \frac{L_1 L_3}{L_1 + L_3}, \quad L_2 C_2 = (L_1 + L_3) C_3, \quad \frac{L_0 + L_2}{L_0 L_2 C_2} = \frac{1}{L_3 C_3},$$

from which

$$L_0 = \frac{L_1 L_3}{L_1 + L_3}, \quad L_2 = \frac{L_1^2}{L_1 + L_3}, \quad C_2 = C_3 \frac{(L_1 + L_3)^2}{L_1^2}; \tag{6.14}$$

$$L_1 = L_0 + L_2, \quad L_3 = \frac{L_0(L_0 + L_2)}{L_2}, \quad C_3 = C_2 \frac{L_2^2}{(L_0 + L_2)^2}. \tag{6.15}$$

In turn, in accordance with (11) and (13) the networks shown in Figs. 8*a* and 10*a* are equivalent if

$$\frac{C_0 + C_2}{C_0 C_2} = \frac{1}{C_3}, \quad L_2(C_0 + C_2) = L_1 C_1, \quad L_2 C_2 = \frac{L_1 C_1 C_3}{C_1 + C_3},$$

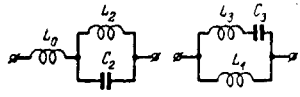
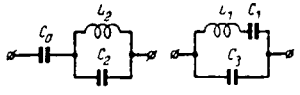
from which

$$C_0 = C_1 + C_3, \quad C_2 = \frac{C_3(C_1 + C_3)}{C_1}, \quad L_2 = L_1 \frac{C_1^2}{(C_1 + C_3)^2}, \tag{6.16}$$

$$C_3 = \frac{C_0 C_2}{C_0 + C_2}, \quad C_1 = \frac{C_0^2}{C_0 + C_2}, \quad L_1 = L_2 \frac{(C_0 + C_1)^2}{C_0^2}. \tag{6.17}$$

Formulae (14)–(17) are given in Table 1 (Refs. 1 and 9).

TABLE 1. EQUIVALENT THREE-ELEMENT NON-DISSIPATIVE TWO-TERMINAL NETWORKS

| Circuit diagrams | Equivalence conditions | |
|---|---|---|
|  | $L_0 = \frac{L_1 L_3}{L_1 + L_3}$ | $L_1 = L_0 + L_2$ |
| | $L_2 = \frac{L_1^2}{L_1 + L_3}$ | $L_3 = \frac{L_0(L_0 + L_2)}{L_2}$ |
| | $C_2 = C_3\left(\frac{L_1 + L_3}{L_1}\right)^2$ | $C_3 = C_2\left(\frac{L_2}{L_0 + L_2}\right)^2$ |
|  | $C_0 = C_1 + C_3$ | $C_3 = \frac{C_0 C_2}{C_0 + C_2}$ |
| | $C_2 = \frac{C_3}{C_1}(C_1 + C_3)$ | $C_1 = \frac{C_0^2}{C_0 + C_2}$ |
| | $L_2 = L_1\left(\frac{C_1}{C_1 + C_3}\right)^2$ | $L_1 = L_2\left(\frac{C_0 + C_2}{C_0}\right)^2$ |

The above formulae are for converting one three-element reactive two-terminal network into another network, equivalent to it.

As distinct from twin-element reactive two-terminal networks with one resonance frequency, three-element reactive two-terminal networks are characterized by having two resonance frequencies, either series resonance or parallel resonance occurring first, depending on the circuit. If the series resonance is first the factor $j\omega$ is contained in the denominator of the

expression for the impedance of the network; and if the parallel resonance is first, the factor $j\omega$ is contained in the numerator.

The number of resonances is generally one less than the number of elements in the given reactive network, series resonance and parallel resonance alternating.

If, when $\omega = 0$, the impedance of the network is zero, i.e. direct current can pass, then parallel resonance occurs first, followed by series resonance.

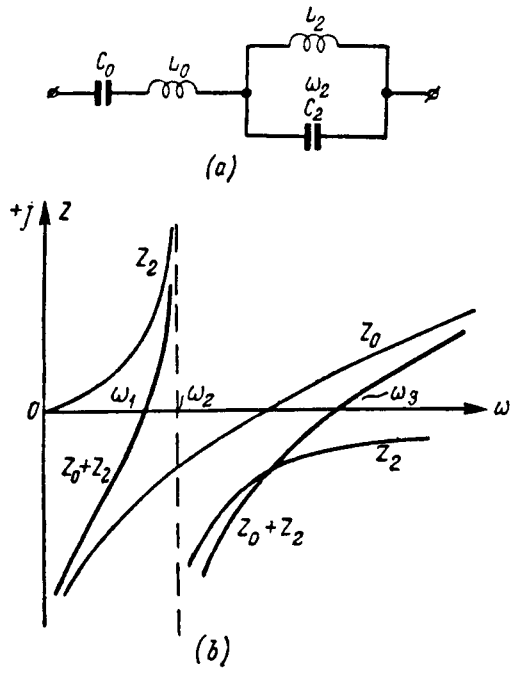



FIG. 11. (*a*) Circuit diagram; (*b*) impedance function of a four-element reactive two-terminal network.

If the opposite is the case, the order of the resonances is reversed: series resonance is first, followed by parallel resonance.

*Four-element* reactive two-terminal networks comprise two inductances and two capacitances. The over-all number of possible circuits for these four-element reactive networks is eight.

The frequency functions of the impedances of all four-element reactive two-terminal networks can be ultimately reduced to the two standard characteristics shown in Figs. 11*b* and 12*b*. These are obtained by adding the impedances of the component networks which, in the circuits shown in Figs. 11*a* and 12*a*, are connected in series.

It is possible to show that the impedances of four-element two-terminal networks are expressed by the following formulae:

for the Fig. 11*a* circuit

$$Z = \frac{L_0}{j\omega} \cdot \frac{(\omega_1^2 - \omega^2)(\omega_3^2 - \omega^2)}{\omega_2^2 - \omega^2}; \tag{6.18}$$

for the Fig. 12*a* circuit

$$Z = j \frac{\omega}{\dfrac{C_2 C_4}{C_2 + C_4}} \frac{\omega_3^2 - \omega^2}{(\omega_2^2 - \omega^2)(\omega_4^2 - \omega^2)}. \tag{6.19}$$

Thus the impedances of four-element networks contain two differences between the squares of the resonance frequency and variable frequency in the numerator and one difference between the squares of the resonance frequency and the variable frequency in the denominator, or vice versa.

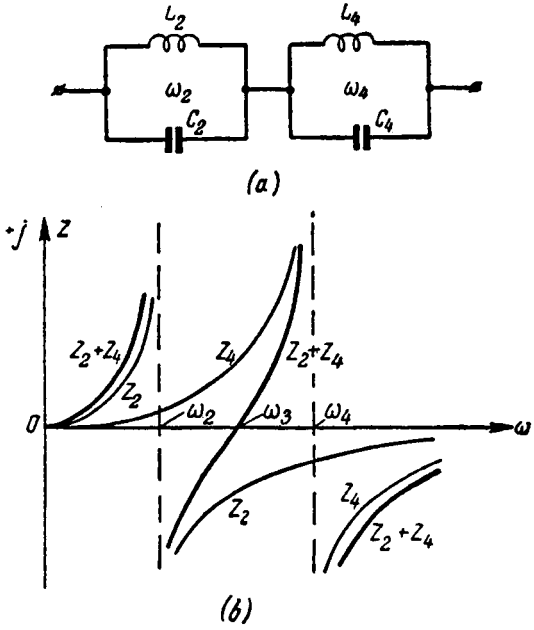



FIG. 12. (*a*) Circuit diagram; (*b*) impedance function if a four-element network.

The resonance frequencies contained in the numerator of expressions (18) and (19) are frequencies of series resonance, and those in the denominator are frequencies of parallel resonance.

If series resonance occurs first, the factor $j\omega$ is contained in the denominator of the expression for impedance; and if parallel resonance occurs first, the factor $j\omega$ is contained in the numerator.

## 5. General Expression for the Impedance of a Passive Dissipationless Multi-element Two-terminal Network

Let us suppose that a zero-impedance voltage source is applied to the input terminals of a passive two-terminal network representing an electrical circuit of any complexity (Fig. 13). This voltage can be regarded as a driving e.m.f. in the input loop of the network. Correspondingly, the current at the input of the network can also be regarded as a loop current.

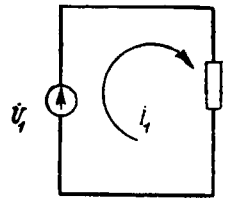


FIG. 13. Passive multi-element two-terminal network excited by a voltage source.

Denoting the input loop of the network by the digit $l$ we can find the input impedance $Z$ of the network as the ratio of the e.m.f. $\dot{U}_1$ to the loop current $\dot{I}_1$

$$Z = \frac{\dot{U}_1}{\dot{I}_1} = \frac{\Delta}{\Delta_{11}}, \tag{6.20}$$

where $\Delta$ is the system determinant, whose elements are the self-impedances $Z_{ii}$ and the mutual impedances $Z_{ik}$ of the loops; and $\Delta_{11}$ is the cofactor (in this case equal to the minor) of element $Z_{11}$ of the system determinant (Chapter IV, equation (2)).

The number of the rows and also of the columns in the determinant of the system is equal to the number of independent loops ($n$) in the network.

We shall multiply the numerator and denominator of the right-hand side of (20) by $(j\omega)^n$

$$Z = \frac{(j\omega)^n \Delta}{(j\omega)^n \Delta_{11}}.$$

Multiplying determinant $\Delta$ by $j\omega$ is equivalent to multiplying all the elements of one line (row or column) by $j\omega$. Correspondingly, multiplying $\Delta_{11}$ by $(j\omega)^{n-1}$ is equivalent to multiplying all the elements of the determinant $\Delta_{11}$ by $j\omega$.

Let us suppose that the network consists of reactive elements, the self-impedance of loop $i$ being

$$Z_{ii} = j\omega L_{ii} + \frac{1}{j\omega C_{ii}},$$

and the mutual impedance of loops $i$ and $k$

$$Z_{ik} = -\left(j\omega L_{ik} + \frac{1}{j\omega C_{ik}}\right).$$

From this

$$j\omega Z_{ii} = (j\omega)^2 L_{ii} + \frac{1}{C_{ii}} \quad \text{and} \quad j\omega Z_{ik} = -\left[(j\omega)^2 L_{ik} + \frac{1}{C_{ik}}\right].$$

Introducing the notation

$$p = j\omega, \tag{6.21}$$

we get

$$\left.\begin{aligned} pZ_{ii} &= p^2 L_{ii} + \frac{1}{C_{ii}}, \\ pZ_{ik} &= -\left[p^2 L_{ik} + \frac{1}{C_{ik}}\right]. \end{aligned}\right\} \tag{6.22}$$

The determinant $p^n\Delta$ of the order $n$ and consisting of elements of the form of (22) is a polynomial of the $2n$-th degree with respect to $p$ or, the same thing, is a polynomial of the $n$-th degree with respect to $p^2$. Correspondingly, the determinant $p^{n-1}\Delta_{11}$ of the order $n-1$ and consisting of analogous elements, is a polynomial in $p$ of degree $2n-2$ or, the same thing, a polynomial in $p^2$ of degree $n-1$.

Thus

$$Z = \frac{a_{2n}p^{2n} + a_{2n-2}p^{2n-2} + \cdots + a_2 p^2 + a_0}{p(b_{2n-1}p^{2n-2} + b_{2n-3}p^{2n-4} + \cdots + b_3 p^2 + b_1)}. \tag{6.23}$$

The polynomial in $p^2$ of degree $n$ in the numerator of (23) has $n$ roots

$$p^2 = p_{2k-1}^2 = -\omega_{2k-1}^2, \tag{6.24}$$

turning it to zero (the number $k$ assumes values from 1 to $n$). As expression (24) shows, these roots have real negative values whereas

$$p_{2k-1} = \pm j\omega_{2k-1}$$

are imaginary numbers.

The polynomial in $p^2$ of degree $n-1$ in the denominator of (23) has $n-1$ roots

$$p^2 = p_{2k}^2 = -\omega_{2k}^2, \tag{6.25}$$

where the index $k$ assumes values from 1 to $n-1$.

Factorizing these polynomials, we can write (23) thus

$$Z = \frac{a_{2n}(p^2 - p_1^2)(p^2 - p_3^2)\ldots(p^2 - p_{2n-1}^2)}{pb_{2n-1}(p^2 - p_2^2)(p^2 - p_4^2)\ldots(p^2 - p_{2n-2}^2)}. \tag{6.26}$$

Denoting

$$H = \frac{a_{2n}}{b_{2n-1}} \tag{6.27}$$

and multiplying the numerator and denominator of (26) by $p$, on the basis of (21) and (24)–(26) we get

$$Z = \frac{j\omega H(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}{(\omega^2 - \omega_0^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)\dots(\omega^2 - \omega_{2n-2}^2)}, \tag{6.28}$$

where $\omega_0 = 0$.

The frequencies with odd indices turning the numerator into zero are called the *zeros of function Z* and the frequencies with even indices turning the denominator into zero are the *poles of function Z*.

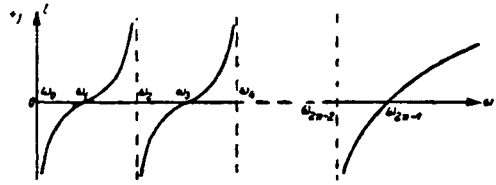


FIG. 14. Impedance function of a reactive two-terminal network not passing direct current.

Formula (28) is known in mathematical literature as Foster's theorem.

In the previous discussion it was assumed that the roots of the equations, i.e. the zeros and poles of function $Z$, were simple (of the first order).

The degree of the numerator with respect to $\omega$ is higher than the degree of the denominator by unity. Function $Z$ has two external poles: at infinity ($\omega_{2n} = \infty$), and at the origin ($\omega_0 = 0$).

This is explained by the fact that both the self-impedance of any loop and the mutual impedance between loops are of the inductive-capacitive type ($L$ and $C$ connected in series). A two-terminal network composed of such circuits will not pass direct current or current of an infinitely high frequency.

On the basis of formula (28) the impedance function of a network is completely specified by its poles, zeros and the scale factor $H$. the latter is easily determined if, in addition to information about the arrangement of the zeros and poles, we know the input impedance of the network for any intermediate frequency (not coinciding with a zero or pole).

The impedance of a reactive two-terminal network, as a function of frequency in accordance with formula (28), is given in Fig. 14.

The zeros and poles of the function alternate (Chapter VI, Section 17)

and, correspondingly, the derivative of $Z/j$, with respect to frequency is positive:

$$0 = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_{2n-2} < \omega_{2n-1} < \infty$$

and

$$\frac{dZ}{j\,d\omega} > 0 \quad \text{when} \quad 0 \leqq \omega < \infty.$$

At each transition through a zero or a pole, the sign of impedance $Z$ changes.

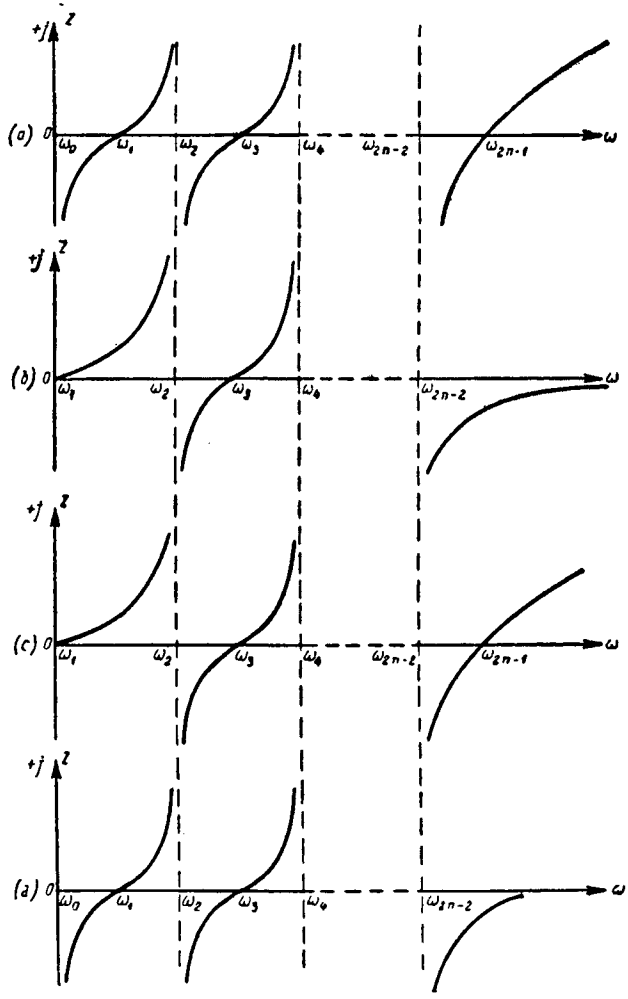



FIG. 15. Four types of impedance function of a reactive two-terminal network: (*a*) external poles; (*b*) external zeros; (*c*) internal zero and pole; (*d*) external pole and zero.

At a certain frequency $\omega$ situated sufficiently near a pole and to the left of it, impedance $Z$ is positive. And if frequency $\omega$ is situated to the right of the pole, impedance $Z$ is negative. In accordance with (28) this is only possible for positive values of $H$. Thus the scale factor $H$ contained in (28) is positive.

Depending on the locations of the zeros and poles, four types of frequency function are possible for a multi-element reactive two-terminal network.

*A frequency function with two external poles* (Fig. 15*a*): when $\omega_0 = 0$ and $\omega_{2n} = \infty$, the impedance of the network equals infinity, i.e. direct current and current of infinitely high frequency cannot pass.

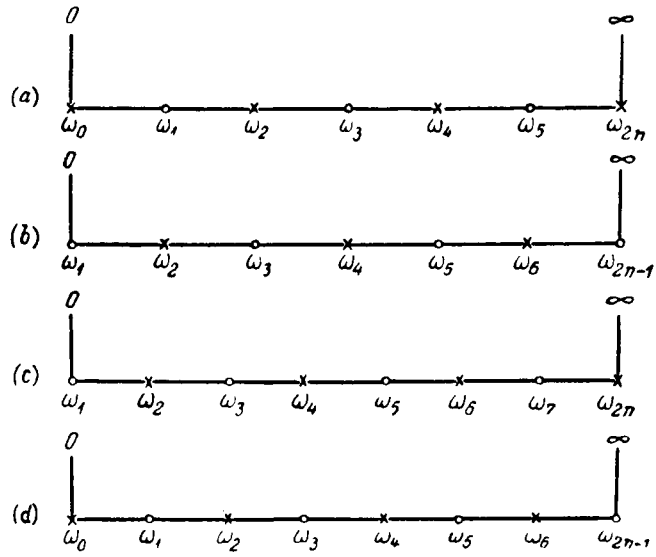



FIG. 16. Frequency pattern: (*a*) external poles; (*b*) external zeros; (*c*) external zero and pole; (*d*) external pole and zero.

A network with this type of characteristic has been examined above (Fig. 14).

*A frequency function with two external zeros* (Fig. 15*b*): when $\omega_1 = 0$ and $\omega_{2n+1} = \infty$, the impedance of the network equals zero, i.e. direct current and current of infinitely high frequencies can pass.

*A frequency function with an external zero,* $\omega_1 = 0$, *and an external pole,* $\omega_{2n} = \infty$ (Fig. 15*c*): direct current can pass but current of infinitely high frequency cannot.

*A frequency function with an external pole,* $\omega_0 = 0$, *and an external zero,* $\omega_{2n-1} = 0$ (Fig. 15*d*): direct current cannot pass but current of infinitely high frequency can.

Here, as in the previous section, the frequencies corresponding to the zeros of function $Z$ are denoted by odd subscripts and those corresponding to the poles of function $Z$ by even subscripts.

If we denote the zeros of the function by circles and the poles by crosses, then we can represent the sequence of the alternation of zeros and poles mentioned above with the aid of Fig. 16. This sketch, called a frequency pattern, contains just as much information as does Fig. 15; only the scale factor $H$ is missing.

## 6. Canonic Forms of Dissipationless Two-terminal Networks

The rational function (28) can be expanded thus:

$$Z = j\omega H\left(1 + \sum_{k=1}^{n} \frac{\alpha_{2k-2}}{\omega^2 - \omega_{2k-2}^2}\right). \tag{6.29}$$

Unity is obtained in the expansion by separating the integer part of rational function (28).

In order to determine the factor $\alpha_{2k-2}$ we multiply both sides of equalities (28) and (29) by $\omega^2 - \omega_{2k-2}^2$ and proceed to the limit $\omega = \omega_{2k-2}$. Then

$$\alpha_{2k-2} = \left[\frac{(\omega^2 - \omega_{2k-2}^2)Z}{j\omega H}\right]_{\omega=\omega_{2k-2}}. \tag{6.30}$$

The sum of the first two terms in the right-hand side of (29) expresses the impedances of a twin-element reactive two-terminal network

$$j\left[\omega H - \frac{1}{\omega\left(-\frac{1}{\alpha_0 H}\right)}\right] = j\left(\omega L_{2n} - \frac{1}{\omega C_0}\right),$$

consisting of an inductance $L_{2n} = H > 0$ connected in series with a capacitance $C_0 = -1/\alpha_0 H$ (Fig. 17*a*).

In accordance with (30) $\alpha_0 < 0$ and, consequently, $C_0 > 0$.

The other terms in the right-hand part of (29) have the form

$$Z_{2k-2} = j\omega \frac{H\alpha_{2k-2}}{\omega^2 - \omega_{2k-2}^2}. \tag{6.31}$$

The given expression corresponds to the impedance of the twin-element reactive two-terminal network shown in Fig. 17*b*:

$$Z_{2k-2} = \frac{j\omega L_{2k-2}}{1 - \omega^2 L_{2k-2} C_{2k-2}} = \frac{j\omega\left(-\frac{1}{C_{2k-2}}\right)}{\omega^2 - \omega_{2k-2}^2}, \tag{6.32}$$

where $\omega_{2k-2} = 1/\sqrt{(L_{2k-2}C_{2k-2})}$ is the resonance frequency of the parallel $L_{2k-2}C_{2k-2}$ circuit.

Equating (31) with (32) we find

$$C_{2k-2} = -\frac{1}{H\alpha_{2k-2}} > 0. \tag{6.33}$$

Substituting (30) in (33) yields

$$C_{2k-2} = -\left[\frac{j\omega}{Z(\omega^2 - \omega_{2k-2}^2)}\right]_{\omega=\omega_{2k-2}} \tag{6.34}$$

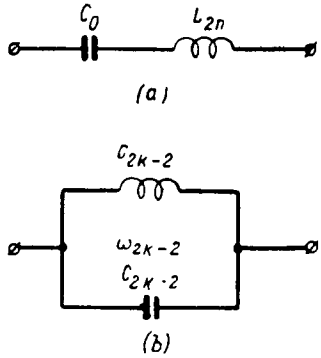



FIG. 17. Twin-element reactive two-terminal network.

For the given resonance frequency $\omega_{k2-2}$ the inductance $L_{2k-2}$ is found from the formula

$$L_{2k-2} = \frac{1}{\omega_{2k-2}^2 C_{2k-2}}. \tag{6.35}$$

The circuit of a generalized reactive two-terminal network, the impedance of which is expressed by formula (28), is given in Fig. 18.

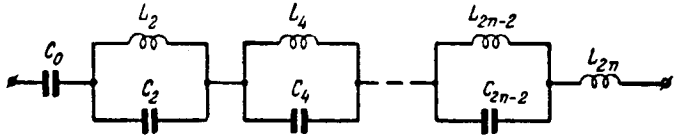



FIG. 18. Generalized reactive two-terminal network consisting of twin-element two-terminal networks connected in series.

This type of circuit is called a *canonic (fundamental) circuit.* The frequency function shown in Fig. 14 corresponds to it.

It should be noted that the Fig. 18 circuit is not the only form of canonic circuit for a two-terminal network. Instead of twin-element reactive two-terminal networks connected in series the circuit of a generalized

two-terminal network may contain twin-element networks connected in parallel (Fig. 19).

In fact, on the basis of (28), the admittance of the network is

$$Y = \frac{\omega^2(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)\dots(\omega^2 - \omega_{2n-2}^2)}{j\omega H(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}. \tag{6.36}$$

The given rational–fractional function may be represented in the form of the sum of the fractions

$$Y = -j\omega H^{-1} \sum_{k=1}^{n} \frac{\beta_{2k-1}}{\omega^2 - \omega_{2k-1}^2}. \tag{6.37}$$

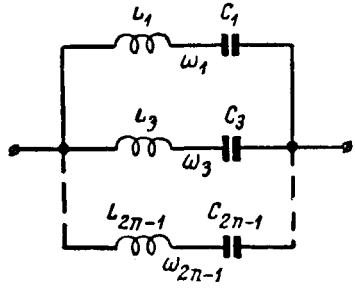



FIG. 19. Generalized reactive two-terminal network consisting of twin-element two-terminal networks connected in parallel.

Multiplying both sides of equalities (36) and (37) by $\omega^2 - \omega_{2k-1}$ and turning $\omega$ to $\omega_{2k-1}$ we find

$$\beta_{2k-1} = \left[\frac{(\omega^2 - \omega_{2k-1}^2)Y}{-j\omega H^{-1}}\right]_{\omega=\omega_{2k-1}}. \tag{6.38}$$

Every term in the right-hand part of (37) expresses the admittance of a twin-element reactive two-terminal network

$$\frac{-j\omega H^{-1}\beta_{2k-1}}{\omega^2 - \omega_{2k-1}^2} = \frac{-j\omega \dfrac{1}{L_{2k-1}}}{\omega^2 - \omega_{2k-1}^2},$$

consisting of an inductance $L_{2k-1} = H/\beta_{2k-1}$ and a capacitance $C_{2k-1} = 1/\omega_{2k-1}L_{2k-1}$ connected in series.

Taking (38) into account

$$L_{2k-1} = -\left[\frac{j\omega Z}{\omega^2 - \omega_{2k-1}^2}\right]_{\omega=\omega_{2k-1}}. \tag{6.39}$$

The admittance of the reactive two-terminal network shown in Fig. 19 is expressed by formula (36) and its impedance, accordingly, by formula (28). Consequently, the networks given in Figs. 18 and 19 are equivalent.

We have been examining two circuits for a generalized reactive two-terminal network with an impedance function of the kind shown in Fig. 15*a*.

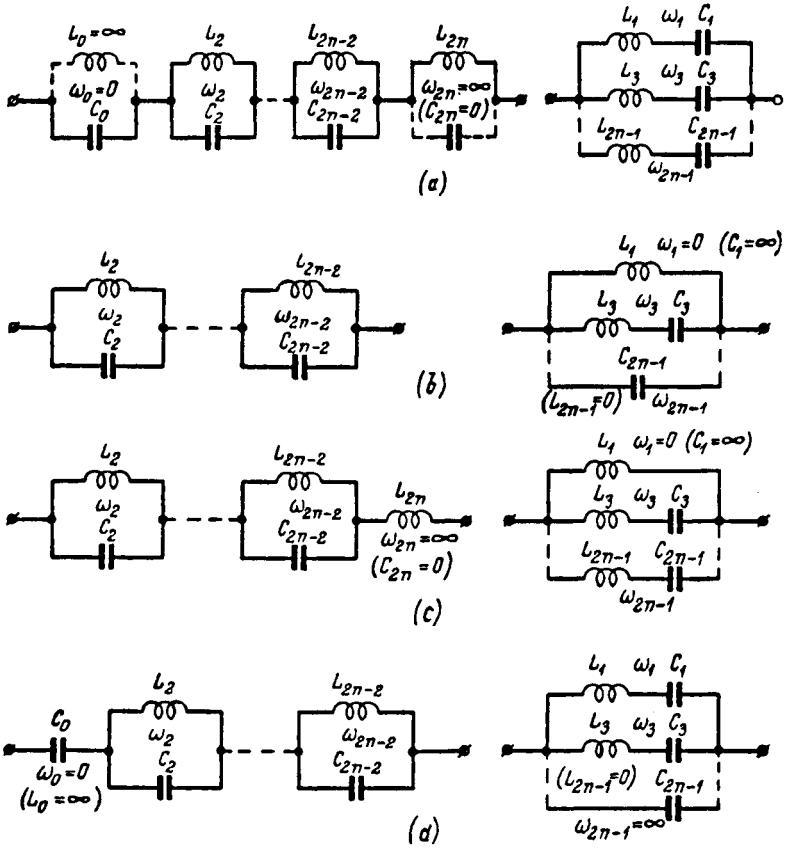



FIG. 20. Canonic forms of a reactive two-terminal network: (*a*) external poles; (*b*) external zeros; (*c*) external zero and pole; (*d*) external pole and zero.

This type of network consists of $2n$ elements (Fig. 20*a*). After deducting the external poles of the frequency function the number of resonance frequencies for this type of network is $2n - 1$.

Frequencies $\omega_0 = 0$ and $\omega_{2n} = \infty$ corresponding to the external poles of function $Z$ can be regarded as the parallel resonance frequencies of the networks $L_0C_0$ and $L_{2n}C_{2n}$ with elements

$$L_0 = \frac{1}{\omega_0^2 C_0} = \infty \quad \text{and} \quad C_{2n} = \frac{1}{\omega_{2n}^2 L_{2n}} = 0 \quad \text{(Fig. 20}a\text{)}.$$

Thus the external poles of function $Z$ can be conventionally counted among the network's parallel resonances.

Let us pause to consider some other particular cases of the canonic circuit of a reactive two-terminal network (Fig. 20).

*A* network with two external zeros (Fig. 15*b*) can be regarded as a network in which $\omega_1 = 0$ and $\omega_{2n-1} = \infty$; then

$$C_1 = \frac{1}{\omega_1^2 L_1} = \infty,$$

$$L_{2n-1} = \frac{1}{\omega_{2n-1}^2 C_{2n-1}} = 0$$

(Fig. 20*b*, right), and there are no $L_0$, $C_0$, $L_{2n}$ or $C_{2n}$ elements (Fig. 20*b*, left).

By analogy with (28) the impedance of such a network is expressed by the formula

$$Z = \frac{jH(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-3}^2)}{\omega(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)\dots(\omega^2 - \omega_{2n-2}^2)}. \tag{6.40}$$

A two-terminal network with an external zero and pole (Fig. 15*c*) can be regarded as a network in which $\omega_1 = 0$ and $\omega_{2n} = \infty$; then $C_1 = \infty$, $C_{2n} = 0$ and there are no elements $L_0$ and $C_0$ (Fig. 20*c*).

The impedance of this type of network is expressed by the formula

$$Z = \frac{j\omega H(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}{(\omega^2 - \omega_2^2)\dots(\omega^2 - \omega_{2n-2}^2)}. \tag{6.41}$$

Finally, a two-terminal network with an external pole and zero (Fig. 15*d*) can be regarded as a network in which $\omega_0 = 0$ and $\omega_{2n} - 1 = \infty$; then

$$L_0 = \frac{1}{\omega_0^2 C_0} = \infty,$$

$$L_{2n-1} = \frac{1}{\omega_{2n-1}^2 C_{2n-1}} = 0$$

and there are no $L_{2n}$ and $C_{2n}$ elements (Fig. 20*d*).

The impedance of this type of network is expressed by the formula

$$Z = \frac{jH(\omega^2 - \omega_1^2)\dots(\omega^2 - \omega_{2n-3}^2)}{\omega(\omega^2 - \omega_2^2)\dots(\omega^2 - \omega_{2n-2}^2)}. \tag{6.42}$$

In formulae (40)–(42), as in formula (28), the number of binomials with differences between the squares of the variable frequency and resonance frequency in the numerator and denominator is the same. By dividing each

binomial in the numerator and denominator by $\omega^2$ we get a binomial of the form $1 = \omega k^2/\omega^2$ which approaches unity when $\omega \to \infty$. This makes it possible to find the scale factor $H$ contained in formulae (28) and (40)–(42).

In accordance with (28), as the frequency increases, $Z \to j\omega H$ and, on the basis of Fig. 20*a*, when $\omega$ increases, $Z \to j\omega L_{2n}$ or

$$Z \to j\omega \left( \frac{1}{L_1} + \frac{1}{L_3} + \cdots + \frac{1}{L_{2n-1}} \right)^{-1},$$

from which

$$H = L_{2n} = \left( \frac{1}{L_1} + \frac{1}{L_3} + \cdots + \frac{1}{L_{2n-1}} \right)^{-1}. \tag{6.43}$$

In accordance with (40), as the frequency increases, $Z \to j\,H/\omega$. On the basis of Fig. 20*b*, when $\omega$ increases

$$Z \to \frac{1}{j\omega \left( \frac{1}{C_2} + \frac{1}{C_4} + \cdots + \frac{1}{C_{2n-2}} \right)^{-1}}$$

or

$$Z \to \frac{1}{j\omega C_{2n-1}},$$

from which

$$H = -\left( \frac{1}{C_2} + \frac{1}{C_4} + \cdots + \frac{1}{C_{2n-2}} \right) = -\frac{1}{C_{2n-1}}. \tag{6.44}$$

In accordance with (41) as the frequency increases, $Z \to j\omega H$ and, on the basis of Fig. 20*c*, as $\omega$ increases

$$Z \to j\omega L_{2n} \quad \text{or} \quad Z \to j\omega \left( \frac{1}{L_1} + \frac{1}{L_3} + \cdots + \frac{1}{L_{2n-1}} \right)^{-1},$$

from which

$$H = L_{2n} = \left( \frac{1}{L_1} + \frac{1}{L_3} + \cdots + \frac{1}{L_{2n-1}} \right)^{-1}. \tag{6.45}$$

In accordance with (42), as the frequency increases $Z \to j\,H/\omega$ and, on the basis of Fig. 20*d*, when $\omega$ increases

$$Z \to \frac{1}{j\omega \left( \frac{1}{C_0} + \frac{1}{C_2} + \cdots + \frac{1}{C_{2n-2}} \right)}$$

or

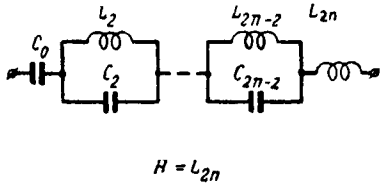
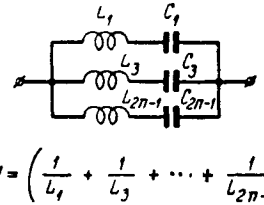
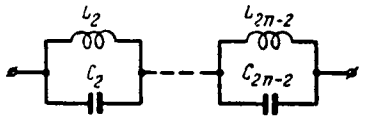
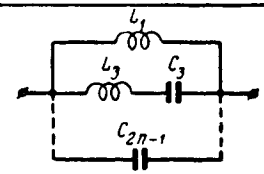
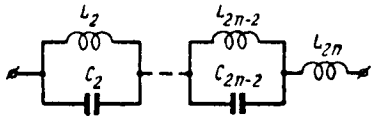
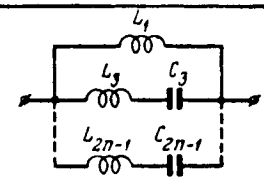
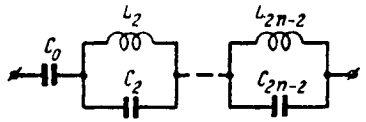
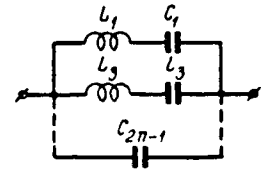
$$Z \to \frac{1}{j\omega C_{2n-1}},$$

from which

$$H = -\left(\frac{1}{C_0} + \frac{1}{C_2} + \cdots + \frac{1}{C_{2n-2}}\right) = -\frac{1}{C_{2n-1}}. \qquad (6.46)$$

The results of determining the scale factor $H$ are given in Table 2.

TABLE 2. SCALE FACTOR $H$

| | |
|---|---|
|   $H = L_{2n}$ |   $H = \left(\frac{1}{L_1} + \frac{1}{L_3} + \cdots + \frac{1}{L_{2n-1}}\right)^{-1}$ |
|   $H = -\left(\frac{1}{C_2} + \frac{1}{C_4} + \cdots + \frac{1}{C_{2n-2}}\right)$ |   $H = -\frac{1}{C_{2n-1}}$ |
|   $H = L_{2n}$ |   $H = \left(\frac{1}{L_1} + \frac{1}{L_3} + \cdots + \frac{1}{L_{2n-1}}\right)^{-1}$ |
|   $H = -\left(\frac{1}{C_0} + \frac{1}{C_2} + \cdots + \frac{1}{C_{2n-2}}\right)$ |   $H = -\frac{1}{C_{2n-1}}$ |

## 7. Sign of the Frequency Derivative of the Reactance or Susceptance Functions of a Dissipationless Two-terminal Network

The impedance of a generalized dissipationless two-terminal network consisting of twin-element two-terminal networks connected in series (Fig. 18) is determined by the sum of the impedances of the component networks:

$$Z = \frac{1}{j\omega C_0} + j\omega L_{2n} + \sum_{k=1}^{n-1} Z_{2k},$$

where

$$Z_{2k} = \frac{j\omega}{C_{2k}(\omega_{2k}^2 - \omega^2)}.$$

Taking the derivative of $Z/j$ with respect to $\omega$, we get:

$$\frac{dZ}{j\,d\omega} = \frac{1}{\omega_2 C_0} + L_{2n} + \sum_{k=1}^{n-1} \frac{(\omega_{2k}^2 - \omega^2) - \omega(-2\omega)}{C_{2k}(\omega_{2k}^2 - \omega^2)^2}$$

$$= \frac{1}{\omega^2 C_0} + L_{2n} + \sum_{k=1}^{n-1} \frac{\omega^2 + \omega_{2k}^2}{C_{2k}(\omega_{2k}^2 - \omega^2)^2}. \tag{6.47}$$

All the terms in the right-hand part of (47) are positive. Consequently, *the frequency derivative of the reactance function of a non-dissipative two terminal network generally has a positive sign.*

It can also be shown analogously that *the frequency derivative of the susceptance function of a non-dissipative two-terminal network is generally positive.*

In fact, on the basis of Fig. 20*a* (right)

$$Y = \sum_{k=1}^{n} \frac{\omega}{jL(\omega^2 - \omega_{2k-1}^2)}. \tag{6.48}$$

From this

$$\frac{dY}{j\,d\omega} = \sum_{k=1}^{n} \frac{-(\omega^2 - \omega_{2k-1}^2) + \omega(2\omega)}{L_{2k-1}(\omega^2 - \omega_{2k-1}^2)^2} = \sum_{k=1}^{n} \frac{\omega^2 + \omega_{2k-1}^2}{L_{2k-1}(\omega^2 - \omega_{2k-1}^2)^2} > 0.$$

From the positive nature of the derivative $dZ/j\,d\omega$ it follows that *the zeros and poles of function Z should alternate*, since if there were two consecutive zeros not separated by a pole there would be a section of the curve with a negative derivative. The same is true of function $Y$.

EXAMPLE 1 (Ref. 78). It is required to construct the canonic form of a multi-element dissipationless two-terminal network with the given values of zero $\omega_3 = 300$, $\omega_5 = 500$, $\omega_7 = 700$ rad/sec, poles $\omega_2 = 200$, $\omega_4 = 400$, $\omega_6 = 600$ rad/sec, and an impedance $Z = j\,1250\Omega$ with $\omega = 50$ rad/sec.

On the basis of (41)

$$Z = \frac{j\omega H(\omega^2 - \omega_3^2)(\omega^2 - \omega)_5^2(\omega^2 - \omega_7^2)}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)}.$$

Canonic circuits, in accordance with Fig. 20*c* (Table 2), are shown in Fig. 21*b* and *c*.

When $\omega = 50$ rad/sec

$$1250 = 50H\frac{(0{\cdot}25 - 9)(0{\cdot}25 - 25)(0{\cdot}25 - 49)}{(0{\cdot}25 - 4)(0{\cdot}25 - 16)(0{\cdot}25 - 36)},$$

from which $H = 5$.

According to (45) $H = L_8$; consequently, $L_8 = 5$ henries.

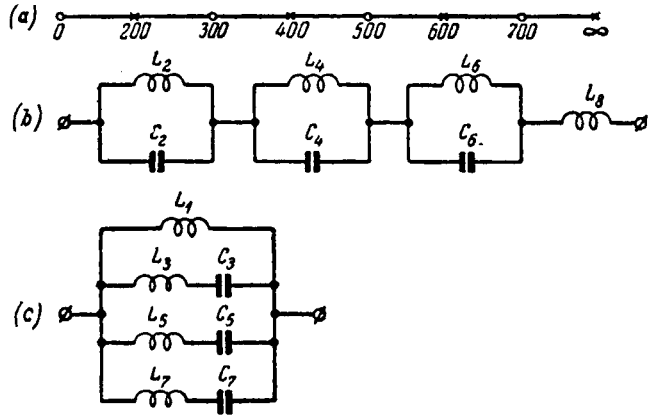



FIG. 21. Example 1.

Using formulae (34) and (35), we find

$$C_2 = -\frac{1}{5}\cdot\frac{(4 - 16)(4 - 36)\,10^{-4}}{(4 - 9)(4 - 25)(4 - 49)} = 1{\cdot}625\,.\,10^{-6}\ \text{F}$$

and

$$L_2 = \frac{1}{4\,.\,10^4\,.\,1{\cdot}625\,.\,10^{-6}} = 15{\cdot}38\ \text{henries}.$$

We can find the remaining parameters of the Fig. 21*b* circuit analogously:

$$C_4 = 2{\cdot}31\,.\,10^{-6}\ \text{F}, \quad L_4 = 2{\cdot}704\ \text{henries},$$

$$C_6 = 3{\cdot}315\,.\,10^{-6}\ \text{F}, \quad L_6 = 0{\cdot}838\ \text{henries}.$$

In accordance with (39)

$$L_1 = 5\frac{9\,.\,25\,.\,49}{4\,.\,16\,.\,36} = 23{\cdot}92\ \text{henries},$$

$$L_3 = 5\frac{9(9 - 25)(9 - 49)}{(9 - 4)(9 - 16)(9 - 25)} = 30{\cdot}5\ \text{henries},$$

$$C_3 = \frac{1}{9\,.\,10^4\,.\,30{\cdot}5} = 0{\cdot}364\,.\,10^{-6}\ \text{F}.$$

We can determine the remaining parameters of the Fig. 21 *c* circuit analogously:

$$C_5 = 0{\cdot}173 \cdot 10^{-6}\ \text{F}, \quad L_5 = 23{\cdot}1\ \text{henries},$$

$$C^7 = 167 \cdot 10^{-6}\ \text{F}, \quad L^7 = 12{\cdot}18\ \text{henries}.$$

The solution can be verified by formula (45):

$$\left(\frac{1}{23{\cdot}92} + \frac{1}{30{\cdot}5} + \frac{1}{23{\cdot}1} + \frac{1}{12{\cdot}18}\right)^{-1} = 5,$$

i.e. the solution is correct.

## 8. Dissipationless Ladder Networks

Together with the circuits of a generalized two-terminal network examined above which respond to expressions for $Z$, there are other canonic forms of the type given in Fig. 22, which can be obtained if the impedance of the network is expressed as a continued fraction

$$Z = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_3 + \cfrac{1}{Y_4 + \cdots}}} \cdots + \cfrac{1}{Z_{n-1} + \cfrac{1}{Y_n}}. \tag{6.49}$$

The rational-fractional function $Z$ expressed by formula (23) is expanded in the form of a continued fraction by one of two methods.

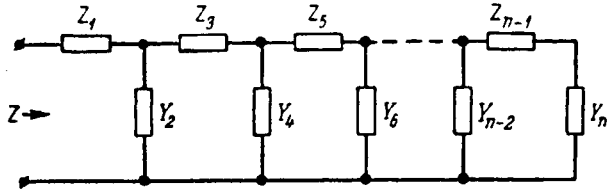



FIG. 22. Ladder structure of a generalized two-terminal network.

(1) Let $Z = M/N$,

where

$$M = a_{2n}p^{2n} + a_{2n-2}p^{2n-2} + \cdots + a_2p^2 + a_0$$

and

$$N = b_{2n-1}p^{2n-1} + b_{2n-3}p^{2n-3} + \cdots + b_3p^3 + b_1p,$$

i.e. $M$ is one degree higher than $N$.

The quotient of dividing the numerator $M$ by denominator $N$ by positive powers of $p$ is in the form of two terms, the first of which contains $p$ to the first power, the second being the remainder:

$$Z = L_1 p + \frac{M'}{N}.$$

Here $M'$ is a polynomial of degree $2n-2$. Hence $N$ is one degree higher than $M'$. Therefore, representing the remainder in the form $1/N/M'$, it is possible to continue the division:

$$Z = L_1 p + \cfrac{1}{C_1 p + \cfrac{N'}{M'}},$$

where $N'$ is a polynomial of degree $2n-3$, i.e. one degree lower than $M'$. Each division and inversion reduces the remaining quotient by one degree.

Continuing thus we get

$$Z = L_1 p + \cfrac{1}{C_1 p + \cfrac{1}{L_2 p + \cfrac{1}{C_2 p \ldots}}} \cdots + \cfrac{1}{L_n p + \cfrac{1}{C_n p}}. \tag{6.50}$$

Comparing expression (49) for the impedance of the circuit with the expansion of function $Z$ in the form of a continued fraction (50), taking into account that $p = j\omega$, we find that

$$Z_k = j\omega K_k, \quad Y_{k+1} = j\omega C_k. \tag{6.51}$$

A circuit diagram, with inductances $L_k$ and $C_k$, corresponding to this case is given in Fig. 23*a*, left.

(2) If $M$ is divided by $N$ with respect to the quantity $p^{-1}$, i.e. for negative powers of $p$, this results in an expansion of the form

$$Z = C_1^{-1} p^{-1} + \cfrac{1}{L_1^{-1} p^{-1} + \cfrac{1}{C_2^{-1} p^{-1} + \cfrac{1}{L_2^{-1} p^{-1} \ldots}}} \cdots + \cfrac{1}{C_n^{-1} p^{-1} + \cfrac{1}{L_n^{-1} p^{-1}}}. \tag{6.52}$$

From a comparison of (49) with (52) it follows that

$$Z_k = \frac{1}{j\omega C_k}, \quad Y_{k+1} = \frac{1}{j\omega L_k}. \tag{6.53}$$

A circuit diagram, with elements $L_k$ and $C_k$, corresponding to this case is shown in Fig. 23*a*, right.

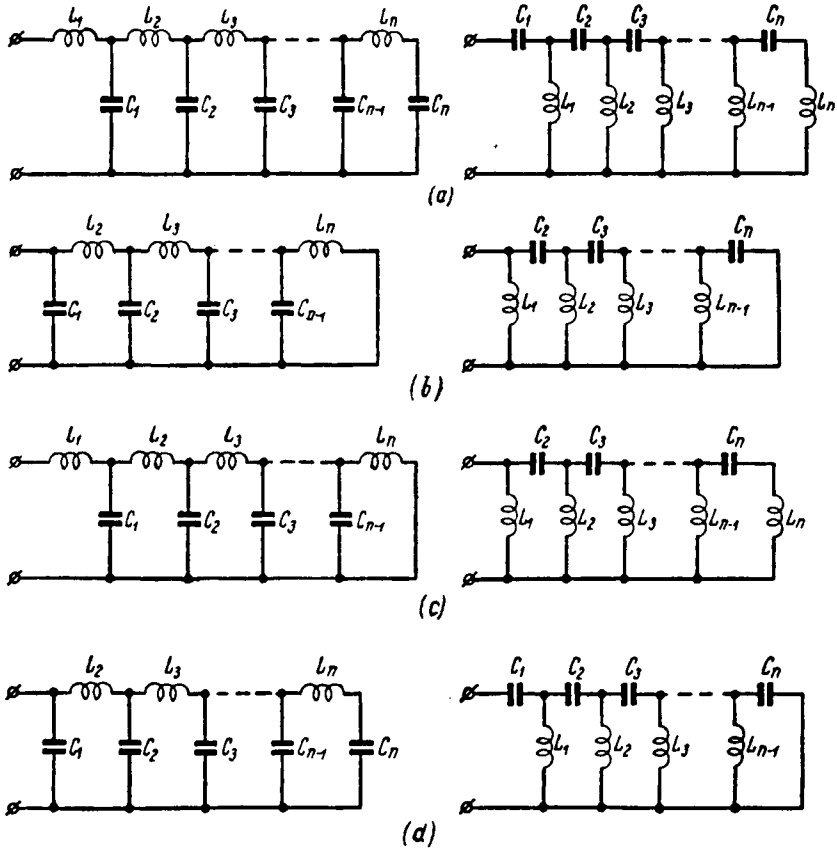



FIG. 23. Ladder structures of a reactive two-terminal network: (*a*) external poles; (*b*) external zeros; (*c*) external zero and pole; (*d*) external pole and zero.

The Fig. 23*a* circuits, equivalent to the Fig. 20*a* circuits, correspond to the case when function $Z$ has two external poles.

If function $Z$ has two external zeros, then elements $L_1$ and $C_n$ are omitted from the left-hand circuit in Fig. 23*a*, and $C_1$ and $L_n$ from the right. The Fig. 23*b* circuits correspond to this case.

If function $Z$ has an external zero and an external pole, there is no capacitance $C_n$ in the left circuit of Fig. 23*c* and no capacitance $C_1$ in the right.

Finally, if function $Z$ has an external pole and an external zero, there is no inductance $L_1$ in the left circuit of Fig. 23*d* and no inductance $L_n$ in the right.

EXAMPLE 2 (Ref. 78). It is required to construct circuit diagrams for a multi-element dissipationless two-terminal network equivalent to that examined in Example 1.

According to the condition

$$Z = \frac{j\omega H(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)(\omega^2 - \omega_7^2)}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)}$$

$$= \frac{5p^7 + 4{\cdot}15 \cdot 10^6 p^5 + 9{\cdot}45 \cdot 10^{11} p^3 + 5{\cdot}51 \cdot 10^{10} p}{p^6 + 5{\cdot}6 \cdot 10^5 p^4 + 7{\cdot}84 \cdot 10^{10} p^2 + 2{\cdot}3 \cdot 10^{15}}.$$

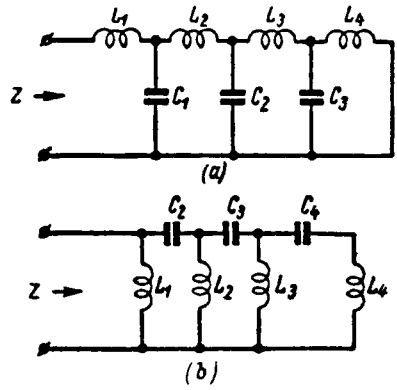



FIG. 24. Example 2.

The given rational-fractional function is expanded in the form of a continued fraction

$$Z = 5p + \cfrac{1}{0{\cdot}74 \cdot 10^{-6} p + \cfrac{1}{9p + \cfrac{1}{1{\cdot}087 \cdot 10^{-6} p + \cfrac{1}{6{\cdot}5p + \cfrac{1}{2{\cdot}68 \cdot 10^{-6} p + \cfrac{1}{3{\cdot}43p}}}}}}.$$

The corresponding circuit diagram for the network is given in Fig. 24*a* where

$$L_1 = 5 \text{ H}, \qquad C_1 = 0{\cdot}74 \cdot 10^{-6} \text{ F},$$

$$L_2 = 9 \text{ H}, \qquad C_2 = 1{\cdot}087 \cdot 10^{-6} \text{ F},$$

$$L_3 = 6{\cdot}5 \text{ H}, \qquad C_3 = 2{\cdot}68 \cdot 10^{-6} \text{ F}.$$

$$L_4 = 3{\cdot}43 \text{ H};$$

The expansion of function $Z$ into a continued fraction in terms of $p^{-1}$ has the form

$$Z = \cfrac{1}{23{\cdot}9^{-1}p^{-1} + \cfrac{1}{(0{\cdot}706\,.\,10^{-6})^{-1}p^{-1} + \cfrac{1}{10{\cdot}2^{-1}p^{-1} + \cfrac{1}{(0{\cdot}227\,.\,10^{-6})^{-1}p^{-1} + \cfrac{1}{18{\cdot}9^{-1}p^{-1} + \cfrac{1}{(0{\cdot}0212\,.\,10^{-6})^{-1}p^{-1} + \cfrac{1}{139^{-1}p^{-1}}}}}}}.$$

The corresponding circuit diagram for the network is given in Fig. 24*b* where

$$L_1 = 23{\cdot}9\ \text{H}, \qquad C_2 = 0{\cdot}706\,.\,10^{-6}\ \text{F},$$
$$L_2 = 10{\cdot}2\ \text{H}, \qquad C_3 = 0{\cdot}227\,.\,10^{-6}\ \text{F},$$
$$L_3 = 18{\cdot}9\ \text{H}, \qquad C_4 = 0{\cdot}0212\,.\,10^{-6}\ \text{F}.$$
$$L_4 = 139\ \text{H};$$

## 9. Potentially Equivalent Two-terminal Networks and the Conditions for their Equivalence

Various canonic forms (Figs. 20 and 23) of equivalent dissipationless two-terminal networks were examined in the previous sections. The condition for their equivalence was the coincidence of the zeros and poles and the equality of the $H$ factors. Only in this case was the equality of the impedances of the networks ensured throughout the whole scale of frequencies, i.e. the networks were *equivalent.*

A numerical change in the resonance frequencies of one of the networks, not affecting the distribution of poles and zeros, or a numerical change in scale factor $H$ contained in the expression for $Z$ violates the requirement of equivalence: the impedances of the networks which are, until then, equivalent, become unequal.

Any two such networks are *potentially equivalent.* This means that they can be equivalent when the appropriate conditions are observed.

Not only dissipationless two-terminal networks but also dissipative two-terminal networks can be potentially equivalent.

As an example let us examine the circuits given in Fig. 25*a* and *b*.

Their impedances are expressed thus:

for the Fig. 25*a* circuit:

$$Z = dZ_1 + \frac{bcZ_1Z_2}{bZ_1 + cZ_2} = \frac{\frac{bd}{c}Z^2 + (b + d)Z_1Z_2}{\frac{b}{c}Z_1 + Z_2}; \qquad (6.54)$$

for the Fig. 25*b* circuit:

$$Z = \frac{aZ_1(Z_1 + Z_2)}{(a+1)Z_1 + Z_2} = \frac{aZ_1^2 + aZ_1Z_2}{(a+1)Z_1 + Z_2}. \tag{6.55}$$

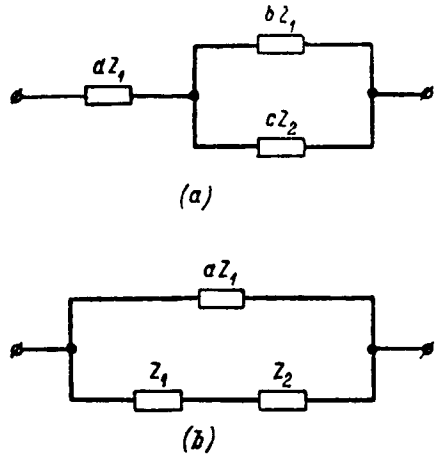



FIG. 25. Potentially equivalent two-terminal networks.

Expressions (54) and (55) are equal in the whole spectrum of frequencies if these conditions hold:

$$a = \frac{bd}{c}, \quad a = b + d, \quad a + 1 = \frac{b}{c}$$

or

$$b = \frac{a^2}{a+1}, \quad c = \left(\frac{a}{a+1}\right)^2, \quad d = \frac{a}{a+1}.$$

These are the conditions for the equivalence of the two-terminal networks shown in Fig. 25*a* and *b*.

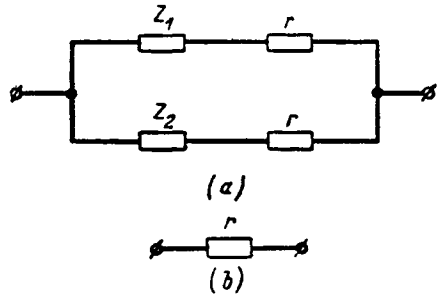



FIG. 26. Potentially equivalent two-terminal networks.

TABLE 3. POTENTIALLY EQUIVALENT TWO-TERMINAL NETWORKS

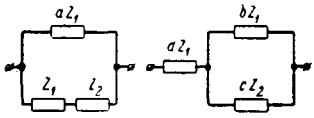
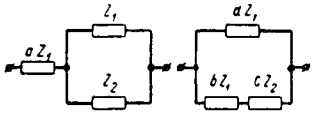
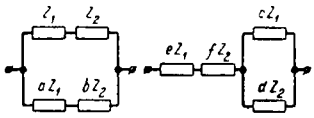
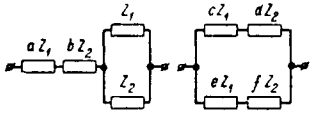
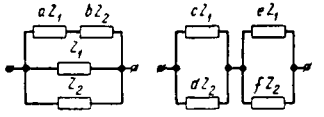
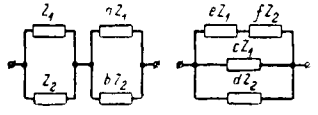
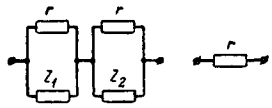
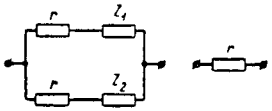
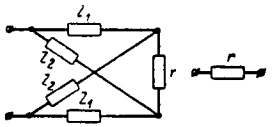
| Circuit diagrams | Equivalence conditions |
|---|---|
|  | $B=\dfrac{a^2}{a+1},\quad c=\left(\dfrac{a}{a+1}\right)^2,\quad d=\dfrac{a}{a+1}$ |
|  | $B=a(a+1),\quad c=(a+1)^2,\quad d=a+1$ |
|  | $c=\dfrac{(a-B)^2}{(a+1)(B+1)^2},\quad d=\dfrac{(a-B)^2}{(a+1)^2(B+1)},$ $e=\dfrac{a}{a+1},\quad f=\dfrac{B}{B+1}$ |
|  | $c=\dfrac{B(A+B)}{A+B-2B},\quad d=\dfrac{2BB}{A+B-2B},$ $e=\dfrac{B(A-B)}{B-A+2B},\quad f=\dfrac{2BB}{B-A+2B},$ where $A=a+B+1\quad B=\sqrt{(A^2-4aB)}$ |
|  | $c=\dfrac{(2B-A+B)(A+B)}{4BB},\quad d=\dfrac{2B-A+B}{2B},$ $e=\dfrac{(A+B-2B)(A-B)}{4BB},\quad f=\dfrac{A+B-2B}{2B}$ where $A=a+B+1,\quad B=\sqrt{(A^2-4aB)}$ |
|  | $c=a+1,\quad d=B+1,$ $e=\dfrac{a(a+1)(B+1)^2}{(a-B)^2}$ $f=\dfrac{B(a+1)^2(B+1)}{(a-B)^2}$ |

TABLE 3. CONTD.

| Circuit diagrams | Equivalence conditions |
|---|---|
|  | $Z_1Z_2 = r^2$ |
|  | $Z_1Z_2 = r^2$ |
|  | $Z_1Z_2 = r^2$ |

Let us examine another example, Fig. 26.

The impedance of the two-terminal network in Fig. 26*a*

$$Z = \frac{(Z_1 + r)(Z_2 + r)}{Z_1 + Z_2 + 2r} = \frac{(Z_1 + Z_2)r + Z_1Z_2 + r^2}{Z_1 + Z_2 + 2r}.$$

When the condition $Z_1Z_2 = r^2$ obtains, we get $Z = r$, i.e. the network in Fig. 26*a* becomes equivalent to the simplest single-element two-terminal network consisting of a resistance $r$ (Fig. 26*b*).

The circuits of some potentially equivalent two-terminal networks and the conditions for their equivalence are given in Table 3 (Refs. 1 and 6).

## 10. Potentially Inverse (Dual) Two-terminal Networks and the Conditions for their Duality

Any two-terminal networks are called *inverse* or *dual* if the product of their impedances $Z_I$ and $Z_{II}$ is a constant independent of frequency

$$Z_IZ_{II} = k^2; \tag{6.56}$$

here $k$ is a constant having the dimension of the resistance.

If two networks are *inverse* when certain conditions are observed they are called *potentially inverse.*

Inverse two-terminal networks are dual circuits (Chapter IV, Section 14) whose impedances satisfy condition (56). These networks are used extensively as component elements of electric wave-filters (Chapter VIII).

As an example, Fig. 27 shows inverse two-terminal networks in the form of circuit diagrams. The impedance function of one of these networks is analogous to the admittance functions of the other.

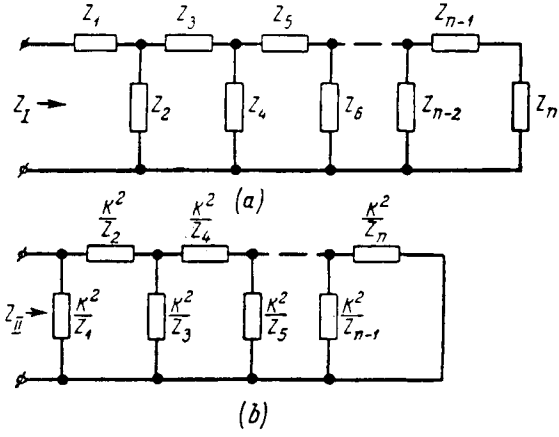



FIG. 27. Reciprocal two-terminal network.

By studying typical frequency functions of the impedances of the dissipationless two-terminal network shown in Fig. 15 it is easy to confirm that the networks with curves *a* and *b* are potentially inverse. The same is true for the networks with curves *c* and *d*. The condition of their being mutually inverse is the coincidence of the zeros and, accordingly, the poles of functions $Z$ and $Y$ of the networks over the whole scale of frequencies.

In accordance with (28) let

$$Z_I = \frac{j\omega H_1(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}{(\omega^2 - \omega_0^2)(\omega^2 - \omega_2^2)\dots(\omega^2 - \omega_{2n-2}^2)}. \qquad (6.57)$$

An inverse two-terminal network has an impedance*

$$Z_{II} = \frac{(\omega^2 - \omega_0^2)(\omega^2 - \omega_2^2)\dots(\omega^2 - \omega_{2n-2}^2)}{j\omega H_2(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}. \qquad (6.58)$$

* Here, in contrast with the previous case, series-resonance frequencies are denoted by even and parallel-resonance frequencies by odd indices.

The product of (57) and (58) satisfies equation (56)

$$Z_{\mathrm{I}}Z_{\mathrm{II}} = \frac{H_1}{H_2} = \text{const.}$$

If, in accordance with (41),

$$Z_{\mathrm{I}} = \frac{j\omega H_1(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}{(\omega^2 - \omega_2^2)\dots(\omega^2 - \omega_{2n-2}^2)}, \tag{6.59}$$

the inverse two-terminal network has an impedance

$$Z_{\mathrm{II}} = \frac{(\omega^2 - \omega_2^2)\dots(\omega^2 - \omega_{2n-2}^2)}{j\omega H_2(\omega^2 - \omega_3^2)\dots(\omega^2 - \omega_{2n-1}^2)}. \tag{6.60}$$

The product of impedances $Z_{\mathrm{I}}$ and $Z_{\mathrm{II}}$, in accordance with (59) and (60), satisfies equation (56)

$$Z_{\mathrm{I}}Z_{\mathrm{II}} = \frac{H_1}{H_2} = \text{const.}$$

## 11. Multi-element Dissipative Two-terminal Networks Containing Two Types of Element

Two types of multi-element dissipative two-terminal network are examined below: a network containing $r$ and $L$ elements, and a network containing $r$ and $C$ elements.

### Two-terminal Network with $r$ and $L$ Elements

If a network consists of $r$ and $L$ elements, i.e. $Z_{ii} = r_{ii} + pL_{ii}$ and $Z_{ik} = -(r_{ik} + pL_{ik})$, the number of independent loops being $n$, the determinant of the system $\Delta$ in (20) contains powers of $p$ from 0 to $n$ and the cofactor $\Delta_{11}$ from 0 to $n-1$. Consequently, the denominator is one degree lower than the numerator.

$$Z = \frac{a_n p^n + a_{n-1}p^{n-1} + \dots + a_1 p + a_0}{b_{n-1}p^{n-1} - b_{n-2}p^{n-2} + \dots + b_1 p + b_0}. \tag{6.61}$$

Fraction (61) is an improper fraction and therefore function $Z$ has a pole at infinity*. The zeros and remaining poles of function $Z$ are real and negative. We shall denote them by $p_1, p_3, \dots, p_{2n-1}$ and $p_2, p_4, \dots, p_{2n-2}$ respectively.

* In view of the fact that the difference on the degrees of the numerator and denominator is unity, the pole at infinity is a simple pole (Appendix I, Section 3).

They alternate on a real semiaxis

$$-\infty \leqq p_{2n-1} < p_{2n-2} < \cdots < p_2 < p_1 \leqq 0.$$

The numerator and denominator of (61) can be factorized linearly, i.e. represented in the form of the products of the differences $p - p_k$

$$Z = \frac{H(p - p_1)(p - p_3) \ldots (p - p_{2n-1})}{(p - p_2)(p - p_4) \ldots (p - p_{2n-2})}. \tag{6.62}$$

By dividing the numerator and denominator of fraction (61) we get

$$Z = \frac{a_0}{b_0} + \frac{a_n}{b_{n-1}} p + \frac{a'_{n-1} p^{n-1} + a'_{n-2} p^{n-2} + \cdots + a'_1 p}{b_{n-1} p^{n-1} + b_{n-2} p^{n-2} + \cdots + b_0},$$

where $a'_{n-1}, a'_{n-2}, \ldots, a'_1$ are new factors obtained as a result of dividing. The last expression can be represented in this form:

$$Z = \frac{a_0}{b_0} + \frac{a_n}{b_{n-1}} p + \sum_{k=2}^{n} \frac{\alpha_{2k-2} p}{p - p_{2k-2}}. \tag{6.63}$$

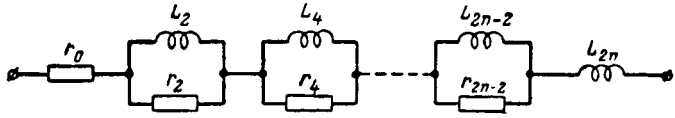



FIG. 28. *rL* two-terminal network corresponding to formula (63).

The first constant term expresses the resistance

$$r = \frac{a_0}{b_0}, \tag{6.64}$$

given in Fig. 28.

The inductive element shown in Fig. 28 corresponds to the second term*, i.e.

$$L_{2n} = \frac{a_n}{b_{n-1}} = H. \tag{6.65}$$

In structure the item $a_{2k-2} p/(p - p_{2k-2})$ corresponds to a two-terminal network consisting of a resistance $r_{2k-2}$ and an inductance $L_{2k-2}$ connected in parallel,

$$Z_{2k-2} = \frac{r_{2k-2} p L_{2k-2}}{r_{2k-2} + p L_{2k-2}} = \frac{r_{2k-2} p}{p + \dfrac{r_{2k-2}}{L_{2k-2}}}.$$

* The second term in expression (63) represents the only term in the so-called "infinite part" of the function at the pole $p = \infty$.

In the present case

$$\left.\begin{aligned} r_{2k-2} &= \alpha_{2k-2}, \\ \frac{r_{2k-2}}{L_{2k-2}} &= -p_{2k-2}. \end{aligned}\right\} \tag{6.66}$$

The problem is thus reduced to determining $\alpha_{2k-2}$.

Multiplying both parts of (63) by $p - p_{2k-2}$ and moving $p$ to $p_{2k-2}$, we get*

$$\alpha_{2k-2} = \left[(p - p_{2k-2})\frac{Z}{p}\right]_{p=p_{2k-2}}. \tag{6.67}$$

As we should expect, $\alpha_{2k-2}$ has the value of the impedance.

Inductance $L_{2k-2}$ is found from expression (66) as

$$L_{2k-2} = -\frac{r_{2k-2}}{p_{2k-2}}. \tag{6.68}$$

Thus the two-terminal network shown in Fig. 28 corresponds to expression (63), its elements being determined from (64) to (68).

If we start with the expression for the network's admittance

$$Y = \frac{(p - p_2)(p - p_4)\ldots(p - p_{2n-2})}{H(p - p_1)(p - p_3)\ldots(p - p_{2n-1})}$$

and expand it down into partial fractions

$$Y = \sum_{k=1}^{n} \frac{\alpha_{2k-1}}{p - p_{2k-1}}, \tag{6.69}$$

every term $\alpha_{2k-1}/(p - p_{2k-2})$ may be regarded as the admittance of a two-terminal network consisting of a resistance $r_{2k-1}$ and an inductance $L_{2k-1}$ connected in series:

$$Y_{2k-1} = \frac{1}{r_{2k-1} + pL_{2k-1}} = \frac{\dfrac{1}{L_{2k-1}}}{p + \dfrac{r_{2k-1}}{L_{2k-1}}}.$$

* The term $\alpha_{2k-2}/(p - p_{2k-2})$ may be regarded as the first term with a negative power in the expansion of function $Z/p$ into a Laurent series in the vicinity of pole $p_{2k-2}$. Therefore $\alpha_{2k-2}$ is the residue of function $Z/p$ at the pole $p_{2k-2}$. If the impedance $Z$ is expressed by the admittance $Y$, the residue can be found from the formula

$$a_{2k-2} = \frac{1}{\left(p\dfrac{dY}{dp}\right)_{p=p_{2k-2}}}.$$

Consequently, in the given case

$$\left.\begin{aligned} \frac{1}{L_{2k-1}} &= \alpha_{2k-1}, \\ \frac{r_{2k-1}}{L_{2k-1}} &= -p_{2k-1}. \end{aligned}\right\} \tag{6.70}$$

The factor $\alpha_{2k-1}$ is found from the formula*

$$\alpha_{2k-1} = [(p - p_{2k-1})Y]_{p=p_{2k-2}} = \frac{1}{\left(\frac{dZ}{dp}\right)_{p=p_{2k-1}}}. \tag{6.71}$$

$$(k = 1, 2, \ldots, n)$$

Thus, the two-terminal network shown in Fig. 29 corresponds to expression (69), its elements being determined on the basis of (70) and (71).

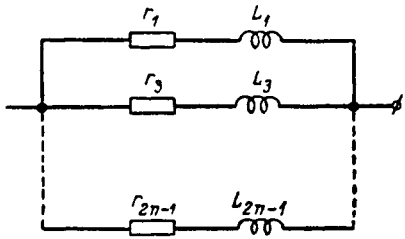



FIG. 29. *rL* two-terminal network corresponding to formula (69).

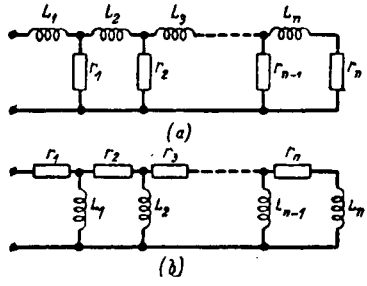



FIG. 30. Ladder structure of *rL* two-terminal network.

A multi-element *rL*-type two-terminal network can also be produced in the form of the circuit diagrams given in Fig. 30*a* and *b*.

* Factor $\alpha_{2k-1}$ equals the residue of function $Y$ at the pole $p_{2k-1}$.

The Fig. 30*a* circuit corresponds to expanding $Z$ into the continued fraction

$$Z = L_1 p + \cfrac{1}{r_1^{-1} + \cfrac{1}{L_2 p + \cfrac{1}{r_2^{-1} + \cdots}}} \quad \cdots + \cfrac{1}{L_n p + \cfrac{1}{r_n^{-1}}}. \tag{6.72}$$

The Fig. 30*b* circuit corresponds to expanding $Z$ into the continued fraction

$$Z = r_1 + \cfrac{1}{L_1^{-1} p^{-1} + \cfrac{1}{r^2 + \cfrac{1}{L_2^{-1} p^{-1} + \cdots}}} \quad \cdots + \cfrac{1}{r_n + \cfrac{1}{L_n^{-1} p^{-1}}}. \tag{6.73}$$

**Two-terminal Network with *r* and *C* Elements**

If a two-terminal network consists of $r$ and $C$ elements, i.e.

$$Z_{ii} = r_{ii} + \frac{1}{pC_{ii}} \quad \text{and} \quad Z_{ik} = -\left(r_{ik} + \frac{1}{pC_{ik}}\right),$$

and the number of independent loops is $n$, then the impedance of the network

$$Z = \frac{\Delta}{\Delta_{11}} = \frac{p^n \Delta}{p^n \Delta_{11}}$$

is the relationship of the polynomials of the $n$-th degree without the constant term in the denominator

$$Z = \frac{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}{b_n p^n + b_{n-1} p^{n-1} + \cdots + b_2 p^2 + b_1 p}. \tag{6.74}$$

The zeros and poles of function $Z$ are real and negative (13). We shall denote them by $p_1, p_3, \ldots, p_{2n-1}$ and $p_2, p_4, \ldots, p_{2n-2}$ respectively.

On the real negative semiaxis they alternate

$$-\infty \leqq p_{2n-1} < p_{2n-2} < \cdots < p_2 < p_1 < p_0 = 0.$$

The numerator and denominator of (74) can be factorized linearly

$$Z = \frac{H(p - p_1)(p - p_3) \ldots (p - p_{2n-1})}{p(p - p_2)(p - p_4) \ldots (p - p_{2n-2})}. \tag{6.75}$$

Separating the constant part and expanding the right-hand fraction into partial fractions, we find

$$Z = \frac{a_n}{b_n} + \sum_{k=1}^{n} \frac{\alpha_{2k-2}}{p - p_{2k-2}}. \tag{6.76}$$

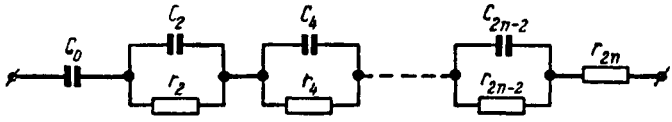



FIG. 31. $rC$ two-terminal network corresponding to formula (76).

The first term of expansion (76) expresses the resistance

$$r_{2n} = \lim_{p \to \infty} Z = \frac{a_n}{b_n}, \tag{6.77}$$

shown in Fig. 31.

The term $\alpha_{2k-2}/(p - p_{2k-2})$ corresponds to the impedance of a two-terminal network consisting of a resistance $r_{2k-2}$ and a capacitance $C_{2k-2}$ connected in parallel:

$$Z_{2k-2} = \frac{r_{2k-2} + \dfrac{1}{pC_{2k-2}}}{r_{2k-2}\dfrac{1}{pC_{2k-2}}} = \frac{\dfrac{1}{C_{2k-2}}}{p + \dfrac{1}{r_{2k-2}C_{2k-2}}}.$$

Consequently, in the given case

$$\left.\begin{aligned} &\frac{1}{C_{2k-2}} = \alpha_{2k-2}, \quad \frac{1}{r_{2k-2}C_{2k-2}} = -p_{2k-2}, \\ &\alpha_{2k-2} = [(p - p_{2k-2})Z]_{p=p_{2k-2}} = \frac{1}{\left(\dfrac{dY}{dp}\right)_{p=p_{2k-2}}} \end{aligned}\right\} \tag{6.78}$$

$$(k = 1, 2, \ldots, n).$$

When $k = 1$ $\alpha_0 = (pZ)_{p=0} = a_0/b_1$ and

$$C_0 = \frac{b_1}{a_0}, \quad r_0 = \infty. \tag{6.79}$$

Thus the two-terminal network shown in Fig. 31 corresponds to expression (76), its elements being determined from (77)–(79).

If we start with the equation for the admittance of the network

$$Y = \frac{p(p - p_2)(p - p_4) \ldots (p - p_{2n-2})}{H(p - p_1)(p - p_3) \ldots (p - p_{2n-1})}$$

and give it in the form of the sum of fractions

$$Y = \sum_{k=1}^{n} \frac{\alpha_{2k-1} p}{p - p_{2k-1}}, \tag{6.80}$$

then every term of $\alpha_{2k-1}p/(p - p_{2k-1})$ may be regarded as the admittance of a two-terminal network consisting of a resistance $r_{2k-1}$ and a capacitance $C_{2k-1}$ connected in series:

$$Y_{2k-1} = \frac{1}{r_{2k-1} + \dfrac{1}{pC_{2k-1}}} = \frac{\dfrac{1}{r_{2k-1}} p}{p + \dfrac{1}{r_{2k-1}C_{2k-1}}}.$$

Consequently, here

$$\left.\begin{aligned} &\frac{1}{r_{2k-1}} = \alpha_{2k-1}, \quad \frac{1}{r_{2k-1}C_{2k-1}} = -p_{2k-1} \\ &\alpha_{2k-1} = \left[(p - p_{2k-1})\frac{Y}{p}\right]_{p=p_{2k-1}} = \frac{1}{\left(p\dfrac{dZ}{dp}\right)_{p=p_{2k-1}}} \end{aligned}\right\} \tag{6.81}$$

$$(k = 1, 2, \ldots, n).$$

The circuit of a two-terminal network corresponding to expressions (80) and (81) is given in Fig. 32.

A multi-element $rC$-type two-terminal network can also be produced in the form of ladder networks (Fig. 33*a* and *b*).

The Fig. 33*a* circuit corresponds to expanding $Z$ into a continued fraction of the form

$$Z = r_1 + \cfrac{1}{C_1 p + \cfrac{1}{r_2 + \cfrac{1}{C_2 p \ldots}}} \quad \cdots + \cfrac{1}{r_n + \cfrac{1}{C_n p}}. \tag{6.82}$$

The Fig. 33*b* circuit corresponds to expanding $Z$ into a continued fraction of the form

$$Z = C_1^{-1} p^{-1} + \cfrac{1}{r_1^{-1} + \cfrac{1}{C_2^{-1} p^{-1} + \cfrac{1}{r_2^{-1} + \cdots}}} \quad \cdots + \cfrac{1}{C_n^{-1} p^{-1} + \cfrac{1}{r_n^{-1}}}. \tag{6.83}$$

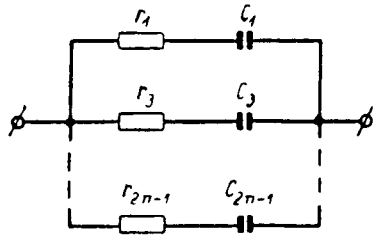



FIG. 32: *rC* two-terminal network corresponding to formula (80).

EXAMPLE 3 (Ref. 78). It is required to construct circuits for an *rC*-type two-terminal network (Figs. 31–33) with given values of zero and given poles of function $Z(p)$: zeros at points $p_1 = -1$, $p_3 = -3$; poles at points $p_0 = 0$, $p_2 = -2$, $p_4 = -4$. It is given that when $\omega = 0$, $\mathrm{Re}\, Z(j\omega) = 1\,\Omega$.

On the basis of expression (75)

$$Z(p) = \frac{H(p+1)(p+3)}{p(p+2)(p+4)} = H\frac{p^2+4p+3}{p^3+6p^2+8p}.$$

When $p = j\omega$

$$Z(j\omega) = H\frac{(2\omega^4+14\omega^2) - j(\omega^5+13\omega^3+24\omega)}{\omega^6+20\omega^4+64\omega^2}.$$

According to the given values

$$H\left(\frac{2\omega^4 + 14\omega^2}{\omega^6 + 20\omega^4 + 64\omega^2}\right)_{\omega=0} = 1,$$

from which

$$H\frac{14}{64} = 1 \quad \text{and} \quad H = \frac{32}{7}.$$

Calculation formulae (77) and (78) are used for constructing a circuit from the Fig. 31 diagram.

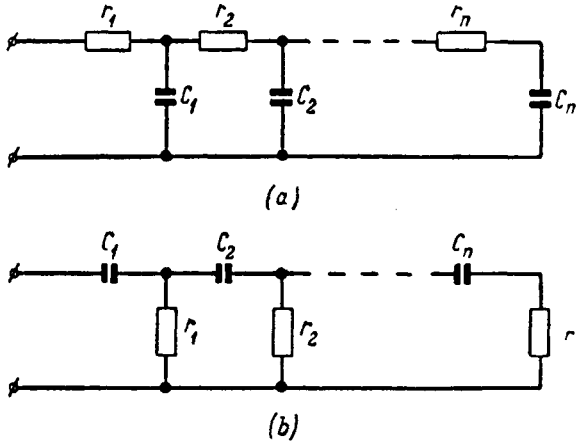



FIG. 33. Ladder structure of $rC$ two-terminal network.

In the case being examined, $Z(p)$ is a proper fraction; consequently,

$$r_{2n} = \lim_{p\to\infty} Z(p) = 0.$$

When $k = 1$ $\alpha_0 = (pZ)_{p=0} = H\frac{3}{8} = \frac{12}{7}$;

when $k = 2$ $\alpha_2 = [(p+2)Z]_{p=-2} = H\frac{1}{4} = \frac{8}{7}$;

when $k = 3$ $\alpha_4 = [(p+4)Z]_{p=-4} = H\frac{3}{8} = \frac{12}{7}$.

Consequently,

$$C_0 = \frac{7}{12}, \quad C_2 = \frac{7}{8}, \quad C_4 = \frac{7}{12}\,\text{F};$$

$$r_2 = \frac{8}{7\cdot 2} = \frac{4}{7}, \quad r_4 = \frac{12}{7\cdot 4} = \frac{3}{7}\,\Omega.$$

Calculation formulae (81) are used for constructing a circuit from the Fig. 32 diagram. If the admittance of the two-terminal network is

$$Y = \frac{p(p+2)(p+4)}{H(p+1)(p+3)},$$

then

$$\alpha_1 = \left[(p+1)\frac{Y}{p}\right]_{p=-1} = \frac{3}{H.2} = \frac{21}{64},$$

$$\alpha_3 = \left[(p+3)\frac{Y}{p}\right]_{p=-3} = \frac{1}{H2} = \frac{7}{64}.$$

Consequently

$$r_1 = \frac{64}{21}, \quad r_3 = \frac{64}{7}, \quad C_1 = \frac{21}{64}, \quad C_3 = \frac{7}{192}\,\text{F}.$$

Capacitance $C_5$ corresponds to a pole at infinity; when $p \to \infty$, $Y \to p/H$ from which

$$C_5 = \frac{1}{H} = \frac{7}{32}\,\text{F}.$$

Then $r_5 = 0$.

The following expansion corresponds to the Fig. 33*a* circuit diagram:

$$Z = \cfrac{1}{\frac{7}{32}p + \cfrac{1}{\frac{16}{17} + \cfrac{1}{\frac{28}{96}p + \cfrac{1}{\frac{48}{7} + \cfrac{1}{\frac{7}{96}p}}}}}.$$

On the basis of expression (82)

$$r_1 = 0, \quad r_2 = \frac{16}{7}, \quad r_3 = \frac{48}{7}\,\Omega; \quad C_1 = \frac{7}{32}, \quad C_2 = \frac{7}{24}, \quad C_3 = \frac{7}{96}\,\text{F}.$$

The following expansion corresponds to the Fig. 33*b* circuit diagram:

$$Z = \left(\frac{7}{12}\right)^{-1} p^{-1} + \cfrac{1}{1 + \cfrac{1}{\left(\frac{11}{28}\right)^{-1} p^{-1} + \cfrac{1}{\left(\frac{12}{121}\right)^{-1} + \cfrac{1}{\left(\frac{77}{24}\right)^{-1} p^{-1}}}}}.$$

On the basis of expression (83)

$$C_1 = \frac{7}{12}, \quad C_2 = \frac{11}{28}, \quad C_3 = \frac{77}{24}\,\text{F}; \qquad r_1 = 1, \quad r_2 = \frac{12}{121}, \quad r_3 = 0\,\Omega.$$

## 12. Real or Imaginary Parts of Input Impedance or Admittance. Sign of Resistance and Conductance

In the expression for the impedance of a two-terminal network $Z(j\omega) = \Delta(j\omega)/\Delta_{11}(j\omega)$, the determinant of the system, when the circuits dissipative, consists of the elements

$$Z_{ii} = r_{ii} + j\left(\omega L_{ii} - \frac{1}{C\omega_{ii}}\right)$$

and

$$Z_{ik} = -\left[r_{ik} + j\left(\omega L_{ik} - \frac{1}{\omega C_{ik}}\right)\right].$$

It can be expanded about the row

$$\Delta = Z_{11}\Delta_{11} + Z_{12}\Delta_{12} + \cdots + Z_{1n}\Delta_{1n},$$

where $\Delta_{ik}$ is the cofactor of element $Z_{ik}$. Continuing the expansion we get the sum of the terms, each of which is a product of the elements of the system determinant of the system. To make the structure of $\Delta$ clear it is sufficient to concentrate on examining the products of the elements of the main diagonal of the determinant (Ref. 90):

$$Z_{11}Z_{22}\ldots Z_{nn} = \left[r_{11} + j\left(\omega L_{11} - \frac{1}{\omega C_{11}}\right)\right] \times$$

$$\times \left[r_2 + j\left(\omega L_{22} - \frac{1}{\omega C_{22}}\right)\right] \ldots \left[r_{nn} + j\left(\omega L_{nn} - \frac{1}{\omega C_{nn}}\right)\right].$$

Consequently,

$$\Delta = (a_0 + a_2\omega^2 + a_{-2}\omega^{-2} + a_4\omega^4 + a_{-4}\omega^{-4} + \cdots)$$

$$+ j(a_1\omega + a_{-1}\omega^{-1} + a_3\omega^3 + a_{-3}\omega^{-3} + \cdots).$$

The real part of the expression obtained contains *even* powers of $\omega$ and the imaginary part *odd* powers.

Thus, in a general form

$$\Delta = F_1(\omega^2) + j\omega F_2(\omega^2),$$

where $F_1(\omega^2)$ is an even and $\omega F_2(\omega^2)$ an odd function of $\omega$.

Analogously,

$$\Delta_{11} = F_3(\omega^2) + j\omega F_4(\omega^2).$$

It is easy to confirm that the ratio $\Delta/\Delta_{11}$ possesses the even character of the real component and the odd character of the imaginary component with respect to frequency $\omega$

$$Z(j\omega) = \frac{F_1(\omega^2) + j\omega F_2(\omega^2)}{F_3(\omega^2) + j\omega F_4(\omega^2)} = \frac{(F_1 + j\omega F_2)(F_3 - j\omega F_4)}{F_3^2 + \omega^2 F_4^2}$$

$$= \frac{F_1F_3 + \omega^2 F_2F_4}{F_3^2 + \omega^2 F_4^2} + j\omega\,\frac{F_2F_3 - F_1F_4}{F_3^2 + \omega^2 F_4^2} = r(\omega) + jx(\omega).$$

Like functions $F_1$, $F_2$, $F_3$ and $F_4$, their conjugate products $F_1F_3$, $F_2F_4$, $F_2F_3$ and $F_1F_4$ are even with respect to $\omega$. Consequently, *the resistance*

*$r(\omega)$, which is the real part of expression $Z(j\omega)$, is an even function of $\omega$, and the reactance $x(\omega)$, which is the imaginary part of $Z(j\omega)$, is an odd function.*

An analogous dependence of frequency also occurs for the conductance and the susceptance of the two-terminal network

$$Y(j\omega) = \frac{1}{r(\omega) + jx(\omega)} = \frac{r}{r^2 + x^2} - j\frac{x}{r^2 + x^2} = g(\omega) - jb(\omega).$$

Consequently, the conductance $g(\omega)$ is an even and the susceptance $b(\omega)$ an odd function of $\omega$.

It follows from the odd character of the reactive components of the impedance and admittance with respect to frequency that the reactance and susceptance of a two-terminal network cannot both be constant values (other than zero) independent of frequency $\omega$.

As $\omega$ approaches either zero or infinity, the resistance $r(\omega)$ will tend to a value proportional to $\omega^{2n}$ (which is constant when $n = 0$) and the reactance $x(\omega)$ will tend to a value proportional to $\omega^{2n+1}$, where $n$ is an integer—positive, negative or zero.

Consequently, $x(\omega)$, unlike the resistance, can tend only to zero or to infinity as the frequency approaches one of the limit values (0 or $\infty$).

The frequency derivative of the resistance possesses the same property: $dr/d\omega$ tends to zero or infinity as $\omega$ tends to one of these limits.

The active power consumed by a passive circuit cannot be negative

$$i^2 \operatorname{Re}[Z(j\omega)] \geqq 0.$$

Consequently, in proportion as $i^2 > 0$,

$$\operatorname{Re}[Z(j\omega)] \geqq 0.$$

Thus, *the resistance $r(\omega)$ of a passive circuit is positive or zero.*

From this

$$\operatorname{Re}[Y(j\omega)] = \operatorname{Re}\left[\frac{1}{Z(j\omega)}\right] = \operatorname{Re}\left[\frac{Z(-j\omega)}{z^2}\right] \geqq 0,$$

i.e. *the conductance $g(\omega)$ of a passive circuit is also positive or zero.*

EXAMPLE 4 (Ref. 76). To construct, in a complex plane, the relationships between the impedance and the admittance of the two-terminal network shown in Fig. 34*a* and frequency.

The impedance is

$$Z = \frac{(r + j\omega L)\frac{1}{j\omega C}}{r + j\omega L + \frac{1}{j\omega C}} = \frac{r + j\omega L}{1 - \omega^2 LC + jr\omega C}.$$

The admittance is

$$Y = \frac{1 - \omega^2 LC + jr\omega C}{r + j\omega L}.$$

When $\omega = 0$, $Z = r$, and $Y = 1/r$. As $\omega$ varies from 0 to $\infty$, $Z$ and $Y$ describe the curves shown in Fig. 34*b*.

As Fig. 34*b* shows, both curves $Z$ and $Y$ lie in the right-hand halfplane. They intersect the real axis when frequency $\omega = \sqrt{[1/LC - (r/L)^2]}$ corresponding to the occurrence of parallel resonance.

When $\omega \to \infty$, $Z \to 0$ and $Y \to j\infty$.

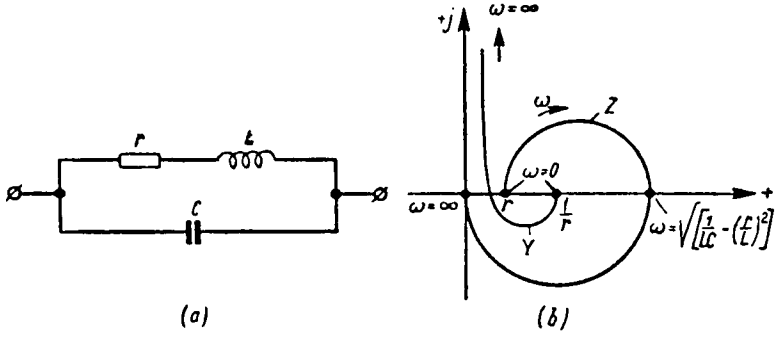



FIG. 34. Example 4.

## 13. Complex Frequency

We can examine the relationships between currents, power and other electrical quantities, and a voltage applied to the circuit in accordance with a definite law, in order to extend our information about the properties of the passive linear electrical circuit.

The voltage applied to the circuit is generally called the total-source or excitation function and the resultant reaction is known as the response of the circuit.

The concept of *complex frequency** $p = c + j\omega$ is introduced in order to examine the general properties of an electrical circuit.

Voltage in the form

$$Ue^{j\psi}e^{pt} = Ue^{ct}e^{j(\omega t + \psi)}$$

is a generalized form of the excitation function.

Depending on the values of $c$ and $\omega$, this function expresses some particular law of the change in voltage (instantaneous values of voltage are determined by the imaginary or real part of this function – Fig. 35).

The condition $\omega \neq 0$ corresponds to the harmonic oscillation with a changing or a constant amplitude: when value $c$ is positive the amplitude of the oscillations increases (Fig. 35*a*) and when it is negative it decreases

* Strictly speaking, the concept of complex frequency belongs more properly to the quantity $p/j = \omega - jc$ which is composed of the real (conventional) frequency and a certain imaginary frequency. In technical literature, however, $p = c + j\omega$ is referred to as the complex frequency, frequency $\omega$ being read off the imaginary axis.

(Fig. 35*b*). The case of $c = 0$ corresponds to an oscillation of constant amplitude (Fig. 35*c*).

The condition $\omega = 0$ corresponds to an exponential function or a fixed quantity: when $c > 0$ the function increases and when $c < 0$ it decreases, with $c = 0$ constant (Fig. 36).

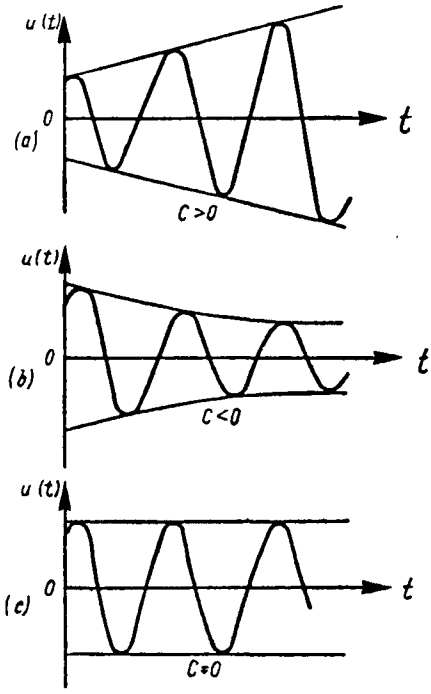



FIG. 35. Harmonic function

Thus, using a complex frequency makes it possible to conduct research with a generalized exponential form of excitation, applicable to different states of the circuit.

Examining, as an example, the response of a circuit with $r$, $L$ and $C$ connected in series (Fig. 37) to an applied voltage $Ve^{j\psi}e^{pt}$, we can write the integro-differential equation

$$rI(t) + L\frac{dI(t)}{dt} + \frac{1}{C}\int I(t)\,dt = Ue^{j\psi}e^{pt}.$$

Let us seek the solution of the forced current in the form $I(t) = Ie^{j\alpha}e^{pt}$. Substituting this expression in the initial integro-differential equation yields

$$\left(r + pL + \frac{1}{pC}\right)Ie^{j\alpha}e^{pt} = Ue^{j\psi}e^{pt}.$$

Denoting $Z(p) = r + pL + 1/pC$, we get

$$Ie^{j\alpha} = \frac{Ue^{j\psi}}{Z(p)},$$

where the generalized impedance $Z(p)$ is obtained from $Z(j\omega)$ by substituting $j\omega$ for $p$. This is equivalent to substituting the inductive reactances $j\omega L$ for $pL$ and the capacitive reactances $1/(j\omega C)$ for $1/(pC)$.

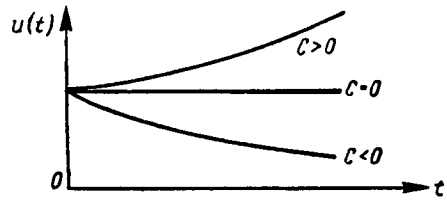



FIG. 36. Exponential function.

Generally, the impedance for a complex frequency is

$$Z(p) = \frac{\Delta(p)}{\Delta_{11}(p)},$$

where $\Delta(p)$ is a polynomial with positive and negative powers of $p$ in the form

$$\Delta(p) = a_0 + a_1 p + a_2 p^2 + \cdots + a_{-1}p^{-1} + a_{-2}p^{-2} + \cdots$$

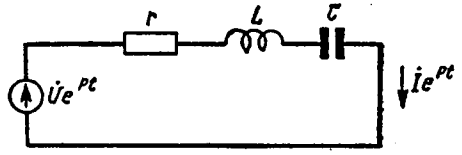



FIG. 37. Circuit with $r$, $L$ and $C$ connected in series.

Free oscillations of current ($U = 0$) are determined by the characteristic equation

$$Z(p) = 0 \quad \text{or} \quad \Delta(p) = 0.$$

The roots of this equation are $p_1, p_2, \ldots, p_n$.

The free current at the terminals of the network is

$$i_f = I_1 e^{j\alpha_1} e^{p_1 t} + I_2 e^{j\alpha_2} e^{p_2 t} + \cdots + I_n e^{j\alpha_n} e^{p_n t}.$$

The conjugate complex root $p_2 = \overset{*}{p}_1 = c_1 + j\omega_1$ corresponds to the complex root $p_1 = c_1 - j\omega_1$, and the conjugate-complex factor $I_1 e^{j\alpha_1}$ corresponds to the complex factor $I_2 e^{j\alpha_2} = I_1 e^{-j\alpha_1}$.

Therefore, the complex terms, in pairs, form the real value

$$I_1 e^{j\alpha_1} e^{p_1 t} + I_2 e^{j\alpha_2} e^{p_2 t} = I_1 e^{c_1 t}[e^{j(\alpha_1+\omega_1 t)} + e^{-j(\alpha_1+\omega_1 t)}]$$

$$= 2I_1 e^{c_1 t} \cos(\omega_1 t + \alpha_1).$$

In a passive circuit the free current does not increase indefinitely, therefore

$$c_k = \operatorname{Re} p_k \leqq 0,$$

i.e. the zeros of function $Z(p)$ have negative real parts.

In the foregoing considerations the use of the loop-current method has been borne in mind. Similarly, starting with the node method, it is possible to show that the zeros of function $Y(p)$ have negative real parts. In view of the fact that $Z(p) = 1/Y(p)$, the zeros of function $Y(p)$ are the poles of function $Z(p)$. Consequently, the zeros and poles of function $Z(p)$ are located on the left side of the complex plane $p$.

## 14. Immittance of a Two-terminal Network as a Positive-real Function

Function $Z(p)$ is called *positive* if its real part is positive for the positive real parts of $p$. The real part of function $Z(p)$ is not negative on the imaginary axis

$$\operatorname{Re}[Z(p)] \geqq 0 \quad \text{when} \quad \operatorname{Re} p \geqq 0.$$

If the positive function has real values for real arguments it is called a *positive-real function.* If, in addition, it is rational, it is called a ***Brune function.***

A positive-real function in the right half plane $p$ is analytic—property (2), expression (15). We shall show that *in the case of a passive two-terminal network the impedance is a positive-real function* (Ref. 65).

Under the effect of a voltage $u(t) = \operatorname{Im}[Ue^{j\psi}e^{pt}]$, applied to the two-terminal network when $t = -\infty$, the current equals $i(t) = \operatorname{Im}[Ie^{j\alpha}e^{pt}]$, and

$$\frac{Ue^{j\psi}e^{pt}}{Ie^{j\alpha}e^{pt}} = \frac{Ue^{j\psi}}{Ie^{j\alpha}} = Z(p).$$

The instantaneous power

$$ui = \operatorname{Im}[Ue^{j\psi}e^{pt}] \cdot \operatorname{Im}[Ie^{j\alpha}e^{pt}]$$

$$= UIe^{2ct} \sin(\omega t + \psi) \sin(\omega t + \alpha)$$

$$= \frac{e^{2ct}}{2} UI[-\cos(2\omega t + \psi + \alpha) + \cos(\psi - \alpha)]$$

$$= \frac{e^{2ct}}{2} \operatorname{Re}\left[-UIe^{j(2\omega t+\psi+\alpha)} + UIe^{j(\psi-\alpha)}\right]$$

$$= \frac{e^{2ct}}{2} I^2 \operatorname{Re}\left[-\frac{U}{I} e^{j(2\omega t+\psi+\alpha)} + \frac{U}{I} e^{j(\psi-\alpha)}\right]$$

$$= \frac{e^{2ct}}{2} I^2 \operatorname{Re}\left[-\frac{U}{I} e^{j(2\omega t+\psi+\alpha)} + Z(p)\right].$$

The total energy supplied to the network from the initial instant ($t = -\infty$) to the instant in question $t$, is positive:

$$\int_{-\infty}^{t} ui \, dt = I^2 \frac{e^{2ct}}{2} \operatorname{Re}\left[-\frac{U}{I} \frac{e^{j(2\omega t+\psi+\alpha)}}{2(c+j\omega)} + \frac{Z(p)}{2c}\right] > 0.$$

This condition also holds in the case where the first term in square brackets is imaginary. Consequently,

$$\operatorname{Re}[Z(p)] > 0 \quad \text{when} \quad c = \operatorname{Re} p > 0.$$

The real elements $r + pL + 1/(pC)$ of determinants $\Delta(p)$ and $\Delta_{11}(p)$ correspond to real values of $p$. The ratio of these determinants is therefore a real quantity.

Thus, $Z(p)$ is a positive-real function.

In view of the fact that in circuit theory impedance (or admittance*) is most frequently regarded as a positive function, henceforth a positive-real function will generally be denoted by the letter $Z$. Being rational it is given in the form of the relationship of two polynomials, i.e. it is a Brune function, as a rule.

When such a function is *physically realizable*, it means that it is possible to construct the circuit of a linear passive two-terminal network whose impedance (or admittance) for the complex frequency is expressed by the given function. As will be shown below (Section 15), a two-terminal network can be constructed for every positive-real function, the latter being its impedance.

As an example, Table 4 refers to expressions for the impedance of certain two-terminal networks in terms of complex frequency $p$ and shows the distribution of zeros and poles in a complex plane.

If function $Z(p)$ is given in the form of a rational fraction, the corresponding circuit can be constructed according to definite rules based on the general properties of positive-real functions.

* In connexion with the study of the general properties of functions expressing the impedance and admittance of electrical circuits, the term "immittance", comprehending the concepts of "impedance" and "admittance", is widely used in present-day literature.

TABLE 4. IMMITTANCE FUNCTIONS AND POLE-ZERO LOCATIONS

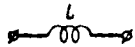
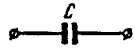
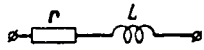
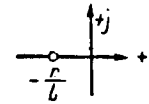
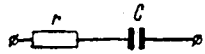
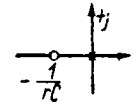
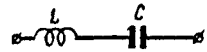
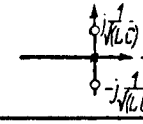
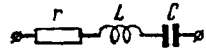
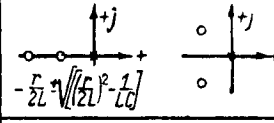
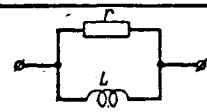
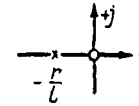
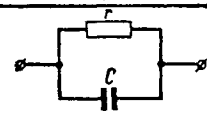
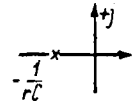
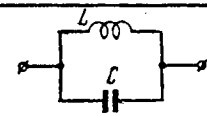
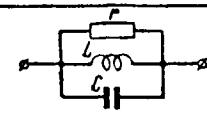
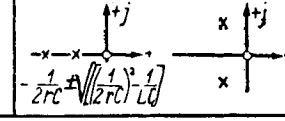
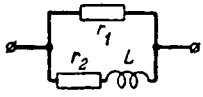
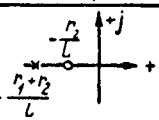
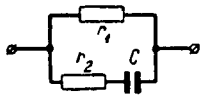
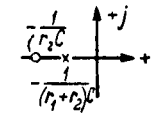
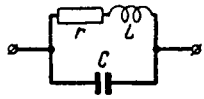
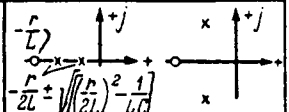
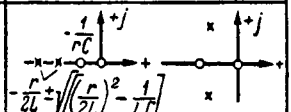
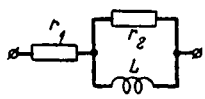
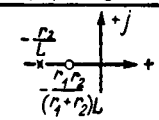
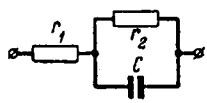
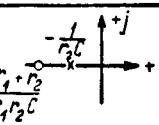
| No. | Circuit | $Z(p)$ | Zeros and poles $z(p)$ |
|---|---|---|---|
| 1 |  | $pL$ | |
| 2 |  | $\frac{1}{pC}$ | |
| 3 |  | $r+pL$ |  |
| 4 |  | $\frac{pr+\frac{1}{C}}{p}$ |  |
| 5 |  | $\frac{p^2L+\frac{1}{C}}{p}$ |  |
| 6 |  | $\frac{p^2L+pr+\frac{1}{C}}{p}$ |  |
| 7 |  | $\frac{pr}{p+\frac{r}{L}}$ |  |
| 8 |  | $\frac{1}{pC+\frac{1}{r}}$ |  |
| 9 |  | $\frac{p}{p^2C+\frac{1}{L}}$ | |
| 10 |  | $\frac{p}{p^2C+p\frac{1}{r}+\frac{1}{L}}$ |  |

TABLE 4. CONTD.

| No | Circuit | $Z(p)$ | Zeros and poles $z(p)$ |
|---|---|---|---|
| 11 |  | $\frac{pr_1L + r_1r_2}{pL + r_1 + r_2}$ |  |
| 12 |  | $\frac{pr_1r_2 + \frac{r_1}{C}}{p(r_1 + r_2) + \frac{1}{C}}$ |  |
| 13 |  | $\frac{pL + r}{p^2LC + prC + 1}$ |  |
| 14 | | $\frac{p^2LCr + pL}{p^2LC + prC + 1}$ |  |
| 15 |  | $\frac{p(r_1 + r_2) + \frac{r_1r_2}{L}}{p + \frac{r_2}{L}}$ |  |
| 16 |  | $\frac{pr_1C + \frac{r_1}{r_2} + 1}{pC + \frac{1}{r_2}}$ |  |

Section 15 is devoted to an examination of these properties and Section 16 is devoted to the general procedure for constructing two-terminal networks from a given positive-real function.

Table 4 can be used for the simplest expressions.

## 15. Properties of Positive-real Functions

According to the definition given above, function $Z(p)$ of the complex variable $p$ is a positive real function if it satisfies two requirements:

(1) values $Z(p)$ located in the right halfplane correspond to the complex numbers $p$ in the right halfplane, i.e. the real part of function $Z(p)$ is positive for the positive-real parts of $p$ (hence the name 'positive' function);

(2) function $Z(p)$ is real values of $p$ (hence the name 'real' function).

On the basis of these general conditions which are imposed on positive-real functions and refer only to the internal points of the right halfplane $p$, conclusions of importance to circuit theory can be drawn with respect to the general properties of positive-real functions.

1. *A quantity which is the reciprocal of a positive-real function is also a positive-real function.*

Let $Z(p) = r(c, \omega) + jx(c, \omega)$, where $p = c + j\omega$. Obviously the reciprocal is

$$Y(p) = \frac{1}{Z(p)} = \frac{1}{r + jx} = \frac{r - jx}{r^2 + x^2}.$$

In the right halfplane $r > 0$ and, consequently, $r/r^2 + x^2 > 0$; the real values $Y(p) = 1/Z(p)$ corresponding to the real values of $p$. Thus the condition formulated above is proved.

In view of the fact that the impedance of any passive two-terminal network is a positive-real function (Section 14) then, in accordance with the above, the admittance of any passive two-terminal network is also a positive-real function.

In addition, in so far as a two-terminal network can be constructed for any positive-real function, the latter being its impedance, the said property establishes the existence of an inverse (dual) network $k^2/Z(p)$.

2. *There cannot be poles and zeros of a positive-real function at the internal points of the right halfplane p.*

Let us suppose that function $Z(p)$ has a pole of order $m$ at point $p = p_0$.

A Laurent expansion around the pole is written in the form

$$Z(p) = \frac{c_{-m}}{(p - p_0)^m} + \frac{c_{-(m-1)}}{(p - p_0)^{m-1}} + \cdots$$

$$+ \frac{c_{-1}}{p - p_0} + \sum_{n=0}^{\infty} c_n (p - p_0)^n.$$

In the rather small vicinity of point $p_0$ the behaviour of function $Z(p)$ is determined by the first term of the expansion:

$$Z(p) \cong \frac{c_{-m}}{(p - p_0)^m}.$$

Let $c_{-m} = Ae^{j\Theta}$ and $p - p_0 = \varrho e^{j\psi}$, where $\varrho$ is a value as small as you like ($\varrho > 0$).

The expression

$$\operatorname{Re} Z(p) \cong \operatorname{Re} \frac{A}{\varrho^m} e^{j(\Theta - m\psi)} = \frac{A}{\varrho^m} \cos(m\psi - \Theta)$$

is obtained for the real part of function $Z(p)$.

Plotting round $p_0$ at a radius $\varrho$, angle $\psi$ changes from 0 to $2\pi$, and value $\Theta$ remains constant.

In this case the sign of the real part of the function is determined by the sign of $\cos(m\psi - \Theta)$ which changes $2m$ times round an $m$-fold pole.

If $Z(p)$ is a positive function, its real part on the right halfplane is positive and cannot change its sign. Consequently, at the internal points of the right halfplane no poles are possible.

By property 1, $Y(p) = 1/Z(p)$ is a positive-real function and, consequently, the condition that there are no poles on the right halfplane is extended to it. But a pole of function $Y(p)$ is a zero of function $Z(p)$. Therefore the positive-real function cannot have right-halfplane zeros on the right halfplane.

The positive-real function and its reciprocal are analytic in the right halfplane.

3. *If there are poles or zeros on the imaginary axis, then they are all simple, the residues at the poles and derivatives at the zeros being real and positive.*

A pole located on the imaginary axis should be simple ($m = 1$), angle $\Theta$ should equal zero and quantity $A$ should be positive, since only on this condition does $\operatorname{Re} Z(p) \cong A/\varrho \cos\psi$ retain the plus sign in the halfplane $p$ where $-\pi/2 < \psi < \pi/2$.

The residue of function $Z(p)$ at this pole should be a real positive number as $c_{-1} = A$ in this case.

On the strength of property 1, property 3 is extended to the function $Y(p) = 1/Z(p)$ and function $Y(p)$, particularly, can only have simple poles on the imaginary axis and the residues of the function $Y(p)$ at these poles should be real and positive. But a simple pole of function $Y(p)$ is a simple zero of function $Z(p)$, and the residue of $Y(p)$ in a simple pole is the reciprocal of derivative $\mathrm{d}Z(p)/\mathrm{d}p$ at a zero of function $Z(p)$. Consequently, zeros can only be simple on the imaginary axis, their derivatives being real and positive.

4. *The zeros and, correspondingly, poles of a rational positive-real function can be real or conjugate-complex.*

The given proposition follows from the fact that the rational function

$$Z(p) = \frac{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0}$$

is a positive-real function only when all the coefficients of the polynomials in the numerator and the denominator are real and have the same sign. For this condition substituting the conjugate quantity $\overset{*}{p}$ for the complex argument $p$ gives a conjugate function $Z(\overset{*}{p}) = \overset{*}{Z}(p)$.

Therefore, if $p_0$ is a complex zero of function $Z(p)$, i.e. if $Z(p_0) = 0$,

the value of $Z(p_0)$ must be the conjugate of zero which is again zero; in other words, $p_0$ is also a zero of function $Z(p)$.

The same condition also extends to the zeros of function $1/Z(p)$, i.e. to the poles of function $Z(p)$.

On the basis of the above, the complex zeros and, correspondingly, the poles of the functions of the impedances or admittances of two-terminal networks are arranged symmetrically with respect to the real axis, i.e. they form conjugate pairs.

5. *The degrees of the polynomials (i.e. the highest powers of p) in the numerator and denominator of a rational positive-real function can differ by at most unity.*

In view of the fact that an infinitely remote point may be considered as lying on the imaginary axis then, on the basis of property 3, the corresponding zero or pole at infinity should be simple. If $p = \infty$ is a simple zero or pole, the numerator is one degree more or less than the denominator (Appendix 1, Section 13). Consequently, $|n - m| \leqq 1$.

6. *The exponents of the lowest powers of p in the numerator and denominator of a rational positive-real function can differ by at most unity.*

For very small values of $|p|$ the terms with powers higher than unity can be disregarded in the expression

$$Z(p) = \frac{a_0 + a_1 p + \cdots}{b_0 + b_1 p + \cdots}$$

If $a_0 = 0$ and $b_0 \neq 0$, then $p$ is a factor of the numerator and function $Z(p) = (a_1/b_0)p + \cdots$ has a simple zero at the point $p = 0$, i.e. at the origin. On the basis of property 3 the derivative at this point, equal to $a_1/b_0$, is real and positive. Consequently, in the given case, $a_1 \neq 0$.

If $a_0 \neq 0$ and $b_0 = 0$, $p$ is a factor of the denominator and function $Z(p) = (a_0/b_1)p^{-1} + \cdots$ has a simple pole at the point $p = 0$. On the basis of property 3 the residue at this point, equal to $a_0/b_1$, is real and positive. Consequently, in the given case, $b_1 \neq 0$.

It follows from the above that the lowest indices of the powers of $p$ in the numerator and denominator cannot differ by more than unity.

7. *The real part of a positive-real function reaches its maximum and minimum on the imaginary axis.*

This property is based on the known principle of the maximum and minimum modulus of an analytic function, from which it follows that the real (and also the imaginary) part of a function which is analytic within a certain region and at its boundary, has a maximum and minimum at the boundary of the region. In the given case the right halfplane of $p$ is the region, and the imaginary axis is the boundary. If poles are located on the imaginary axis, they can be circumscribed by semicircles with a radius as

small as is required, and these considerations can be related to the remainder of the region.

By applying the said principle to the real part of function $Z(p)$ and taking into account that point $p = \infty$ can be on the imaginary axis, we can confirm that $r_{\min} < \operatorname{Re} Z(p) < r_{\max}$ here, $r_{\min}$ and $r_{\max}$ are the minimum and maximum values of $\operatorname{Re} Z(p)$ on the imaginary axis.

An important conclusion can be drawn from this: if a certain function $F(p)$ is analytic on the right halfplane $p$ and its real part on the imaginary axis is not negative, i.e. $\operatorname{Re} F(j\omega) \geqq 0$, then $F(p)$ is a positive function: $\operatorname{Re} F(p) > 0$ when $\operatorname{Re} p > 0$.

Similarly it can be confirmed that a function $Z(p)$ which is analytic on the right halfplane and satisfies the condition $\operatorname{Re} Z(j\omega) \geqq 0$ is positive and real.

On this basis it is also possible to conclude that if a positive-real function $Z(p)$ is given, with a certain quantity $r_1$ less than or equal to the minimum of the real part $Z(j\omega)$, then $Z_1(p) = Z(p) - r_1$ is also a positive-real function.

8. *If $Z(p)$ is a rational positive-real function with poles $\pm j\omega_k$, 0 and $\infty$, then*

$$Z(p) = \alpha_\infty p + \frac{\alpha_0}{p} + \sum_{k=1}^{q} \frac{2\alpha_k p}{p^2 + \omega_k^2} + Z_1(p), \tag{6.84a}$$

where $Z_1(p)$ is a rational positive-real function; $\alpha_k$ is the residue of $Z(p)$ at poles $\pm j\omega_k$; $\alpha_0$ is the residue at $p = 0$.

A pole results at point $p = \infty$ when the power of the numerator exceeds the power of the denominator by unity, i.e.

$$Z(p) = \frac{a_{m+1}p^{m+1} + a_m p^m + \cdots + a_1 p + a_0}{b_m p^m + \cdots + b_1 p + b_0}.$$

By dividing we get

$$Z(p) = \frac{a_{m+1}}{b_m} p + \frac{a'_m p^m + \cdots + a'_1 p + a_0}{b_m p^m + \cdots + b_1 p + b_0} = \alpha_\infty p + Z_0(p).$$

Here $\alpha_\infty = a_{m+1}/b_m$ is a positive-real quantity.

The real part of $\alpha_\infty p$ on the imaginary axis equals zero. Therefore

$$\operatorname{Re} Z_0(j\omega) = \operatorname{Re} Z(j\omega) \geqq 0.$$

Thus, function $Z_0(p)$ obtained from $Z(p)$ after separating $\alpha_\infty p$ is not negative on the imaginary axis; on the basis of property 7, $Z_0(p)$ is a positive-real function.

If $Z(p)$ and, correspondingly, $Z_0(p)$ have poles $\pm j\omega_k$ and 0 on the imaginary axis with residues $\alpha_k$ and $\alpha_0$, then expanding $Z_0(p)$ into elementary fractions gives*

$$Z_0(p) = \frac{\alpha_0}{p} + \sum_{k=1}^{q} \frac{\alpha_k}{p + j\omega_k} + \sum_{k=1}^{q} \frac{\alpha_k}{p - j\omega_k} + Z_1(p)$$

$$= \frac{\alpha_0}{p} + \sum_{k=1}^{q} \frac{2\alpha_k p}{p^2 + \omega_k^2} + Z_1(p).$$

On the imaginary axis, the expressions $2\alpha_k p/(p^2 + \omega_k^2)$ and $\alpha_0/p$ take imaginary values. Therefore $\operatorname{Re} Z_1(j\omega) = \operatorname{Re} Z_0(j\omega) = \operatorname{Re} Z(j\omega) \geqq 0$ and, consequently, $Z_1(p)$ is a positive-real function.

Function $Z_1(p)$ obtained from $Z_0(p)$ after separating all the poles lying on the imaginary axis can have poles in the left halfplane. If all the poles $Z_0(p)$ lie on the imaginary axis then, in accordance with Liouville's theorem (Appendix 1, Section 17), $Z_1(p)$ is a positive constant.

9. *If $Z(p)$ is a rational positive-real function with zeros $\pm j\omega_k$ and $\infty$, then*

$$\frac{1}{Z(p)} = \alpha_\infty p + \frac{\alpha_0}{p} + \sum_{k=1}^{s} \frac{2\alpha_k p}{p^2 + \omega_k^2} + \frac{1}{Z_2(p)}, \qquad (6.84\text{b})$$

where $Z_2(p)$ is a rational positive-real function, and $\alpha_k$ and $\alpha_0$ are the residues of $1/Z(p)$ at $\pm j\omega_k$ and 0.

The proof of this proposition is analogous to that of the previous proposition.

If all the zeros of function $Z(p)$ lie on the imaginary axis, $Z_2(p)$ is a positive constant.

### Tests for Positive-real Properties

A given rational function of $p$ is positive and real if all the following conditions obtain at the same time (Ref. 112):

(1) all the coefficients of the polynomials must be real (real function condition);

(2) there are no poles in the right halfplane (analytic function condition);

(3) the function behaves as $\alpha_\infty p$, $\alpha$ or $\alpha_0/p$ at infinity (property 5);

(4) the poles on the imaginary axis must be simple and the residues real and positive (property 3);

(5) the real part of the function on the imaginary axis must not be negative† for any value of $\omega$ (property 7).

* Residues $\alpha_k$ at conjugate imaginary poles are equal because they are real (property 3).

† It is simplest to calculate from formula (109).

The above conditions are necessary and sufficient for the given rational function to be positive and real. If only one of these conditions does not obtain the given function will not be positive and real.

EXAMPLE 5. To prove that function $F(p) = (p^2 + p + 2)/(p + 1)$ is positive and real.

(1) The coefficients of the polynomials are real;
(2) there are no right halfplane poles;
(3) the function behaves like $p$ at infinity;
(4) there are no imaginary poles;
(5) $\operatorname{Re} F(j\omega) = 2/(1 + \omega^2) \geqq 0$.

The proof shows that the given function is a positive-real function.

## 16. Construction of a Two-terminal Network from a given Positive-real Function

Finding the circuit of a two-terminal network which corresponds to a prescribed positive-real function belongs to the *synthesis* of electrical circuits. As distinct from the problem of analysis which leads to a unique result (the circuit's response to an applied excitation), a synthesis problem can have a number of solutions. It can be reduced to finding a certain combination of circuit elements based on the application of the known properties of the given function and the component elements of the circuit.

In problems of synthesis function $Z(p)$ can be given directly or obtained from the frequency characteristic of the circuit $Z(j\omega)$ by substituting $p = j\omega$.

Function $Z(p)$ is considered physically realizable if a corresponding two-terminal network can be composed of circuit elements with positive-real parameters: resistances, inductances and capacitances.

The condition that the function is positive and real is sufficient for the network to be physically realizable.

In order to construct a two-terminal network from a given positive-real function the method of the successive removal of zeros and poles or the method of expanding into a continued fraction are used (Refs. 10, 65, 78 and 81).

### Method of Successive Removal of Zeros and Poles

If all the zeros and poles of a given rational function $Z(p)$ lie on an imaginary axis, then they are all simple; on the basis of property 8

$$Z(p) = \alpha_\infty p + \frac{\alpha_0}{p} + \sum_{k=1}^{q} \frac{2\alpha_k p}{p^2 + \omega_k^2} + \alpha. \tag{6.85}$$

The positive constant $\alpha$ contained in expression (85) corresponds to resistance $r$. The remainder of (85), which is analytic when $p = j\omega$ (29),

can be realized with a canonic form for a reactive two-terminal network (Figs. 18 and 19).

There is a pole $p = \infty$ when there is an inductance $L_{2n} = \alpha_\infty$ in the Fig. 18 circuit, and there is the possibility of a pole $p = 0$ where there is a capacitance $C_0 = 1/\alpha_\infty$ in the same circuit, where $\alpha_0$ is the residue of $Z(p)$ at the pole $p = 0$.

Thus, function $Z(p)$ given in expression (85) is physically realizable with the circuit of a reactive two-terminal network connected in series with a

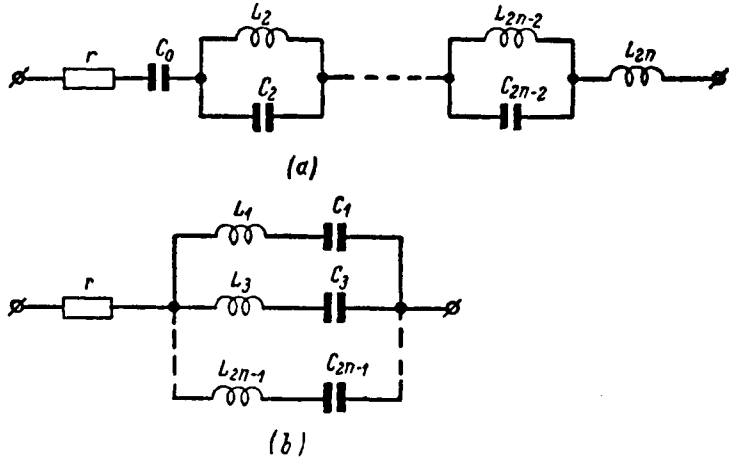



FIG. 38. Two-terminal network corresponding to formula (85).

resistance (Fig. 38). A reactive two-terminal network can also be constructed from the ladder network in Fig. 23.

In turn, from property 9 the admittance

$$Y(p) = \alpha_\infty p + \frac{\alpha_0}{p} + \sum_{k=1}^{s} \frac{2\alpha_k p}{p^2 + \omega_k^2} + \alpha \tag{6.86}$$

is physically practicable from the Fig. 39 circuits which represent the reciprocal two-terminal networks of those in Fig. 38.

If all the zeros and poles of the given rational function $Z(p)$ lie on the negative real semiaxis then, in accordance with (84a),

$$Z(p) = \alpha_\infty p + Z_1(p), \tag{6.87}$$

where $Z_1(p)$ is a positive-real function with zeros and poles on the open left halfplane. In view of the rational nature of $Z_1(p)$, the number of poles is finite. Function $Z_1(p)$ is analytic at infinity since the power of the numerator does not exceed the denominator in degree.

If the constant term in the numerator $Z_1(p)$ is zero, then $p = 0$ is a zero of function $Z_1(p)$. In this case $Z_1(p)$ is expanded into the sum of the fractions

$$Z_1(p) = \sum_{k=0}^{q} \frac{\alpha_k p}{p + \beta_k}, \tag{6.88}$$

where $\beta_k$ assumes real positive values.

When $\beta_0 = 0$, expansion (88) contains the constant term $\alpha_0$.

By analogy with (63), function (87) can be realized, taking (88) into account, from the circuits in Figs. 28, 29 or 30.

If the constant term in the numerator $Z_1(p)$ does not equal zero, $Z_1$ is expanded into partial fractions

$$Z_1(p) = \sum_{k=0}^{q} \frac{\alpha_k}{p + \beta_k} + \alpha, \tag{6.89}$$

where $\alpha$ is a positive constant.

The condition $\beta_0 = 0$ obtains when there is a pole at point $p = 0$, the constant term in the denominator $Z_1(p)$ being zero.

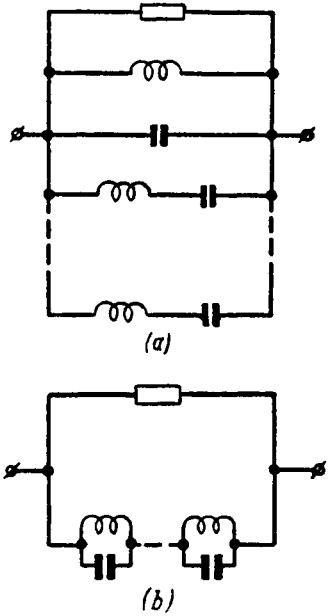



FIG. 39. Two-terminal network corresponding to formula (86).

By analogy with (76), function (89) can be realized from the circuits in Figs. 31, 32 or 33.

In general, the poles and zeros of a given positive-real function can lie on the imaginary axis and the left halfplane. This function is realized step by step by extracting the elements which are immediately realizable.

First, all the poles of the function of the impedance $Z(p)$ lying on the imaginary axis can be removed. Then, in accordance with (84a), a new, simpler function $Z_1(p)$ is obtained, which no longer has poles on the imaginary axis; this is called the minimum reactance function.

Thus, after the poles lying on the imaginary axis have been extracted, the problem is reduced to realizing the minimum reactance function $Z_1(p)$.

If the zeros and poles of $Z_1(p)$ lie on the negative real semiaxis, $Z_1(p)$ is realized with the aid of $rL$- and $rC$-type networks in accordance with (87) and (89). And if $Z_1(p)$ is a constant, it is realized as resistance $r$.

If $Z_1(p)$ has zeros on the imaginary axis, they are removed with the aid of (84b), as a result of which we obtain an admittance function $Y_2(p) = 1/Z_2(p)$, which has no poles on the imaginary axis. This is called the minimum susceptance function. Consequently, after the zeros $Z_1(p)$ lying on the imaginary axis have been removed, the problem is reduced to realizing the minimum susceptance function $Y_2(p)$.

The poles and zeros are removed until immediately realizable elements are obtained.

In the above considerations, the realizing of the given function began with property 8—by removing the poles lying on the imaginary axis. But the same function can be realized from property 9, starting with the removing of the zeros lying on the imaginary axis. In this case a circuit different from the previous one is obtained, and one or other of the circuits can be selected in accordance with actual conditions and requirements.

**Continued Fraction Method**

The series of examples given in Sections 8 and 11 illustrate the construction of a two-terminal network from a given rational function $Z(p)$ expanded into a continued fraction. The expansion into a continued fraction is performed by dividing the numerator by the denominator, the division continuing until the expression contained in the remainder represents one impedance or admittance of a certain known two-terminal network. This network is a finite arm in a ladder structure corresponding to the resulting expansion into a continued fraction.

It should be noted that continued fractions and their corresponding ladder networks differ depending on whether the polynomials in the numerator and denominator of the given rational function are arranged in ascending or descending powers of $p$. In addition, instead of function $Z(p)$, function $Y(p) = 1/Z(p)$ can be expanded into a continued fraction. Its numerator and denominator can also be arranged in ascending or descending powers of $p$. As a result, additional variants of the ladder networks can be obtained.

Finally, it is possible to expand into a continued fraction in which first

one, and then another, order of the powers of $p$ is observed in the dividing process.

In individual cases only certain, or even none, of the variants of the resulting expansion into a continued fraction, may permit the network to be realized.

There are also several other methods of constructing two-terminal networks with given frequency functions: the methods of Darlington, Bot–Daffin *et al.* (Refs. 10, 81 and 112).

EXAMPLE 6 (Ref. 7). It is required to realize the positive-real function

$$Z(p) = \frac{2p^2 + p + 1}{p^3 + p^2 + p + 1}.$$

The given function has poles on the imaginary axis at points $\pm j$.
The residue of function $Z(p)$ with respect to pole $j$ (or $-j$) is

$$\operatorname{Res} Z(p) = \left[\frac{2p^2 + p + 1}{3p^2 + 2p + 1}\right]_{p=j} = \frac{-2 + j + 1}{-3 + 2j + 1} = \frac{1}{2}.$$

Consequently, in accordance with (84a)

$$Z(p) = \frac{p}{p^2 + 1} + Z_1(p),$$

where $Z_1(p)$ is the minimum reactance function

$$Z_1(p) = \frac{2p^2 + p + 1}{p^3 + p^2 + p + 1} - \frac{p}{p^2 + 1} = \frac{1}{p + 1}.$$

The fraction $p/(p^2 + 1)$ is realized with the aid of an inductance $L = 1$ and a capacitance $C = 1$ connected in parallel, and fraction $1/(p+1)$ is realized with the aid of a resistance $r = 1$ and a capacitance $C = 1$ connected in parallel, the networks being connected in series.

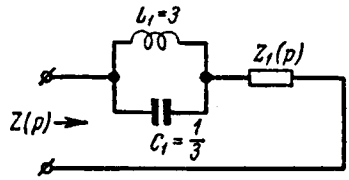



FIG. 40. Example 7.

EXAMPLE 7 (Ref. 108). It is required to realize the positive-real function

$$Z(p) = \frac{p^5 + 5p^4 + 9p^3 + 18p^2 + 8p + 4}{(p^2 + 1)(p^3 + 2p^2 + 4p + 2)}.$$

The given function has poles at points $\pm j$ on the imaginary axis.
The residue of function $Z(p)$ with respect to $j$ (or $-j$) is

$$\operatorname{Res} Z(p) = \lim_{p=j} (p - j)Z(p) = \frac{3}{2}.$$

Consequently, from (84a), $Z(p) = 3p/(p^2 + 1) + Z_1(p)$ i.e. $Z(p)$ is realized by the Fig. 40 circuit, where $Z_1(p)$ is the minimum reactance function:

$$Z_1(p) = Z(p) - \frac{3p}{p^2 + 1} = \frac{p^3 + 2p^2 + 2p + 4}{p^3 + 2p^2 + 4p + 2} = \frac{(p^2 + 2)(p + 2)}{p^3 + 2p^2 + 4p + 2}.$$

$Z_1(p)$ has zeros at points $\pm j\sqrt{2}$ on the imaginary axis. The residue of function $Y_1(p) = 1/Z_1(p)$ with respect to pole $j\sqrt{2}$ (or $-j\sqrt{2}$) is

$$\text{Res } Y_1(p) = \lim_{p \to j\sqrt{2}} (p - j\sqrt{2}) Y_1(p) = \frac{1}{2}.$$

Consequently, from (84b)

$$Y_1(p) = \frac{p}{p^2 + 2} + Y_2(p),$$

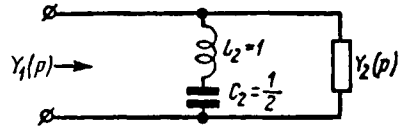



FIG. 41. Example 7.

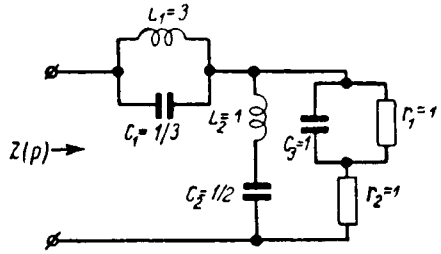



FIG. 42. Example 7.

i.e. $Y_1(p)$ is realized by the Fig. 41 circuit, where $Y_2(p)$ is the minimum susceptance function.

$$Y_2(p) = Y_1(p) - \frac{p}{p^2 + 2} = \frac{p + 1}{p + 2} = \frac{1}{1 + \dfrac{1}{p + 1}}.$$

The first term in the denominator of expression $Y_2(p)$ denotes the resistance and the second the capacitance and resistance connected in parallel; these components are connected in series.

Thus, function $Z(p)$ is realized in the form of the two-terminal network shown in Fig. 42.

EXAMPLE 8 (Ref. 107). It is required to realize the positive-real function

$$Z(p) = \frac{p^2 + p + 1}{p^3 + p^2 + 2p + 1}.$$

The given function is expanded into a continued fraction

$$Z(p) = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p+1}}}.$$

The ladder network shown in Fig. 43 corresponds to this expression.

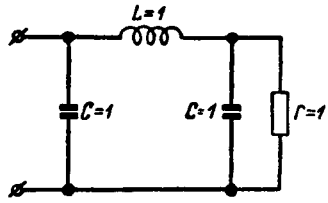



FIG. 43. Example 8.

EXAMPLE 9 (Ref. 107). It is required to realize the positive-real function

$$Z(p) = \frac{2p^3 + 3p^2 + 2p + 1}{2p^2 + 2p + 1}.$$

The numerator is higher in degree than the denominator and consequently the function has a pole at infinity, which indicates the presence of a series inductance.

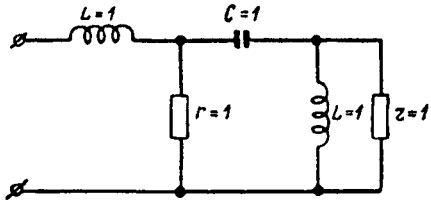



FIG. 44. Example 9.

Commencing with the arrangement of the polynomials in ascending powers of $p$ and varying the order of the powers of $p$ in the dividing process, we can represent the given function in this form:

$$Z(p) = p + \cfrac{1}{\cfrac{1 + 2p + 2p^2}{1 + p + p^2}} = p + \cfrac{1}{1 + \cfrac{1}{\cfrac{1 + p + p^2}{p + p^2}}} = p + \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{p} + \cfrac{1}{\cfrac{1}{p} + 1}}}.$$

The circuit of the two-terminal network given in Fig. 44 corresponds to the continued function thus obtained.

EXAMPLE 10 (Ref. 107). It is required to realize the positive-real function

$$Z(p) = \frac{p^4 + 2p^3 + 3p^2 + p + 1}{p^3 + p^2 + 2p}.$$

The given function has no zeros or poles on the imaginary axis, with the exception of $p = 0$ and $p = j\infty$. A capacitance, therefore, can act as the first series element. Dividing by decreasing powers of $p$, we can isolate the pole at infinity

$$Z(p) = p + \frac{p^3 + p^2 + p + 1}{p^3 + p^2 + 2p} = p + \frac{(p+1)(p^2+1)}{p(p^2+p+2)}.$$

The function obtained in the remainder has a pole at the origin and zeros on the imaginary axis at points $\pm j$. These coincide with the poles of function $p(p^2 + p + 2)/(p + 1)(p^2 + 1)$ which can be represented in the form of the sum of the fractions $p/(p^2 + 1) + p/(p + 1)$. The term $p/(p^2 + 1)$ corresponds to the admittance of the $L$ and $C$ elements connected in series. The term $p/(p + 1)$ does not contain complex

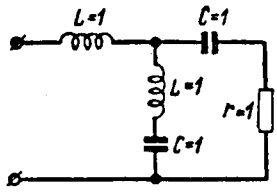



FIG. 45. Example 10.

zeros and poles at infinity, with the exception of a zero at the origin. The reciprocal function $(p + 1)/p$ corresponds to the impedance with a pole at the origin, which denotes the presence of a series capacitance. Thus

$$Z(p) = p + \cfrac{1}{\cfrac{p}{p^2+1} + \cfrac{1}{\cfrac{1}{p} + 1}},$$

to which the circuit shown in Fig. 45 corresponds.

## 17. Effect of Pole–Zero Locations on the Frequency Properties of Two-terminal Networks

The impedance (or admittance) of a two-terminal network which is a rational function $Z(p)$ is the ratio of two polynomials which do not simultaneously become zero for any value of $p$:

$$Z(p) = \frac{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0}. \tag{6.90}$$

Denoting the zeros of the numerator (their number equals the degree of the polynomial $n$) by $p_1, p_3, \ldots, p_{2n-1}$, and the zeros of the denominator

(their number equals the degree of the polynomial $m$), we can factorize the numerator and denominator

$$Z(p) = H \frac{(p - p_1)(p - p_3) \dots (p - p_{2n-1})}{(p - p_2)(p - p_4) \dots (p - p_{2m})}; \qquad (6.91)$$

here $H = a_n/b_m$ is a constant (scale factor);

$p_{2k-1}$ are the zeros of function $Z(p)$, i.e. the values of $p$ for which $Z(p)$ equals zero;

$p_{2k}$ are the poles of function $Z(p)$, i.e. the values of $p$ for which $Z(p)$ equals infinity.

The factors of the polynomials are related to the zeros and poles of the function by the known algebraic correlations:

the quantity $-a_{n-1}/a_n$ equals the sum of the zeros;
the quantity $-b_{m-1}/b_m$ equals the sum of the poles;
the quantity $a_{n-2}/a_n$ equals the sum of the paired products of the zeros;
the quantity $(-1)^n a_0/a_n$ equals the product of the zeros;
the quantity $(-1)^m b_0/b_m$ equals the product of the poles.

The coefficients of the polynomials in the numerator and denominator of (90) are positive and real since each is determined by some combinations of positive and real $r$, $L$ and $C$. The zeros of these polynomials are real or conjugate-complex, with negative real parts, i.e. all the zeros and poles of function $Z(p)$ lie in the left halfplane*. This proposition agrees with the conclusions of the previous sections.

As was shown above in Section 15, the multiple zeros and poles are confined to the left halfplane, excluding the imaginary axis; the zeros and poles on the imaginary axis can only be simple.

The degrees of the polynomials in the numerator and denominator of (90) do not differ by more than unity. Only then can the zero and pole at infinity be simple.

In the case of reactive two-terminal networks, the polynomials can contain only even or only odd powers of $p$: the impedance and admittance are then represented in the form of the ratio of the polynomials, one of which contains the even and the other the odd powers of $p$.

In general, when the network is dissipative the polynomials contain both even and odd powers of $p$.

* When all the zeros of a polynomial having positive and real coefficients lie within the left halfplane, the polynomial must contain all powers of $p$. If some of the zeros lie on the imaginary axis of the $p$ plane, some powers of $p$ may be missing. When all the zeros are restricted to the imaginary axis, the polynomial must have only even powers of $p$ or only odd powers of $p$.

The zeros of function $Z(p)$ are the poles of function $Y(p)$ and the poles of function $Z(p)$ are the zeros of $Y(p)$.

Expression (91) can be rewritten in exponential form. Denoting the modulus and argument (phase angle) of $p - p_{2k-1}$ by $M_{2k-1}$ and $\alpha_{2k-1}$, and the modulus and argument of $p - p_{2k}$ by $M_{2k}$ and $\alpha_{2k}$, we get

$$Z(p) = H \frac{M_1 e^{j\alpha_1} M_3 e^{j\alpha_3} \dots M_{2m-1} e^{j\alpha_{2n-1}}}{M_2 e^{j\alpha_2} M_4 e^{j\alpha_4} \dots M_{2m} e^{j\alpha_{2n}}}$$

or

$$Z(p) = H \frac{M_1 M_3 \dots M_{2m-1}}{M_2 M_4 \dots M_{2m}} e^{j[(\alpha_1+\alpha_3+\dots+\alpha_{2n-1})-(\alpha_2+\alpha_4+\dots+\alpha_{2m})]}. \quad (6.92)$$

Consequently, for the selected value of $p$, the modulus of function $Z(p)$ is determined by the ratio of the product of the moduli of the differences between $p$ and the zeros to the product of the moduli of the differences between $p$ and the poles. The argument of function $Z(p)$ is determined by the difference between two sums: the sum of the arguments of the differences between $p$ and the zeros and the sum of the arguments of the differences between $p$ and the poles.

When studying the frequency characteristics of a two-terminal network in steady-state conditions, the present variable $p$ is taken on the imaginary axis ($p = j\omega$).

*By knowing the pole–zero location on a complex plane, it is possible to determine the frequency characteristics of a two-terminal network: the mode*

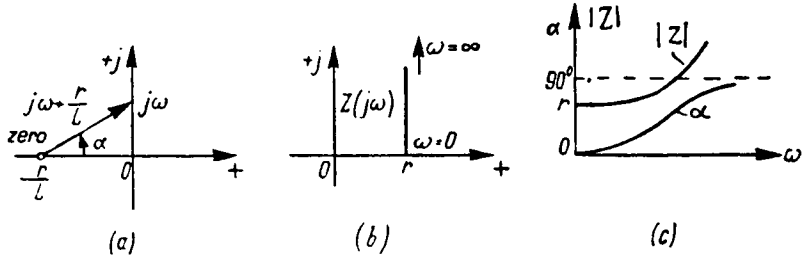



FIG. 46. (*a*) Zero of function $Z(p)$; (*b*) locus of $Z(j\omega)$; (*c*) plots against the frequency of the modulus and argument of $Z(j\omega)$ of an *rL* two-terminal network.

*of the change in the modulus and the argument of the impedance or admittance of the network, depending on frequency, the value of the resonance frequency, etc.* (Refs. 107 and 114).

Let us suppose that the network consists of $r$ and $L$ connected in series. In this case

$$Z(p) = L\left(p + \frac{r}{L}\right).$$

Comparing the given expression with (91) shows that $H = L$. The function $Z(p)$ has zero at point $-r/L$. At real frequencies the difference $p - (-r/L) = j\omega + r/L$ is represented by a phasor which originates at the zero of function $Z(p)$ and terminates at the variable point of the $j\omega$-axis (Fig. 46*a*). As $\omega$ varies from 0 to $\infty$ the modulus of this phasor changes from $r/L$ to $\infty$, and the argument from 0 to 90°. Figure 46*b* and *c* shows

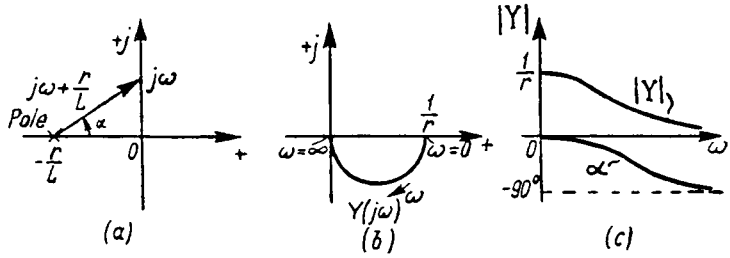



FIG. 47. (*a*) Pole of function $Y(p)$; (*b*) locus of $Y(j\omega)$; (*c*) plots against the frequency of the modulus and argument of $Y(j\omega)$ of an *rL* two-terminal network.

the locus of the phasor $Z(j\omega)$ and the plots of the modulus and argument of $Z(j\omega)$ against frequency.

From the case in question function $Y(p) = [1/(p + r/L)]/L$ has a pole at point $-r/L$ (Fig. 47*a*). As $\omega$ varies from 0 to $\infty$ the modulus of the phasor $1/(j\omega + r/L)$ changes from $L/r$ to 0 and the argument from 0 to $-90°$.

Figures 47*b* and *c* show the locus of the phasor $Y(j\omega)$ and the plots of the modulus and the argument of $Y(j\omega)$.

Let us now examine the case when the two-terminal network consists of *r*, *L* and *C* connected in series. Function

$$Z(p) = L\,\frac{p^2 + p\,\dfrac{r}{L} + \dfrac{1}{LC}}{p}$$

has zeros at points $p_{1,3} = r/2L \pm \sqrt{[(r/2L)^2 - 1/LC]}$ and a pole at point $p_2 = 0$.

Depending on the sign of the radicand, the zeros of the function are real or conjugate-complex. As an example, Fig. 48*a* shows the case of conjugate-complex zeros. For a certain intermediate value $\omega$ which is shown in Fig. 48*a*, angles $\alpha_1 < 0$, $\alpha_3 > 0$ and $\alpha_2 = 90°$.

In accordance with (92), when $p = j\omega$

$$Z(j\omega) = L\,\frac{M_1 M_3}{M_2}\,e^{j(\alpha_1 + \alpha_3 - 90°)}.$$

As the frequency varies from 0 to $\omega_0$, $M_1M_3/M_2$ decreases from $\infty$ to $r/L$ and as $\omega$ changes from $\omega_0$ to $\infty$, it increases from $r/L$ to $\infty$. Then angle $\alpha_1 + \alpha_3 - 90°$ changes from $-90°$ to $90°$ passing through zero, which corresponds to the resonance frequency $\omega_0 = 1/\sqrt{(LC)}$.

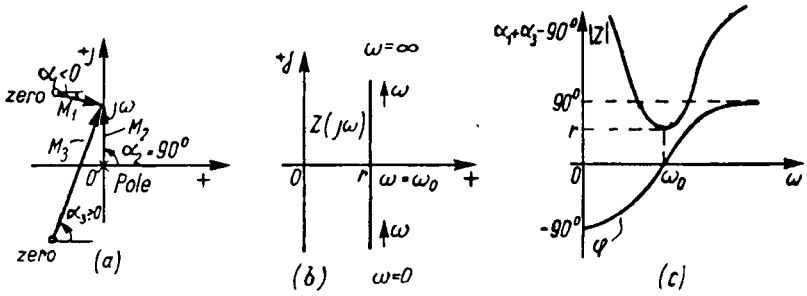



FIG. 48. (*a*) Zeros and pole of function $Z(p)$; (*b*) locus of $Z(j\omega)$; (*c*) plots against the frequency of the modulus and argument of $Z(j\omega)$ of an *rLC* two-terminal network.

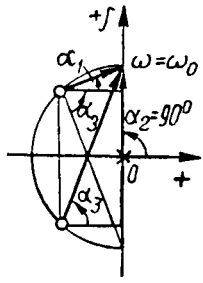



FIG. 49. Finding the resonance frequency $\omega_0$, depending on the distribution of zeros on a complex plane.

Figure 48*b* shows the locus $Z(j\omega)$ and Fig. 48*c* shows the modulus and argument of $Z(j\omega)$ in a rectangular system of coordinates, as functions of frequency.

In order to find the resonance frequency $\omega_0$ for a given arrangement of conjugate-complex zeros of function $Z(p)$, it is sufficient to draw a circle, with its centre at the origin, through the points corresponding to zeros (Fig. 49). The given circle intersects the imaginary axis at points $\pm\omega_0$. This is because angle $\alpha_1 + \alpha_3$, on the diameter, equals 90° and, consequently, $\alpha_1 + \alpha_3 - 90° = 0$.

Function $Y(p)$ has poles at points $p_{1,3} = -r/2L \pm \sqrt{[(r/2L)^2 - 1/LC]}$ and a zero at point $p_2 = 0$ (Fig. 50*a*). Then

$$Y(j\omega) = \frac{1}{L}\,\frac{M_2}{M_1M_3}\,e^{j[90°-(\alpha_1+\alpha_3)]}.$$

As the frequency changes from 0 to $\omega_0$, $M_2/M_1M_3$ increases from 0 to $L/r$, and then as $\omega$ increases to $\infty$, decreases to zero.

Angle $90° - \alpha_1 + \alpha_3$ varies from 90° to $-90°$ passing through a zero value corresponding to the resonance frequency $\omega_0 = 1/\sqrt{(LC)}$ (Fig. 50*b* and *c*). In order to find the resonance frequency for a given arrangement of conjugate-complex poles, a graphic procedure, analogous to that given in Fig. 49, can be adopted.

Examples of the disposition of zeros and poles in a complex plane for various functions are given in Table 4.

If, in two-terminal networks of different design, the zeros and, correspondingly, the poles of function $Z(p)$ coincide and factors $H$ are equal, then these networks are equivalent: their frequency characteristics are identical (Section 9).

On the other hand, if the zeros and poles of function $Z(p)$ of one network coincide, respectively, with the zeros and poles of function $Y(p)$ of another, then these are dual networks (Section 10).

It is possible to determine the presence or absence of energy losses in a two-terminal network from the disposition of zeros and poles.

When a two-terminal network consists of reactive elements only, i.e. there are no energy losses, the argument of the impedance or admittance is

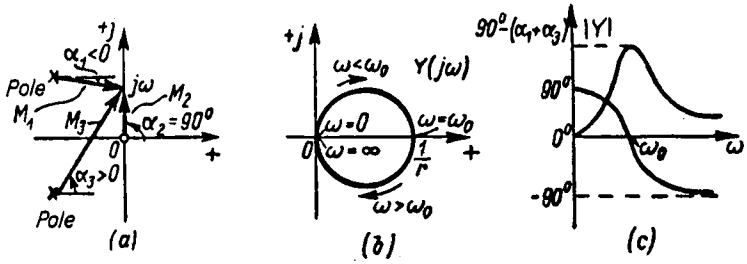



FIG. 50. (*a*) Zero and poles of function $Y(p)$; (*b*) locus of $Y(j\omega)$; (*c*) plots against the frequency of the modulus and argument of $Y(j\omega)$ of an *rLC* two-terminal network.

$\pm 90°$. In accordance with (92), a right angle is only possible on the condition that all the zeros and poles lie on the imaginary axis. As we said above, these zeros and poles must be simple (Section 15).

The over-all number of zeros and poles is odd (the difference between the number of zeros and poles is one); consequently, at the origin (at point $p = 0$) there must be a zero or a pole.

The zeros and poles must alternate. If a zero were followed by another zero, or a pole by a pole, then in accordance with (92) at certain frequencies the angle would equal $-180°$, which violates the condition that the phase angle between the voltage and current across the terminals of a passive two-terminal network should not exceed an absolute value of 90°.

Thus, the arrangement of poles shown in Fig. 51 is typical for reactive two-terminal networks. The cases given here are analogous to the four variants shown in Fig. 16.

When a two-terminal network consists of $r$ and $L$ or $r$ and $C$ elements, the zeros and poles of functions $Z(p)$ and $Y(p)$ are located along the real axis (in the left halfplane). The argument of the impedance and admittance

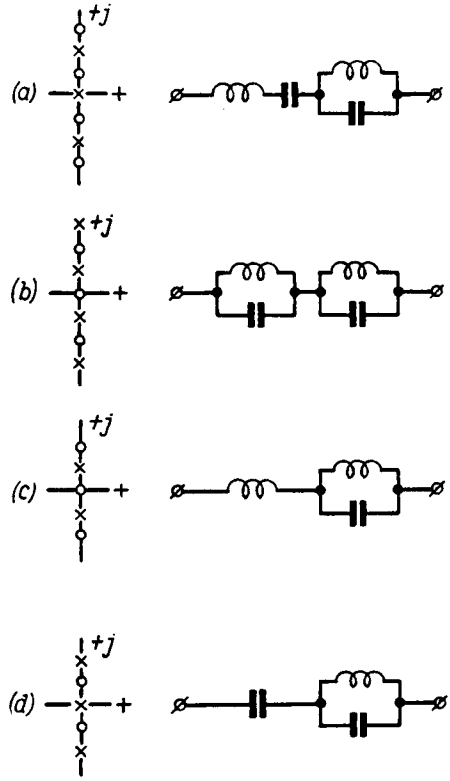



FIG. 51. Pole-zero distribution for non-dissipative two-terminal network.

of such a network does not attain $\pm 90°$; the zeros and poles, therefore, should not only be simple, but should alternate. In view of the fact that the imaginary axis passes through the origin, the zero or pole at the origin should be simple. The difference between the number of zeros and the number of poles cannot be more than unity.

At zero frequency, the impedance if an $rL$-type two-terminal network is zero or has a real value, depending on its design. Therefore the zero $Z(p)$ of such a network either coincides with the origin (Fig. 52$a$) or is displaced to the left along the real axis (Fig. 52$a$—a zero, not a pole, will

be closest to the origin). At an infinite frequency the impedance of an *rL*-type two-terminal network is infinitely great or real.

At zero frequency, the impedance of an *rC*-type two-terminal network is infinite (Fig. 52*d*) or has a real value (Fig. 52*c*). In the latter case a pole will be closest to the origin. At an infinitely high frequency the impedance turns to zero or becomes real.

By reversing the positions of the zeros and poles of $Z(p)$ we get a picture of the pole–zero distribution of $Y(p)$.

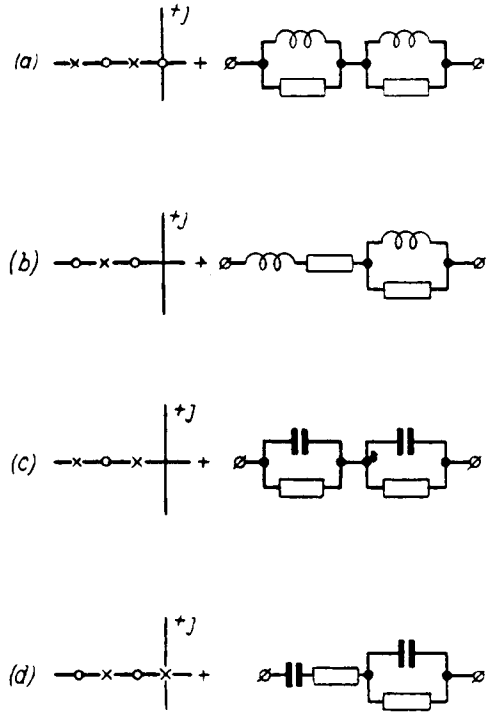



FIG. 52. Pole–zero distribution for *rL* (*a* and *b*) and *rC* (*c* and *d*) two-terminal networks.

It should be noted that when there are losses and the zeros and poles are real or complex values, the impedance and admittance cannot assume zero or infinitely great values at frequencies lying on the imaginary axis, except for frequencies $\omega = 0$ and $\omega = \infty$. But at the frequency corresponding to the projection of pole $p_k$ on to the imaginary axis (i.e. the imaginary part of the pole), when the modulus of the difference $j\omega - p_k$ falls to the minimum, the absolute value of the function being examined passes through its maximum, and at a frequency corresponding to the projection of zero on to the imaginary axis, it passes through the maximum.

The symmetrical disposition of zeros and poles with respect to the real axis, which is essential for all linear two-terminal networks, confirms, on the basis of (92), that the moduli of functions $Z(j\omega)$ and $Y(j\omega)$ are even with respect to $\omega$, and that arguments are odd with respect to $\omega$ (Section 12).

## 18. Relationship Between Resistance and Reactance (Conductance and Susceptance)

The real and imaginary parts of the impedance and admittance of a physically realizable passive two-terminal network are *mutually dependent* frequency functions (Refs. 7, 28, 76 and 112). We are now considering those networks for which $Z(j\omega)$ or $Y(j\omega)$ do not have poles on the imaginary axis, i.e. on the axis of frequency $\omega$.

The impedances of these two-terminal networks are *minimum reactance* functions and the admittances *minimum susceptance* functions (Section 16).

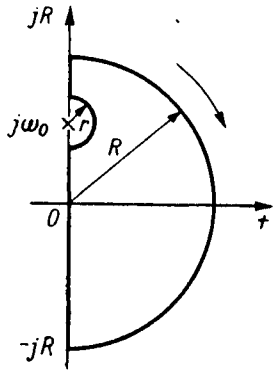



FIG. 53. Path of integration.

In order to establish the analytic relationship which exists between the frequency characteristics of the resistance and the reactance of a two-terminal network, let us examine the contour integral

$$\oint \frac{Z(p)}{p - j\omega_0}\,dp,$$

taken on the plane of the complex variable $p = c + j\omega$ along the closed curve indicated in Fig. 53. Here $Z(p)$ is a positive-real function with no singularities on the imaginary axis or the right halfplane, and $\omega_0$ is an arbitrary real frequency.

Point $p = j\omega_0$ is the only pole of the integrand. We describe a semicircle of radius $r$ around this pole, in such a way that the pole is outside.

Therefore, according to Cauchy's theorem, the integral for a closed loop is zero

$$\oint \frac{Z(p)}{p - j\omega_0}\,dp = 0$$

or

$$\lim_{R\to\infty}\left[\int_{-jR}^{j(\omega_0-r)} + \int_{j(\omega_0+r)}^{jR} + \int_{(r)} + \int_{(R)}\right] = 0. \tag{6.93}$$

Let us suppose that when $p \to \infty$, $Z(p)/p - j\omega_0 \to 0$. Consequently, the last integral in (93) tends to zero when $p \to 0$. Thus there is no contribution from the infinite arc.

The penultimate integral can be evaluated for $r \to 0$, taking into account that

$$\left.\begin{aligned} &p - j\omega_0 = re^{j\varphi} \quad \text{and} \quad dp = jre^{j\varphi}\,d\varphi: \\ &\lim_{r\to 0}\int_{(r)} \frac{Z(p)}{p - j\omega_0}\,dp = Z(j\omega_0)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{jre^{j\varphi}}{re^{j\varphi}}\,d\varphi = j\pi Z(j\omega_0). \end{aligned}\right\} \tag{6.94}$$

When $R \to \infty$ the sum of the first two integrals is the principal value of the integral with infinite limits, excluding point $\omega_0$.

From (93) and (94)

$$\int_{-j\infty}^{+j\infty} \frac{Z(p)}{p - j\omega_0}\,dp = -j\pi Z(j\omega_0).$$

Regarding $Z(j\omega)$ as the limit to which $Z(p)$ tends from $p \to j\omega$, we get

$$\int_{-\infty}^{\infty} \frac{Z(j\omega)}{\omega - \omega_0}\,d\omega = -j\pi Z(j\omega_0),$$

from which

$$\int_0^{\infty} \frac{Z(j\omega)}{\omega - \omega_0}\,d\omega - \int_0^{-\infty} \frac{Z(j\omega)}{\omega - \omega_0}\,d\omega$$

$$= \int_0^{\infty} \frac{Z(j\omega)}{\omega - \omega_0}\,d\omega - \int_0^{\infty} \frac{Z(-j\omega)}{\omega + \omega_0}\,d\omega$$

$$= \int_0^{\infty} \frac{\omega[Z(j\omega) - Z(-j\omega)] + \omega_0[Z(j\omega) + Z(-j\omega)]}{\omega^2 - \omega_0^2}\,d\omega = -j\pi Z(j\omega_0).$$

Let

$$Z(j\omega) = r(\omega) + jx(\omega).$$

In view of the even resistance and the odd reactance

$$Z(j\omega) + Z(-j\omega) = 2r(\omega),$$

$$Z(j\omega) - Z(-j\omega) = j2x(\omega).$$

Consequently,

$$\int_0^\infty \frac{2\omega_0 r(\omega) + j\,2\omega x(\omega)}{\omega^2 - \omega_0^2}\,d\omega = -j\pi[r(\omega_0) + jx(\omega_0)].$$

Separating the real and imaginary parts, we find

$$\left.\begin{aligned} r(\omega_0) &= -\frac{2}{\pi}\int_0^\infty \frac{\omega x(\omega)}{\omega^2 - \omega_0^2}\,d\omega, \\ x(\omega_0) &= \frac{2\omega_0}{\pi}\int_0^\infty \frac{r(\omega)}{\omega^2 - \omega^2}\,d\omega. \end{aligned}\right\} \tag{6.95}$$

he resulting formulae express *the connexion between the frequency characteristics of the resistance and reactance of a passive two-terminal network* satisfying the conditions given above. It should then be remembered that the integrals of (95) are taken as being the principal value with the exception of point $\omega = \omega_0$. The resistance and reactance are calculated for any frequency $\omega_0$, in accordance with (95).

Taking into account that $\ln Z = \ln ze^{j\varphi} = \ln z + j\varphi$, relationships in the form of (95) can be applied to the frequency characteristics of the natural logarithm of the modulus and, correspondingly, the angle of the network's impedance.

Together with (95) there exist other analytic relationships connecting the real and imaginary components of the network's impedance.

Among these relationships are the resistance integral and the reactance integral.

Let $Z(p)$ be a function which is analytic in the right halfplane, including points $p = 0$ and $p = \infty$. Let us expand $Z(p)$ into a Taylor series in the vicinity of $p = 0$

$$Z(p) = A_0 + B_0 p + A_1 p^2 + B_1 p^3 + \cdots \tag{6.96}$$

and in the vicinity of $p = \infty$

$$Z(p) = A_\infty - \frac{B'_\infty}{p} + \frac{A'_1}{p^2} - \frac{B'_1}{p^3} + \cdots \tag{6.97}$$

Along the imaginary axis, i.e. when $p = j\omega$, we get:

$$Z(j\omega) = r(\omega) + jx(\omega), \tag{6.98}$$

where $r(\omega)$ and $x(\omega)$ are the resistance and reactance of the network.

It is evident from comparing (97) with (98) that

$$B'_\infty = \lim_{\omega \to \infty} [\omega x(\omega)]. \tag{6.99}$$

Let us integrate function $Z(p) - A_\infty$ along the closed contour composed of the imaginary axis and a semicircle of infinite radius in the right half-plane (Fig. 54).

In view of the fact that there are no singularities in the enclosed region, the integral equals zero

$$\oint [Z(p) - A_\infty]\, dp = 0,$$

from which, taking (97) and (98) into account

$$\int_{-j\infty}^{+j\infty} [r(\omega) + jx(\omega) - A_\infty]\, d(j\omega) + \int_L \left[ -\frac{B'_\infty}{p} + \frac{A'_1}{p^2} - \frac{B'_1}{p^3} + \cdots \right] dp = 0. \tag{6.100}$$

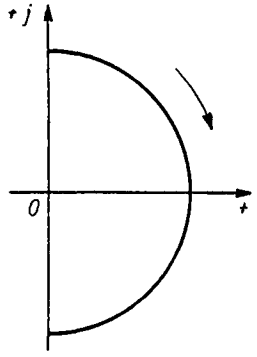



FIG. 54. Path of integration.

Function $x(\omega)$ is odd with respect to $\omega$. In addition, along the semicircular arc $p = Re^{j\varphi}$ and $dp = jRe^{j\varphi}\, d\varphi$, where $R \to \infty$. Consequently,

$$\int_L \left( -\frac{B'_\infty}{p} \right) dp = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{-B'_\infty jRe^{j\varphi} d\varphi}{Re^{j\varphi}} = j\pi B'_\infty$$

and

$$\int_L \frac{dp}{p^n} = 0 \quad \text{when} \quad n > 1.$$

From (100)

$$\int_{-\infty}^{+\infty} [r(\omega) - A_\infty]\, d\omega + \pi B'_\infty = 0. \tag{6.101}$$

Function $r(\omega)$ is even with respect to $\omega$; therefore

$$\int_{-\infty}^{+\infty} [r(\omega) - A_\infty]\,d\omega = 2\int_0^{\infty} [r(\omega) - A_\infty]\,d\omega. \tag{6.102}$$

Thus, from (99), (101) and (102)

$$\int_{\infty}^{\infty} [r(\omega) - A_\infty]\,d\omega = -\frac{\pi}{2} B'_\infty. \tag{6.103}$$

This integral, which is called *the resistance integral* establishes the connexion between $r(\omega)$ and the limit value $B'_\infty = \lim_{\omega\to\infty} [\omega x(\omega)]$ and shows that the area enclosed by curve $r(\omega)$ depends only on the magnitude of $B'_\infty$ and not on the intermediate values of the network's reactance.

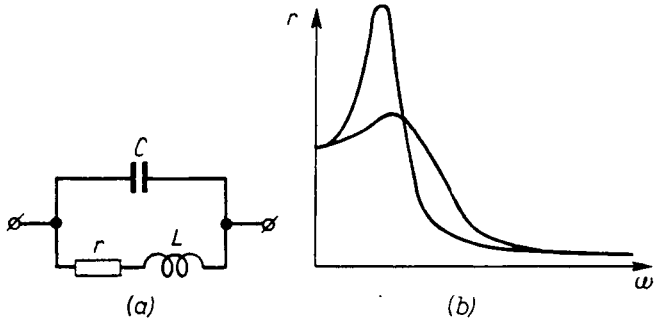



FIG. 55. Example 11: (*a*) circuit diagram of two-terminal network; (*b*) function $r(\omega)$.

EXAMPLE 11 (Ref. 7). To prove the truth of formula (103), for the two-terminal network given in Fig. 55*a*.

The impedance of the given two-terminal network

$$Z = \frac{(r + j\omega L)\dfrac{1}{j\omega C}}{r + j\omega L + \dfrac{1}{j\omega C}} = \frac{r + j\omega L}{1 - \omega^2 LC + jr\omega C}$$

or

$$Z = \frac{r}{(1 - \omega^2 LC)^2 + (r\omega C)^2} + j\,\frac{\omega L(1 - \omega^2 LC) - r^2\omega C}{(1 - \omega^2 LC)^2 + (r\omega C)^2}.$$

Consequently

$$r(\omega) = \frac{r}{(1 - \omega^2 LC)^2 + (r\omega C)^2},$$

$$x(\omega) = \frac{\omega L(1 - \omega^2 LC) - r^2\omega C}{(1 - \omega^2 LC)^2 + (r\omega C)^2}.$$

From this

$$A_\infty = 0, \quad B'_\infty = \lim_{\omega\to\infty} \frac{\omega^2 L(1-\omega^2 LC) - r^2\omega^2 C}{(1-\omega^2 LC)^2 + (r\omega C)^2} = -\frac{1}{C}.$$

On the basis of (103)

$$\int_0^\infty \frac{r}{(1-\omega^2 LC)^2 + (r\omega C)^2}\,\mathrm{d}\omega = \frac{\pi}{2C}.$$

This can be confirmed by integration. Expanding the integrand into partial fractions, we get:

$$\frac{r}{(1-\omega^2 LC)^2 + (r\omega C)^2} = \frac{a}{a-b}\cdot\frac{r}{1+a\omega^2} + \frac{b}{b-a}\cdot\frac{r}{1+b\omega^2},$$

where

$$a = \tfrac{1}{2}\left[r^2C^2 - 2LC + rC\sqrt{(r^2C^2 - 4LC)}\right],$$

$$b = \tfrac{1}{2}\left[r^2C^2 - 2LC - rC\sqrt{(r^2C^2 - 4LC)}\right].$$

Integrating the above function term-by-term, we find

$$\frac{a}{a-b}\int_0^\infty \frac{r\,\mathrm{d}\omega}{1+a\omega} + \frac{b}{b-a}\int_0^\infty \frac{r\,\mathrm{d}\omega}{1+b\omega^2}$$

$$= \frac{r\sqrt{a}}{a-b}\int_0^\infty \frac{\mathrm{d}(\sqrt{a}\,.\,\omega)}{1+(\sqrt{a}\,.\,\omega)^2} + \frac{r\sqrt{b}}{b-a}\int_0^\infty \frac{\mathrm{d}(\sqrt{b}\,.\,\omega)}{1+(\sqrt{b}\,.\,\omega)^2}$$

$$= \frac{r\sqrt{a}}{a-b}\left|\arctan\sqrt{a}\,.\,\omega\right|_0^\infty + \frac{r\sqrt{b}}{b-a}\left|\arctan\sqrt{b}\,.\,\omega\right|_0^\infty$$

$$= \frac{r\sqrt{a}}{a-b}\cdot\frac{\pi}{2} + \frac{r\sqrt{b}}{b-a}\cdot\frac{\pi}{2} = \frac{r\pi}{2}\cdot\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\pi}{2C},$$

since $(\sqrt{a}+\sqrt{b})^2 = a + 2\sqrt{ab} + b = r^2C^2$.

Figure 55*b* shows the typical characteristics $r(\omega)$ of a pair of two-terminal networks made according to the Fig. 55*a* diagram, with identical capacitances $C$. The areas enclosed by the characteristics $r(\omega)$ are identical.

The *reactance integral* is deduced from the equation

$$\oint \frac{Z(p)}{p}\,\mathrm{d}p = 0 \tag{6.104}$$

the path of integration being a contour by-passing a pole at the origin (Fig. 56).

From (104)

$$\int_0^{-j\infty} \frac{r(\omega) + jx(\omega)}{j\omega}\,\mathrm{d}(j\omega) + \int_{L_1} \frac{Z(p)}{p}\,\mathrm{d}p$$

$$+ \int_{+j\infty}^{0} \frac{r(\omega) + jx(\omega)}{j\omega}\,\mathrm{d}(j\omega) + \int_{L_2} \frac{Z(p)}{p}\,\mathrm{d}p = 0. \tag{6.105}$$

On the strength of $r(\omega)$ being even and $x(\omega)$ being odd with respect to $\omega$

$$\int_{-\infty}^{0} \frac{r(\omega)\,d\omega}{\omega} = -\int_{0}^{\infty} \frac{r(\omega)}{\omega}\,d\omega,$$

$$\int_{-\infty}^{0} \frac{x(\omega)\,d\omega}{\omega} = \int_{0}^{\infty} \frac{x(\omega)}{\omega}\,d\omega.$$

Further,

$$\int_{L_1} \frac{Z(p)}{p}\,dp = \int_{L_1} \frac{A_0 + B_0 p + A_1 p^2 + \cdots}{p}\,dp = jA_0\pi,$$

$$\int_{L_2} \frac{Z(p)}{p}\,dp = \int_{L_2} \frac{A_\infty - \frac{B'_\infty}{p} + \frac{A'_1}{p^2} - \cdots}{p}\,dp = -jA_\infty\pi.$$

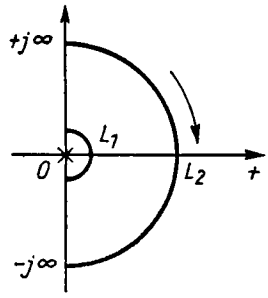



FIG. 56. Path of integration by-passing the origin.

In accordance with this, equation (105) is transformed thus:

$$2j\int_{0}^{\infty} \frac{x(\omega)}{\omega}\,d\omega + jA_0\pi - jA_\infty\pi = 0$$

or

$$\int_{0}^{\infty} \frac{x(\omega)}{\omega}\,d\omega = \frac{\pi}{2}(A_\infty - A_0). \tag{6.106}$$

This integral is called the reactance integral.

Supposing that $u = \ln \omega$, we find $du = d\omega/\omega$ from which, on the basis of (106)

$$\int_{-\infty}^{\infty} x(u) = \frac{\pi}{2}(A_\infty - A_0). \tag{6.107}$$

The formula obtained is analogous to (106) with the difference that the integration is carried out on a logarithmic scale of frequencies. This formula establishes that the area enclosed by the reactance curve (when

plotted against logarithmic frequency) depends only on the difference between the values of the active components of the impedance for zero frequency and infinite frequency, and does not depend on the intermediate values of the resistance.

EXAMPLE 12 (Ref. 7). To prove the truth of formula (106) for the circuit of a two-terminal network given in Example 11.

Using the expressions obtained above, we get:

$$x(\omega) = \frac{\omega L(1 - \omega^2 LC) - r^2 \omega C}{(1 - \omega^2 LC)^2 + (r\omega C)^2}, \quad A_0 = r, \quad A_\infty = 0.$$

Consequently, from (106)

$$\int_0^\infty \frac{L(1 - \omega^2 LC) - r^2 C}{(1 - \omega^2 LC)^2 + (r\omega C)^2} \, d\omega = -\frac{\pi}{2} r.$$

This can be confirmed by integration. Figure 57 illustrates the equality of the areas of the reactance curves plotted against logarithmic frequency for identical values of $r$.

Together with the formulae given above, there are also other relationships between the frequency characteristics $r(\omega)$ and $x(\omega)$ of physically realizable two-terminal networks. These relationships make it possible to find characteristic $r(\omega)$ from characteristic $x(\omega)$ throughout the whole spectrum of frequencies and, vice versa, to find characteristic $x(\omega)$ from characteristic $r(\omega)$. In addition, it is possible to find $r(\omega)$ and $x(\omega)$ for the whole spectrum of frequencies if $r(\omega)$ is given for one part and $x(\omega)$ for the rest of the spectrum (Ref. 7).

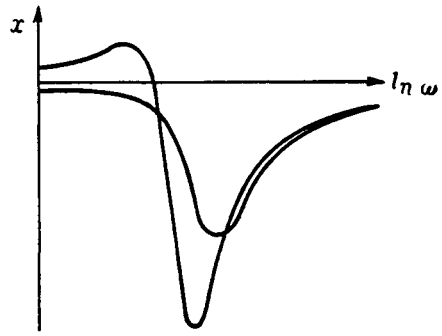



FIG. 57. Example 12.

By analogy with the formulae given above relating $r(\omega)$ and $x(\omega)$ it is possible to deduce the relationships between the frequency characteristics of the conductance and susceptance of a two-terminal network.

Besides the methods given above for establishing the connexion between the real and imaginary parts of an immittance, it is also possible to

make use of the fact that there is no current in a real two-terminal network until the instant the voltages are applied (Ref. 78).

The formulae obtained above show that the conductance and susceptance of a physically realizable two-terminal network are interrelated and cannot be selected independently of each other.

For practical calculations it is convenient to use the algebraic method of calculating the coefficients of the polynomials expressing the immittance as a function of frequency for a given real part (Ref. 75). This method is based on the following.

Representing the impedance function in the form of the ratio of two polynomials

$$Z(p) = \frac{A(p)}{B(p)} = \frac{a_0 + a_1 p + a_2 p^2 + \cdots}{b_0 + b_1 p + b_2 p^2 + \cdots}, \tag{6.108}$$

it is possible to separate the parts of the denominator with even and odd powers of $p$:

$$A_e(p) = a_0 + a_2 p^2 + a_4 p^4 + \cdots$$

$$A_0(p) = a_1 p + a_3 p^3 + a_5 p^5 + \cdots$$

$$B_e(p) = b_0 + b_2 p^2 + b_4 p^4 + \cdots$$

$$B_0(p) = b_1 p + b_3 p^3 + b_5 p^5 + \cdots$$

Let us transform the expression for $Z(p)$:

$$Z(p) = \frac{A(p)B(-p)}{B(p)B(-p)} = \frac{[A_e(p) + A_0(p)]\,[B_e(-p) + B_0(-p)]}{B(p)B(-p)},$$

from which, taking into account the equality $B_e(-p) = B_e(p)$, $B_0(-p) = -B_0(p)$,

$$Z(p) = \frac{A_e(p)B_e(p) - A_0(p)B_0(p)}{B(p)\,.\,B(-p)} + \frac{A_0(p)B_e(p) - A_e(p)B_0(p)}{B(p)\,.\,B(-p)}.$$

Where $p = j\omega$ the first term which is even with respect to frequency $\omega$ expresses the resistance of the network, and the second network, odd with respect to this frequency, expresses the reactance.

Thus

$$\operatorname{Re} Z(j\omega) = \left[\frac{A_e(p)B_e(p) - A_0(p)B_0(p)}{B(p)\,.\,B(-p)}\right]_{p=j\omega}. \tag{6.109}$$

Let us take as given

$$\operatorname{Re} Z(j\omega) = \left[\frac{c_0 + c_2 p^2 + c_4 p^4 + \cdots}{d_0 + d_2 p^2 + d_4 p^4 + \cdots}\right]_{p=j\omega}. \tag{6.110}$$

Equating the numerators of (109) and (110) we get the equation

$$c_0 + c_2p^2 + c_4p^4 + \cdots = (a_0 + a_2p^2 + \cdots)(b_0 + b_2p^2 + \cdots)$$
$$- (a_1p + a_3p^3 + \cdots)(b_1p + b_3p^3 + \cdots), \qquad (6.111)$$

in which the $c$'s are given data. The $b$'s are determined from the locations of the poles. It should then be borne in mind that we select only those poles of the function

$$\frac{c_0 + c_2p^2 + c_4p^4 + \cdots}{d_0 + d_2p^2 + d_4p^4 + \cdots}$$

that appear in the left halfplane, as those belonging to $Z(p)$.

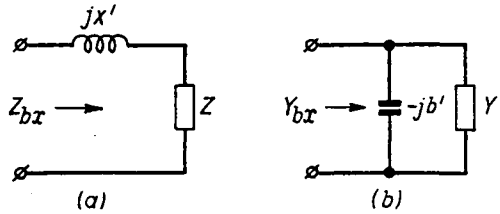



FIG. 58. Illustration of functions with a pole at infinity: (*a*) input impedance; (*b*) input admittance.

The unknown $a$'s are found by equating the coefficients for identical powers of $p$ in the right and left parts of equation (111).

The method described is only applicable for functions with no poles at infinity. Figure 58 shows a two-terminal network with a series reactance $jx'$ (Fig. 58*a*) or with a parallel susceptance $-jb'$ (Fig. 58*b*) chosen arbitrarily. Let us suppose that $Z = r + jx$ and $Y = g - jb$. Obviously, the input impedance and admittance are, respectively,

$$Z_{in} = r + j(x + x'), \quad Y_{in} = g - j(b + b').$$

In view of the fact that the terms $x'$ and $b'$ do not have any effect on $r$ and $g$, the real parts of $Z_{in}$ and $Y_{in}$, the method described does not take these arbitrary terms into account: it is assumed that $x' = 0$ and $b' = 0$, i.e. minimum reactance circuits and minimum susceptance networks are being considered.

EXAMPLE 13 (Ref. 107). To determine function $Z(p)$ and represent the circuit of the two-terminal network if the resistance as a function of frequency obeys the law

$$\operatorname{Re} Z(j\omega) = \frac{1}{1 + \omega^4}.$$

Substituting $p = j\omega$ gives:

$$\frac{1}{1+p^4} = \frac{1}{(p-p_1)(p-\overset{*}{p}_1)(p-p_2)(p-\overset{*}{p}_2)},$$

where $p_2 = -p_1$, $p_1$ and $\overset{*}{p}_1$ and conjugate-complex quantities;

$$p_1, \overset{*}{p}_1 = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}; \quad p_2, \overset{*}{p}_2 = \frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}.$$

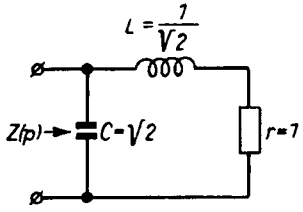



FIG. 59. Example 13.

Only poles with negative real parts, namely $p_1$ and $\overset{*}{p}_1$, belong to function $Z(p)$. As there is no pole at infinity, the number of zeros is equal to the number of poles, i.e. two. Thus

$$Z(p) = \frac{a_0 + a_1 p + a_2 p^2}{(p-p_1)(p-\overset{*}{p}_1)} = \frac{a_0 + a_1 p + a_2 p^2}{p^2 + \sqrt{(2p)} + 1}.$$

Consequently, $b_0 = b_2 = 1$ and $b_1 = \sqrt{2}$. From the data $c_0 = 1$ and $c_2 = c_4 = 0$. From (111)

$$1 = (a_0 + a_2 p^2)(1 + p^2) - a_1 p \sqrt{2} \cdot p$$

or, the same thing,

$$1 = a_0 + a_2 p^2 + a_0 p^2 + a_2 p^4 - \sqrt{2} \cdot a_1 p^2.$$

From this

$$a_0 = 1, \quad a_0 + a_2 - \sqrt{2} \cdot a_1 = 0, \quad a_2 = 0,$$

and consequently

$$a_1 = \frac{1}{\sqrt{2}}.$$

Finally we get

$$Z(p) = \frac{1 + \frac{1}{\sqrt{2}} p}{p^2 + \sqrt{2} \cdot p + 1} = \frac{1}{\sqrt{2}} \frac{p + \sqrt{2}}{p^2 + \sqrt{2} \cdot p + 1}.$$

The two-terminal network shown in Fig. 59 answers to this expression.

CHAPTER VII

# FOUR-TERMINAL NETWORKS

## 1. Basic Definitions and Classification of Four-terminal Networks

Any part of an electrical circuit with two pairs of terminals may be represented as a *four-terminal network** (Fig. 1). Thus a line, a filter, a transformer, an amplifier, an attenuator and any other device with two terminal pairs joining a source and a load, may be referred to as a four-terminal network.

The terminals of a four-terminal network to which an energy source is connected, are called the *input terminals* and those to which the load is connected are called the *output terminals*. For brevity, the terms *input* and *output* are used.

In practice it is possible for both terminal pairs to be inputs (when fed from both sides) or outputs (when the network contains an internal source

FIG. 1. Four-terminal network.

and is terminated in a load at both ends). But only four-terminal networks with one pair of input and one pair of output terminals are examined below.

Four-terminal networks can be classified by various features.

According to the linearity of the elements which they contain, they are divided into *linear* and *non-linear* networks. Only linear networks are considered below.

According to their internal structures, they are divided into the L- (Fig. 2*a*), the T- (Fig. 2*b*), the pi- (Fig. 2*c*), the lattice or bridge (Fig. 2*d*), the bridged T (Fig. 2*e*), and other types of network. Sections 12–16 are devoted to these.

There are *active* and *passive* four-terminal networks.

* The names 'four-pole', 'two-port' and 'two-terminal pair' are also used for such networks.

A four-terminal network is called active if it contains non-compensated energy sources. Then, if these sources are independent, when the four-terminal network is disconnected from the rest of the circuit a voltage appears at its open terminals (from one end or from both). This type of active four-terminal network is called an *autonomous four-terminal network.*

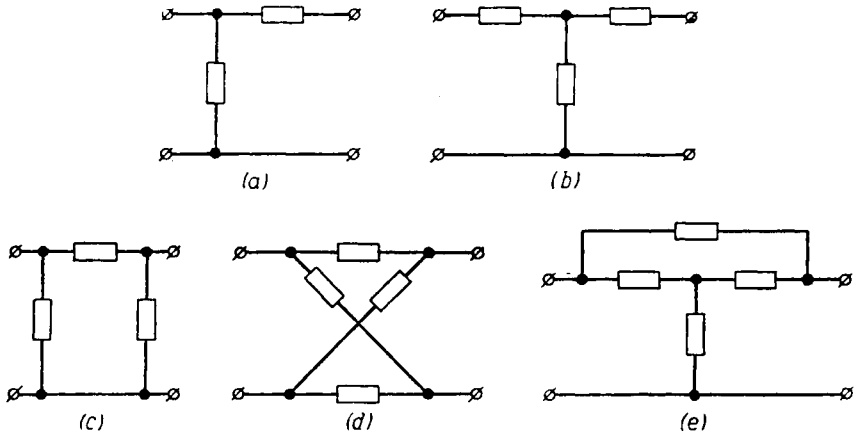



FIG. 2. Different types of circuit diagram for four-terminal networks.

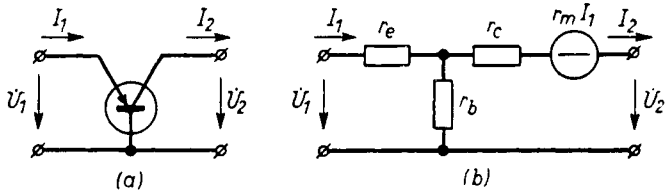



FIG. 3. (*a*) Common-base transistor; (*b*) its equivalent circuit; $r_e$—emitter resistance; $r_c$—collector resistance; $r_b$—base resistance; $r_m$—parameter of a dependent source.

When the sources in a four-terminal network are dependent as, for example, in the equivalent circuits of electron tubes, transistors or magnetic amplifiers, no voltages appear on its open terminals after it has been disconnected from the rest of the circuit. This type of active network is called a non-autonomous four-terminal network. The equivalent circuit of a common-base transistor is given in Fig. 3 (the e.m.f. source $r_m I_1$ is dependent).

A *passive* four-terminal network can either contain no sources or contain mutually compensated sources. Figure 4*a* and *b* shows the electrical circuits of two four-terminal networks which are passive although they contain sources.

The four-terminal network shown in Fig. 4*a* is passive since any two e.m.f.s in the circuit are mutually compensated (these e.m.f.s are cancelled in Kirchhoff's equations).

The four-terminal network shown in Fig. 4*b* is passive because the voltage across the terminals of the branch with the sources is zero.

If all the e.m.f.s in the Fig. 4 circuits are adjusted to zero, the resulting four-terminal networks are equivalent to the original four-terminal networks in the electrical sense of the word. By the *equivalence* of two four-terminal networks of different structure we mean the possibility of substituting one for the other in an electrical circuit without changing the currents and voltages in the remainder of the circuit.

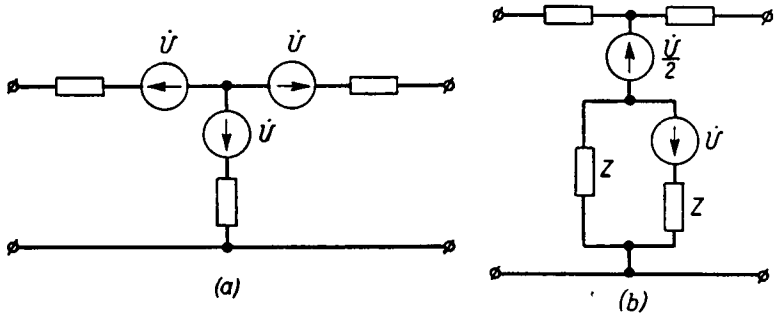



FIG. 4. Passive four-terminal networks with mutually compensating sources.

A four-terminal network is called *reciprocal* or *bilateral* if it obeys the reciprocity theorem, i.e. the ratio of input voltage to the output current of the network or, the same thing, the transfer impedance remains unchanged if the input and output terminal pairs are interchanged. In the opposite case a four-terminal network is called *non-reciprocal* or *unilateral* (e.g. the non-autonomous active four-terminal network shown in Fig. 3).

Passive linear four-terminal networks are reciprocal. Active four-terminal networks can be reciprocal in certain cases.

We distinguish between *symmetrical* and *unsymmetrical* four-terminal networks.

A four-terminal network is *symmetrical* if the currents and voltages in the circuit to which it is connected do not vary when its input and output terminals are reversed. In the opposite case the network is *unsymmetrical.* A symmetrical four-terminal network is always reciprocal.

The principal significance of four-terminal network theory is that, by using certain generalized network parameters, it is possible to examine and relate analytically the voltages and currents at its input and output without calculating the currents and voltages in the interior of the network itself.

In turn, a complex electrical network (e.g. a multistage amplifier or a communication channel) with input and output terminals can be repre-

sented in the form of a combination of component four-terminal networks connected in series or in any other manner. By knowing the analytic relationships between the electrical quantities at the input and output of the component networks, it is possible to obtain relationships between the voltages and currents at the input and output of the over-all network.

The generalized parameters used in the theory of four-terminal networks thus make it possible to find the currents and voltages at the input and output of complex electrical devices without resorting to an examination of the processes within the given network.

The electrical quantities at the input and output of the network which are thus obtained make it possible to estimate the operating conditions for transmission as a whole. The use of generalized network parameters then makes it possible to compare and estimate correctly the transfer properties of circuits of different types and design.

The theory of four-terminal networks also makes it possible to solve problems of synthesis, i.e. to find the structure and elements of a four-terminal network for a given frequency (amplitude-phase) characteristic or for given electrical values at the input and output of the network.

The Soviet scientists V. I. Kovalenkov and E. V. Zelyakh (Refs. 27 and 24) have performed a considerable service in compiling the theory of active four-terminal networks.

V. I. Kovalenkov successfully applied this theory to the analysis of magnetic circuits, antennae, etc. He examined circuits with distributed parameters and sources connected in series.

E. V. Zelyakh extended Kovalenkov's theory to systems with sources with parallel and series-parallel connexion. In addition he developed the theory of active four-terminal networks with lumped parameters and independent sources.

Four-terminal network theory was further developed in the works of R. A. Voronov, G. Ye. Pukhov and V. P. Sigorskii (Refs. 13, 48 and 50). These works are devoted to a more general study of four-terminal networks, connected in various ways to an electrical circuit.

The principles of the theory of four-terminal networks have been developed abroad by F. Strecker, R. Feldkeller, T. E. Shea, W. Cauer *et al.*

## 2. Sets of Equations

The circuit in Fig. 5*a* shows the reference directions assumed for the currents and voltages at the input and output of a four-terminal network which does not contain independent sources. As Fig. 5*a* shows, current $\dot{I}_1$ flows to the upper input terminal of the network and current $\dot{I}_2$ flows from the upper output terminal. Accordingly, each of the voltages $\dot{U}_1$ and $\dot{U}_2$ at

the input and output of the network is determined by the potential difference between the upper terminal and the lower. According to the compensation theorem, the load at the output can be replaced by an e.m.f. $\dot{U}_2$ (Fig. 5*b*).

The initial calculation circuit is thus a four-terminal network of arbitrary design, with e.m.f. sources $\dot{U}_1$ and $\dot{U}_2$ connected to its input and output terminals; the currents $\dot{I}_1$ and $\dot{I}_2$ can be regarded as loop currents and the voltages $\dot{U}_1$ and $\dot{U}_2$ as loop e.m.f.s in the input and output loops. In the following examination, electrical quantities with the subscripts 1 and 2 refer to the left and right terminals respectively.

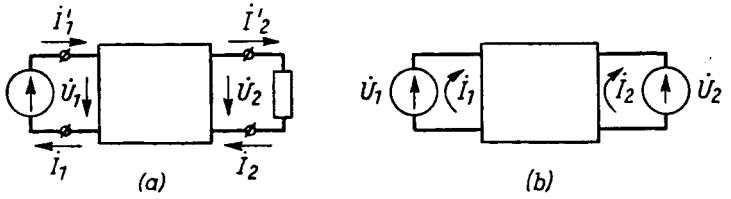



FIG. 5. Reference directions of currents and voltages (forward transmission).

The relationships between the four electrical quantities $\dot{U}_1$, $\dot{U}_2$, $\dot{I}_1$ and $\dot{I}_2$ can be expressed with the aid of various sets of equations. The following six possible forms of equations for four-terminal networks are examined below:

(1) the form $[Y]$: $\dot{I}_1$ and $\dot{I}_2$ are expressed in terms of $\dot{U}_1$ and $\dot{U}_2$;
(2) the form $[Z]$: $\dot{U}_1$ and $\dot{U}_2$ are expressed in terms of $\dot{I}_1$ and $\dot{I}_2$;
(3) the form $[A]$: $\dot{U}_1$ and $\dot{I}_1$ are expressed in terms of $\dot{U}_2$ and $\dot{I}_2$;
(4) the form $[A]^{-1}$: $\dot{U}_2$ and $\dot{I}_2$ are expressed in terms of $\dot{U}_1$ and $\dot{I}_1$;
(5) the form $[H]$: $\dot{U}_1$ and $\dot{I}_2$ are expressed in terms of $\dot{I}_1$ and $\dot{U}_2$;
(6) the form $[H]^{-1}$: $\dot{I}_1$ and $\dot{U}_2$ are expressed in terms of $\dot{U}_1$ and $\dot{I}_2$.

The initial equations of a four-terminal network with no independent sources are the linear equations for the loop currents

$$\left.\begin{aligned} \dot{I}_1 &= \frac{\Delta_{11}}{\Delta}\dot{U}_1 - \frac{\Delta_{21}}{\Delta}\dot{U}_2, \\ \dot{I}_2 &= \frac{\Delta_{12}}{\Delta}\dot{U}_1 - \frac{\Delta_{22}}{\Delta}\dot{U}_2. \end{aligned}\right\} \tag{7.1}$$

Here $\Delta$ is a determinant, its power being equal to the number of independent loops in the network, taking the input and output loops into account; $\Delta_{11}$, $\Delta_{12}$, $\Delta_{21}$ and $\Delta_{12}$ are the corresponding cofactors. In general, all these quantities are functions of frequency.

In the case of a reciprocal four-terminal network the condition

$$\Delta_{12} = \Delta_{21} \tag{7.2}$$

obtains and the network can be completely characterized by three independent parameters: $\Delta_{11}/\Delta, \Delta_{12}/\Delta$ and $\Delta_{22}/\Delta$. Instead of these parameters any other three parameters derived from them may be used.

If, in addition, the four-terminal network is symmetrical, the condition

$$\Delta_{11} = \Delta_{22}. \tag{7.3}$$

obtains, together with (2).

In this case two factors are sufficient to describe the network, e.g. $\Delta_{11}$ and $\Delta_{12}$, or any other two parameters derived from them.

Returning to the general case and denoting

$$Y_{11} = \frac{\Delta_{11}}{\Delta}, \quad Y_{12} = -\frac{\Delta_{21}}{\Delta},$$

$$Y_{21} = \frac{\Delta_{12}}{\Delta}, \quad Y_{22} = -\frac{\Delta_{22}}{\Delta}, \tag{7.4}$$

we get a set of equations for a four-terminal network in the form $[Y]$:

$$\left.\begin{aligned} \dot{I}_1 &= Y_{11}\dot{U}_1 + Y_{12}\dot{U}_2, \\ \dot{I}_2 &= Y_{21}\dot{U}_1 + Y_{22}\dot{U}_2. \end{aligned}\right\} \tag{7.5}$$

The factors $Y_{11}$, $Y_{12}$, $Y_{21}$ and $Y_{22}$ are generally complex and dependent on frequency; they have the dimension of admittance and can be determined thus:

$Y_{11} = (\dot{I}_1/\dot{U}_1)_{U_2=0}$ is the input admittance at the terminals *1* when the output terminals *2* are short-circuited;

$-Y_{22} = (-\dot{I}_2/\dot{U}_2)_{U_1=0}$ is the input admittance at the terminals *2* when the terminals *1* are short-circuited;

$-Y_{21} = (-\dot{I}_2/\dot{U}_1)_{U_2=0}$ is the transfer admittance when the terminals *2* are short-circuited;

$Y_{12} = (\dot{I}_1/\dot{U}_2)_{U_1=0}$ is the transfer admittance when the terminals *1* are short-circuited.

The $Y$'s are called short-circuit admittances.

In the case of a reciprocal four-terminal network

$$Y_{12} = -Y_{21}, \tag{7.6}$$

i.e. only three factors in equations (5) are independent.

If the network is symmetrical, the condition

$$Y_{11} = -Y_{22} \tag{7.7}$$

obtains, together with (6).

In this case the number of independent parameters is two (e.g. $Y_{11}$ and $Y_{12}$).

The minus sign in (6) and (7) corresponds to the reference directions of the currents and voltages assumed in Fig. 5.

Equations (5) can be solved with respect to $\dot{U}_1$ and $\dot{U}_2$; a set of equations for the network is then obtained in the form $[Z]$:

$$\left.\begin{aligned} \dot{U}_1 &= Z_{11}\dot{I}_1 + Z_{12}\dot{I}_2, \\ \dot{U}_2 &= Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2. \end{aligned}\right\} \tag{7.8}$$

Here

$$Z_{11} = \frac{Y_{22}}{|Y|}, \quad Z_{12} = -\frac{Y_{12}}{|Y|},$$

$$Z_{21} = -\frac{Y_{21}}{|Y|}, \quad Z_{22} = \frac{Y_{11}}{|Y|}, \tag{7.9}$$

$$|Y| = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix} = Y_{11}Y_{22} - Y_{12}Y_{21}. \tag{7.10}$$

The factors $Z_{11}$, $Z_{12}$, $Z_{21}$ and $Z_{22}$ are generally complex and depend on frequency; they have the dimension of impedance and can be determined thus:

$Z_{11} = (\dot{U}_1/\dot{I}_1)_{I_2=0}$ is the input impedance at the terminals *1* with the terminals *2* open;

$-Z_2 = (\dot{U}_2/\dot{I}_2)_{I_1=0}$ is the input impedance at the terminals *2* with the terminals *1* open;

$Z_{21} = (\dot{U}_2/\dot{I}_2)_{I_2=0}$ is the transfer impedance with the terminals *2* open;

$-Z_{12} = (\dot{U}_1/-\dot{I}_2)_{I_1=0}$ is the transfer impedance with the terminals *1* open.

The $Z$'s are called open-circuit impedances.

In the case of a reciprocal four-terminal network

$$Z_{12} = -Z_{21}, \tag{7.11}$$

i.e. only three factors in equations (8) are independent.

If the network is symmetrical, the condition

$$Z_{11} = -Z_{22} \tag{7.12}$$

obtains, together with (11).

In this case only two independent parameters remain (e.g. $Z_{11}$ and $Z_{12}$).

The minus sign in (11) and (12) corresponds to the reference directions assumed for the currents and voltages in Fig. 5.

The solution of equations (5), or, correspondingly (8), with respect to $\dot{U}_1$ and $\dot{I}_1$ results in equations for a four-terminal network in the form [$A$]. Thus, on the basis of (5)

$$\dot{U}_1 = -\frac{Y_{22}}{Y_{21}}\dot{U}_2 + \frac{1}{Y_{21}}\dot{I}_2,$$

$$\dot{I}_1 = Y_{11}\left(-\frac{Y_{22}}{Y_{21}}\dot{U}_2 + \frac{1}{Y_{21}}\dot{I}_2\right) + Y_{12}\dot{U}_2$$

$$= -\frac{|Y|}{Y_{21}}\dot{U}_2 + \frac{Y_{11}}{Y_{21}}\dot{I}_2. \tag{7.13}$$

Analogously, starting with (8)

$$\left.\begin{aligned} \dot{I}_1 &= \frac{1}{Z_{21}}\dot{U}_2 - \frac{Z_{22}}{Z_{21}}\dot{I}_2, \\ \dot{U}_1 &= \frac{Z_{11}}{Z_{21}}\dot{U}_2 - \frac{|Z|}{Z_{21}}\dot{I}_2, \end{aligned}\right\} \tag{7.14}$$

where

$$|Z| = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = Z_{11}Z_{22} - Z_{12}Z_{21}. \tag{7.15}$$

Supposing

$$\left.\begin{aligned} A_{11} &= -\frac{Y_{22}}{Y_{21}} = \frac{Z_{11}}{Z_{21}}, \quad A_{12} = \frac{1}{Y_{21}} = -\frac{|Z|}{Z_{21}}, \\ A_{21} &= -\frac{|Y|}{Y_{21}} = \frac{1}{Z_{21}}, \quad A_{22} = \frac{Y_{11}}{Y_{21}} = -\frac{Z_{22}}{Z_{21}}, \end{aligned}\right\} \tag{7.16}$$

we obtain a set of equations for the four-terminal network in the form $A$:

$$\left.\begin{aligned} \dot{U}_1 &= A_{11}\dot{U}_2 + A_{12}\dot{I}_2, \\ \dot{I}_1 &= A_{21}\dot{U}_2 + A_{22}\dot{I}_2. \end{aligned}\right\} \tag{7.17}$$

Factors $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ are generally complex and dependent on frequency; $A_{11}$ and $A_{22}$ are dimensionless; $A_{12}$ has the dimension of impedance; and $A_{21}$ has the dimension of admittance.

These factors can be determined thus:

$A_{11} = (\dot{U}_1/\dot{U}_2)_{I_2=0}$ is the relationship of the voltages for an open-circuited output;

$A_{22} = (\dot{I}_1/\dot{I}_2)_{U_2=0}$ is the relationship of the currents for a short-circuited output;

$A_{12} = (\dot{U}_1/\dot{I}_2)_{U_2=0}$ is the transfer impedance for a short-circuited output;

$A_{21} = (\dot{I}_1/\dot{U}_2)_{I_2=0}$ is the transfer admittance for an open-circuited output.

The $A$'s are called cascade parameters.

In the case of a reciprocal four-terminal network, taking (6) and (11) into account,

$$A_{11}A_{22} - A_{12}A_{21} = 1, \tag{7.18}$$

i.e. only three factors in equations (17) are independent; the fourth is related to the others by expression (18). Thus, given any three of the parameters, the fourth is defined by the above equation.

If the network is symmetrical, according to (7) or (12)

$$A_{11} = A_{22}, \tag{7.19}$$

i.e. the number of independent parameters is two (e.g. $A_{11}$ and $A_{12}$).

The solution of equations (17) with respect to $\dot{U}_2$ and $\dot{I}_2$ results in network equations in the form $[A]^{-1}$:

$$\left.\begin{aligned} \dot{U}_2 &= \frac{A_{22}}{|A|}\dot{U}_1 - \frac{A_{12}}{|A|}\dot{I}_1, \\ \dot{I}_2 &= -\frac{A_{21}}{|A|}\dot{U}_1 + \frac{A_{11}}{|A|}\dot{I}_1. \end{aligned}\right\} \tag{7.20}$$

Taking (18) into account, $|A| = 1$; consequently

$$\left.\begin{aligned} \dot{U}_2 &= A_{22}\dot{U}_1 - A_{12}\dot{I}_1, \\ \dot{I}_2 &= -A_{21}\dot{U}_1 + A_{11}\dot{I}_1. \end{aligned}\right\} \tag{7.21}$$

Finally, the network equations in the form $[H]$ have the appearance

$$\left.\begin{aligned} \dot{U}_1 &= H_{11}\dot{I}_1 + H_{12}\dot{U}_2, \\ \dot{I}_2 &= H_{21}\dot{I}_1 + H_{22}\dot{U}_2, \end{aligned}\right\} \tag{7.22}$$

and in the form $[H]^{-1}$:

$$\left.\begin{aligned} \dot{I}_1 &= \frac{H_{22}}{|H|}\dot{U}_1 - \frac{H_{12}}{|H|}\dot{I}_2, \\ \dot{U}_2 &= -\frac{H_{21}}{|H|}\dot{U}_1 + \frac{H_{11}}{|H|}\dot{I}_2. \end{aligned}\right\} \tag{7.23}$$

The $H$'s are called hybrid parameters.

In the case of a reciprocal four-terminal network

$$H_{12} = H_{21}. \tag{7.24}$$

If the network is symmetrical

$$|H| = -1. \tag{7.25}$$

The resulting relationships between the factors of the four-terminal network are given in Table 1, which facilitates the conversion of one form of equation into another.

Table 2 gives the expressions for the determinants $|A|$, $|Z|$, $|Y|$ and $|H|$.

When the direction of the energy supply varies, and particularly when the energy is transmitted from right to left, the reference directions of the

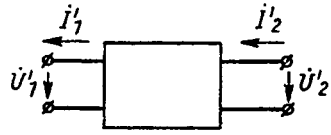



FIG. 6. Altered reference directions of currents (reverse transmission).

currents is as in Fig. 6. The subscripts *1* and *2* remain in accordance with the notation adopted above for the terminals of a four-terminal network (*1* corresponds to the left terminals and *2* to the right).

The currents and voltages in Fig. 6 are noted by primed letters to emphasize that they generally differ from the currents and voltages shown in Fig. 5.

In this case the equations for the network are analogous to those given above, the one difference being that the signs for the currents are reversed. The network equations then have the following form:

$$\left.\begin{aligned} \dot{U}'_2 &= A_{22}\dot{U}'_1 + A_{12}\dot{I}'_1, \\ \dot{I}'_2 &= A_{21}\dot{U}'_1 + A_{11}\dot{I}'_1. \end{aligned}\right\} \tag{7.26}$$

By comparing them with equations (17) which correspond to the direction of the transmission of energy from left to right, we can conclude that as the direction of the supply varies, the factors $A_{11}$ and $A_{22}$, contained in the set of equations, change places.

TABLE 1. PARAMETER INTERRELATIONS

| Form | $[A]$ | $[Z]$ | $[Y]$ | $[H]$ |
|---|---|---|---|---|
| $[A]$ | $A_{11}$ $A_{12}$<br>$A_{21}$ $A_{22}$ | $\frac{Z_{11}}{Z_{21}}$ $-\frac{\lvert Z\rvert}{Z_{21}}$<br>$\frac{1}{Z_{21}}$ $-\frac{Z_{22}}{Z_{21}}$ | $-\frac{Y_{22}}{Y_{21}}$ $\frac{1}{Y_{21}}$<br>$-\frac{\lvert Y\rvert}{Y_{21}}$ $\frac{Y_{11}}{Y_{21}}$ | $-\frac{\lvert H\rvert}{H_{21}}$ $\frac{H_{11}}{H_{21}}$<br>$-\frac{H_{22}}{H_{21}}$ $\frac{1}{H_{21}}$ |
| $[A]^{-1}$ | $\frac{A_{22}}{\lvert A\rvert}$ $-\frac{A_{12}}{\lvert A\rvert}$<br>$-\frac{A_{21}}{\lvert A\rvert}$ $\frac{A_{11}}{\lvert A\rvert}$ | $\frac{Z_{22}}{Z_{12}}$ $-\frac{\lvert Z\rvert}{Z_{12}}$<br>$\frac{1}{Z_{12}}$ $-\frac{Z_{11}}{Z_{12}}$ | $-\frac{Y_{11}}{Y_{12}}$ $\frac{1}{Y_{12}}$<br>$-\frac{\lvert Y\rvert}{Y_{12}}$ $\frac{Y_{22}}{Y_{12}}$ | $\frac{1}{H_{12}}$ $-\frac{H_{11}}{H_{12}}$<br>$\frac{H_{22}}{H_{12}}$ $-\frac{\lvert H\rvert}{H_{12}}$ |
| $[Z]$ | $\frac{A_{11}}{A_{21}}$ $-\frac{\lvert A\rvert}{A_{21}}$<br>$\frac{1}{A_{21}}$ $-\frac{A_{22}}{A_{21}}$ | $Z_{11}$ $Z_{12}$<br>$Z_{21}$ $Z_{22}$ | $\frac{Y_{22}}{\lvert Y\rvert}$ $-\frac{Y_{12}}{\lvert Y\rvert}$<br>$-\frac{Y_{21}}{\lvert Y\rvert}$ $\frac{Y_{11}}{\lvert Y\rvert}$ | $\frac{\lvert H\rvert}{H_{22}}$ $\frac{H_{12}}{H_{22}}$<br>$-\frac{H_{21}}{H_{22}}$ $\frac{1}{H_{22}}$ |
| $[Y]$ | $\frac{A_{22}}{A_{12}}$ $-\frac{\lvert A\rvert}{A_{12}}$<br>$\frac{1}{A_{12}}$ $-\frac{A_{11}}{A_{12}}$ | $\frac{Z_{22}}{\lvert Z\rvert}$ $-\frac{Z_{12}}{\lvert Z\rvert}$<br>$-\frac{Z_{21}}{\lvert Z\rvert}$ $\frac{Z_{11}}{\lvert Z\rvert}$ | $Y_{11}$ $Y_{12}$<br>$Y_{21}$ $Y_{22}$ | $\frac{1}{H_{11}}$ $-\frac{H_{12}}{H_{11}}$<br>$\frac{H_{21}}{H_{11}}$ $\frac{\lvert H\rvert}{H_{11}}$ |
| $[H]$ | $\frac{A_{12}}{A_{22}}$ $\frac{\lvert A\rvert}{A_{22}}$<br>$\frac{1}{A_{22}}$ $-\frac{A_{21}}{A_{22}}$ | $\frac{\lvert Z\rvert}{Z_{22}}$ $\frac{Z_{12}}{Z_{22}}$<br>$-\frac{Z_{21}}{Z_{22}}$ $\frac{1}{Z_{22}}$ | $\frac{1}{Y_{11}}$ $-\frac{Y_{12}}{Y_{11}}$<br>$\frac{Y_{21}}{Y_{11}}$ $\frac{\lvert Y\rvert}{Y_{11}}$ | $H_{11}$ $H_{12}$<br>$H_{21}$ $H_{22}$ |
| $[H]^{-1}$ | $\frac{A_{21}}{A_{11}}$ $\frac{\lvert A\rvert}{A_{11}}$<br>$\frac{1}{A_{11}}$ $-\frac{A_{12}}{A_{11}}$ | $\frac{1}{Z_{11}}$ $-\frac{Z_{12}}{Z_{11}}$<br>$\frac{Z_{21}}{Z_{11}}$ $\frac{\lvert Z\rvert}{Z_{11}}$ | $\frac{\lvert Y\rvert}{Y_{22}}$ $\frac{Y_{12}}{Y_{22}}$<br>$-\frac{Y_{21}}{Y_{22}}$ $\frac{1}{Y_{22}}$ | $\frac{H_{22}}{\lvert H\rvert}$ $-\frac{H_{12}}{\lvert H\rvert}$<br>$-\frac{H_{21}}{\lvert H\rvert}$ $\frac{H_{11}}{\lvert H\rvert}$ |

TABLE 2. DETERMINANT INTERRELATIONS

| Parameters of four-terminal network / Determinants | $A$ | $Z$ | $Y$ | $H$ |
|---|---|---|---|---|
| $\lvert A\rvert$ | $A_{11}A_{22}-A_{12}A_{21}$ | $-\frac{Z_{12}}{Z_{21}}$ | $-\frac{Y_{12}}{Y_{21}}$ | $\frac{H_{12}}{H_{21}}$ |
| $\lvert Z\rvert$ | $-\frac{A_{12}}{A_{21}}$ | $Z_{11}Z_{22}-Z_{12}Z_{21}$ | $\frac{1}{\lvert Y\rvert}$ | $\frac{H_{11}}{H_{22}}$ |
| $\lvert Y\rvert$ | $-\frac{A_{21}}{A_{12}}$ | $\frac{1}{\lvert Z\rvert}$ | $Y_{11}Y_{22}-Y_{12}Y_{21}$ | $\frac{H_{22}}{H_{11}}$ |
| $\lvert H\rvert$ | $-\frac{A_{11}}{A_{22}}$ | $\frac{Z_{11}}{Z_{22}}$ | $\frac{Y_{22}}{Y_{11}}$ | $H_{11}H_{22}-H_{12}H_{21}$ |

### 3. Open-circuit and Short-circuit Parameters

When the factors $Y$ and $Z$ were examined in the previous section, it was shown $Y_{11}$ and $-Y_{22}$ are the input admittances of a four-terminal network, measured from left and right with opposite terminals short-circuited. Correspondingly, $Z_{11}$ and $-Z_{22}$ are the input impedances with open terminals. The reference directions of current and voltage are then taken in accordance with Fig. 5.

If the input impedances or admittances measured for forward transmission, i.e. from the network's left terminals, are in accordance with the Fig. 5 diagrams and the input impedances or admittances measured for reverse transmission, i.e. from the network's right terminals, are in accordance with the reference directions assumed in the Fig. 6 diagram, then, introducing the notations '$s$' and '$o$' to denote short-circuit (terminals closed) and open-circuit conditions, we get:

$$\left.\begin{aligned} Z_{1s} &= \frac{1}{Y_{11}} = \frac{\Delta}{\Delta_{11}}, \quad Z_{2o} = -\frac{1}{Y_{22}} = \frac{\Delta}{\Delta_{22}}, \\ Z_{1o} &= Z_{11} = \frac{\Delta_{22}\Delta}{\Delta_{11}\Delta_{22}-\Delta_{12}\Delta_{21}}, \\ Z_{2o} &= -Z_{22} = \frac{\Delta_{11}\Delta}{\Delta_{11}\Delta_{22}-\Delta_{12}\Delta_{21}}. \end{aligned}\right\} \tag{7.27}$$

If necessary, the equations for a reciprocal four-terminal network can be expressed by these parameters. Open-circuit and short-circuit parameters are not sufficient for writing the equations of a non-reciprocal four-terminal network since only three of these parameters are independent.

Actually, from (27)

$$\frac{Z_{1s}}{Z_{1o}} = \frac{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{12}} \quad \text{and} \quad \frac{Z_{2s}}{Z_{2o}} = \frac{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}},$$

from which

$$\frac{Z_{1s}}{Z_{1o}} = \frac{Z_{2s}}{Z_{2o}}. \tag{7.28}$$

The open-circuit and short-circuit parameters expressed by formulae (27) are thus necessarily related by equation (28). In the case of a symmetrical four-terminal network

$$Z_{1s} = Z_{2s} = Z_s, \quad Z_{1o} = Z_{2o} = Z_o$$

and, consequently, the symmetrical four-terminal network is described by only two parameters.

Open-circuit and short-circuit parameters are expressed by the $A$ parameters as follows:

$$\left.\begin{aligned} Z_{1s} &= \frac{A_{12}}{A_{22}}, \quad Z_{1o} = \frac{A_{11}}{A_{21}}, \\ Z_{2s} &= \frac{A_{12}}{A_{11}}, \quad Z_{2o} = \frac{A_{22}}{A_{21}}. \end{aligned}\right\} \tag{7.29}$$

Where necessary the $A$ parameters of a reciprocal four-terminal network can be expressed, in their turn, by the open-circuit and short-circuit parameters

$$\left.\begin{aligned} A_{11} &= \sqrt{\left(\frac{Z_{1s}Z_{1o}}{Z_{2s}(Z_{1o} - Z_{1s})}\right)}, \\ A_{12} &= \sqrt{\left(\frac{Z_{1s}Z_{1o}Z_{2s}}{Z_{1o} - Z_{1s}}\right)}, \\ A_{21} &= \sqrt{\left(\frac{1}{Z_{2o}(Z_{1o} - Z_{1s})}\right)}, \\ A_{22} &= \sqrt{\left(\frac{Z_{2o}}{Z_{1o} - Z_{1s}}\right)}. \end{aligned}\right\} \tag{7.30}$$

In the case of a symmetrical four-terminal network

$$\left.\begin{aligned} A_{11} &= A_{22} = \sqrt{\left(\frac{Z_o}{Z_o - Z_s}\right)}, \\ A_{12} &= Z_s \sqrt{\left(\frac{Z_o}{Z_o - Z_s}\right)}, \\ A_{21} &= \frac{1}{\sqrt{Z_o(Z_o - Z_s)}}. \end{aligned}\right\} \tag{7.31}$$

If, in addition to the network's two open-circuit parameters $Z_{1o}$ and $Z_{2o}$, the transfer impedance when the output terminals are open $Z_{21}$ for an

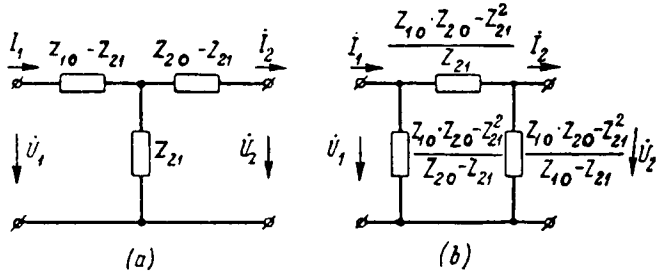



FIG. 7. T- and pi-networks equivalent to any given passive four-terminal net work.

open-circuit output is selected as the third principal parameter, the network equations (8), taking (11) into account, have the appearance:

$$\left.\begin{aligned} \dot{U}_1 &= Z_{1o}\dot{I}_1 - Z_{21}\dot{I}_2, \\ \dot{U}_2 &= Z_{21}\dot{I}_1 - Z_{2o}\dot{I}_2. \end{aligned}\right\} \tag{7.32}$$

The equivalent T-network given in Fig. 7 *a* corresponds to equations (32). In other words, any reciprocal four-terminal network can be replaced by an equivalent T-network with the same parameters as the original network.

The impedances of the parallel arm and series arm of the T-network are determined by (16) and (29) from the expressions

$$Z_{21} = \frac{1}{A_{21}}, \quad Z_{1o} - Z_{21} = \frac{A_{11} - 1}{A_{21}},$$

$$Z_{2o} - Z_{21} = \frac{A_{22} - 1}{A_{21}}.$$

Quantity $Z_{21}$ can also be expressed by open-circuit and short-circuit parameters, in accordance with the Fig. 7*a* diagram

$$Z_{1s} = Z_{1o} - Z_{21} + \frac{(Z_{2o} - Z_{21})Z_{21}}{Z_{2o}}$$

or

$$Z_{1s}Z_{2o} = Z_{1o}Z_{2o} - Z_{21}Z_{2o} + Z_{2o}Z_{21} - Z_{21}^2,$$

from which

$$Z_{21} = \pm \sqrt{Z_{2o}(Z_{1o} - Z_{1s})}. \tag{7.33}$$

In accordance with (33), the solution obtained is ambiguous.

This can be explained physically by the fact that the open-circuit and short-circuit tests give the same results regardless of the crossing of the output terminals.

The impedances connected in a T-network form a three-ray star and the impedances in a pi-network (Fig. 7*b*) form a triangle. Therefore, if we apply the rule for converting a three-ray star into an equivalent triangle, it is possible to replace any T-network by an equivalent pi-network. Its impedances will equal, correspondingly,

$$\frac{A_{12}}{A_{22} - 1} = \frac{Z_{1o}Z_{2o} - Z_{21}^2}{Z_{2o} - Z_{21}}$$

and

$$\frac{A_{12}}{A_{11} - 1} = \frac{Z_{1o}Z_{2o} - Z_{21}^2}{Z_{1o} - Z_{21}}$$

in the parallel arm, and

$$A_{12} = \frac{Z_{1o}Z_{2o} - Z_{21}^2}{Z_{21}}$$

in the series arm. Taking (33) into account

$$Z_{1o}Z_{2o} - Z_{21}^2 = Z_{2o}Z_{1s}.$$

In practice, it is not always possible to realize an equivalent T- or pi-network for a given four-terminal network with prescribed open-circuit or short-circuit parameters (see Section 18).

### 4. Image Parameters

If, during forward transmission, an unsymmetrical four-terminal network is loaded with an impedance $Z_2$, its input impedance, i.e. the ratio of the input voltage to the input current, is determined from the expression

$$Z_1 = \frac{\dot{U}_1}{\dot{I}_1} = \frac{A_{11}\dot{U}_2 + A_{12}\dot{I}_2}{A_{21}\dot{U}_2 + A_{22}\dot{I}_2} = \frac{A_{11}Z_2 + A_{12}}{A_{21}Z_2 + A_{22}}, \tag{7.34}$$

where

$$Z_2 = \frac{\dot{U}_2}{\dot{I}_2}.$$

If, during reverse transmission, this network is loaded with an impedance $Z_1'$, its input impedance from the right is determined from the expression

$$Z_2' = \frac{\dot{U}_2'}{\dot{I}_2'} = \frac{A_{22}U_1' + A_{12}\dot{I}_1'}{A_{21}\dot{U}_1' + A_{11}\dot{I}_1'} = \frac{A_{22}Z_1' + A_{12}}{A_{21}Z_1' + A_{11}}, \tag{7.35}$$

where

$$Z_1' = \frac{\dot{U}_1'}{\dot{I}_1'}.$$

The expressions (29) given in the previous section can be obtained from (34) and (35) as the particular cases when $Z_2 = 0$, $Z_2 = \infty$, $Z_1' = 0$ and $Z_1' = \infty$.

Let us now suppose that impedances $Z_2$ and $Z_1'$ are matched in such a way that $Z_1 = Z_1'$ and $Z_2 = Z_2'$. In other words we shall assume that there

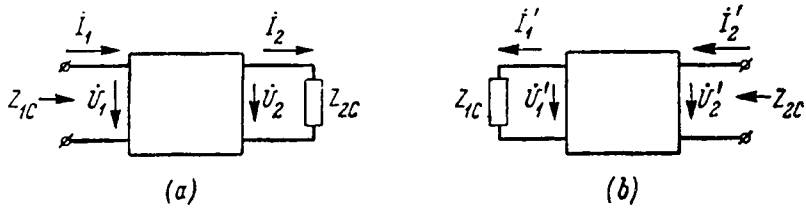



FIG. 8. Image impedances of an unsymmetrical four-terminal network.

are two impedances $Z_{1c}$ and $Z_{2c}$ which satisfy the following condition: for forward transmission (Fig. 8*a*) the input impedance of a four-terminal network loaded with an impedance $Z_{2c}$ is $Z_{1c}$; and for reverse transmission (Fig. 8*b*) the input impedance of a four-terminal network loaded from the left with an impedance $Z_{1c}$ equals $Z_{2c}$.

These two impedances are called the *characteristic or image impedances* of an unsymmetrical four-terminal network.

The condition when a four-terminal network is loaded with the appropriate characteristic impedance is called matched loading.

Substituting $Z_1 = Z_1' = Z_{1c}$ and $Z_2 = Z_2' = Z_{2c}$ in (34) and (35) we get

$$Z_{1c} = \frac{A_{11}Z_{2c} + A_{12}}{A_{21}Z_{2c} + A_{22}}, \quad Z_{2c} = \frac{A_{22}Z_{1c} + A_{12}}{A_{21}Z_{1c} + A_{11}}. \tag{7.36}$$

The joint solution of these equations with respect to the unknown quantities $Z_{1c}$ and $Z_{2c}$ gives

$$Z_{1c} = \sqrt{\left(\frac{A_{11}A_{12}}{A_{21}A_{22}}\right)} \tag{7.37}$$

and

$$Z_{2c} = \sqrt{\left(\frac{A_{22}A_{12}}{A_{21}A_{11}}\right)}. \tag{7.38}$$

Equation (18) of a reciprocal four-terminal network can be represented in the form

$$[\sqrt{(A_{11}A_{22})}]^2 - [\sqrt{(A_{12}A_{21})}]^2 = 1. \tag{7.39}$$

Let us introduce a new parameter $g$ for this network, which satisfies the condition

$$\cosh g = \sqrt{(A_{11}A_{22})} \tag{7.40}$$

and is, in general, complex

$$g = b + ja. \tag{7.41}$$

It should be noted that the hyperbolic cosine of the complex argument can have any given value determined by the right-hand side of (40).

A comparison of (39) and (40) with the known formula

$$\cosh^2 g - \sinh^2 g = 1 \tag{7.42}$$

shows that

$$\sinh g = \sqrt{(A_{12}A_{21})}. \tag{7.43}$$

Parameter $g$ is called the network's *propagation function.* This is the third image parameter of a reciprocal four-terminal network.

The real part of the propagation function ($b$) is called the *image attenuation function,* and the imaginary part ($a$) the *image phase function.* The physical significance of these functions will be explained later.

From (37) and (38)

$$\sqrt{\left(\frac{Z_{1c}}{Z_{2c}}\right)} = \sqrt{\left(\frac{A_{11}}{A_{22}}\right)} \tag{7.44}$$

and

$$\sqrt{(Z_{1c}Z_{2c})} = \sqrt{\left(\frac{A_{12}}{A_{21}}\right)}. \tag{7.45}$$

Multiplying (40) by (44) gives

$$A_{11} = \sqrt{\left(\frac{Z_{1c}}{Z_{2c}}\right)} \cosh g, \tag{7.46}$$

and multiplying (43) by (45)

$$A_{12} = \sqrt{(Z_{1c}Z_{2c})} \sinh g. \tag{7.47}$$

Dividing (40) by (44) gives

$$A_{22} = \sqrt{\left(\frac{Z_{2c}}{Z_{1c}}\right)} \cosh g, \tag{7.48}$$

and dividing (43) by (45)

$$A_{21} = \frac{1}{\sqrt{(Z_{1c}Z_{2c})}} \sinh g. \tag{7.49}$$

As a result of substituting (46)–(49) in (17) we get equations for an unsymmetrical four-terminal network in hyperbolic form

$$\left.\begin{aligned} \dot{U}_1 &= \sqrt{\left(\frac{Z_{1c}}{Z_{2c}}\right)} \left[\dot{U}_2 \cosh g + Z_{2c}\dot{I}_2 \sinh g\right], \\ \dot{I}_1 &= \sqrt{\left(\frac{Z_{2c}}{Z_{1c}}\right)} \left[\dot{I}_2 \cosh g + \frac{1}{Z_{2c}} \dot{U}_2 \sinh g\right]. \end{aligned}\right\} \tag{7.50}$$

With accordant connexion

$$Z_{2c}\dot{I}_2 = \dot{U}_2 \quad \text{and} \quad \frac{1}{Z_{2c}} \dot{U}_2 = \dot{I}_2;$$

taking into account that $\cosh g + \sinh g = e^g$, equations (50) can be simplified:

$$\left.\begin{aligned} \dot{U}_1 &= \sqrt{\left(\frac{Z_{1c}}{Z_{2c}}\right)} \dot{U}_2 e^g, \\ \dot{I}_1 &= \sqrt{\left(\frac{Z_{2c}}{Z_{1c}}\right)} \dot{I}_2 e^g. \end{aligned}\right\} \tag{7.51}$$

It follows from this that the moduli of the voltages and, correspondingly, the currents at the input and output of the network are related by the equations

$$\left.\begin{aligned} U_1 &= \sqrt{\left(\frac{Z_{1c}}{Z_{2c}}\right)}\, U_2 e^b, \\ I_1 &= \sqrt{\left(\frac{Z_{2c}}{Z_{1c}}\right)}\, I_2 e^b. \end{aligned}\right\} \tag{7.52}$$

The factor $\sqrt{(Z_{1c}/Z_{2c})}\, e^b$ equals the ratio of the amplitudes (or effective values) of the voltages at the input and output of the network for a matched load. In turn, the factor $\sqrt{(Z_{2c}/Z_{1c})}\, e^b$ is equal to the ratio of the amplitudes (or effective values) of the currents.

If the arguments (angles) of the characteristic impedances $Z_{1c}$ and $Z_{2c}$ are denoted by $\varphi_{1c}$ and $\varphi_{2c}$, the phase angle of the voltage at the input with respect to the voltage at the output is determined by the quantity $a + (\varphi_{1c} - \varphi_{2c})/2$ and the phase angle of the input current with respect to the output current, by the quantity $a + (\varphi_{2c} - \varphi_{1c})/2$. When angles $\varphi_{1c}$ and $\varphi_{2c}$ are equal, angle $a$ is the phase difference between the voltages or, correspondingly, between the currents of the network, with matched loading.

The image parameters $Z_{1c}$, $Z_{2c}$ and $g$ can be expressed by open-circuit and short-circuit parameters, namely from (29), (40), and (43)

$$\left.\begin{aligned} Z_{1c} &= \sqrt{(Z_{1o}Z_{1s})}, \\ Z_{2c} &= \sqrt{(Z_{2o}Z_{2s})}, \\ \tanh g &= \sqrt{\left(\frac{Z_{1s}}{Z_{1o}}\right)} = \sqrt{\left(\frac{Z_{2s}}{Z_{2o}}\right)}. \end{aligned}\right\} \tag{7.53}$$

In its turn

$$\left.\begin{aligned} Z_{1s} &= Z_{1c}\tanh g, \quad Z_{1o} = Z_{1c}\coth g, \\ Z_{2s} &= Z_{2c}\tanh g, \quad Z_{2o} = Z_{2c}\coth g. \end{aligned}\right\} \tag{7.54}$$

Substituting (40) and (43) in the formula $e^g = \cosh g + \sinh g$ results in an expression relating the image parameter $g$ with the $A$ parameters:

$$e^g = e^b e^{ja} = \sqrt{(A_{11}A_{22})} + \sqrt{(A_{12}A_{21})} = m + jn, \tag{7.55}$$

from which

$$g = \ln\left[\sqrt{(A_{11}A_{22})} + \sqrt{(A_{12}A_{21})}\right]. \tag{7.56}$$

The image attenuation constant $b$ is determined by the natural logarithm of the modulus of the right-hand side of (56):

$$b = \ln\left|\sqrt{(A_{11}A_{12})} + \sqrt{(A_{12}A_{21})}\right|. \tag{7.57}$$

The image phase function $a$ is determined by the inverse tangent of the ratio of the imaginary part of (55) to the real part of (55)

$$a = \arctan \frac{n}{m}. \tag{7.58}$$

It should be noted that parameter $g$ can also be obtained as half of the natural logarithm of the ratio of the products of the voltage and current sinors at the input and output of the network, with matched loading. In fact, from (51)

$$\frac{\dot{U}_1 \dot{I}_1}{\dot{U}_2 \dot{I}_2} = e^{2g} = e^{2b} e^{j2a},$$

from which

$$g = \frac{1}{2} \ln \frac{\dot{U}_1 \dot{I}_1}{\dot{U}_2 \dot{I}_2}. \tag{7.59}$$

The network's image attenuation constant $b$ can be obtained as half of the natural logarithm of the ratio of the apparent power at the input to that at the output of the network, with matched loading

$$b = \frac{1}{2} \ln \frac{U_1 I_1}{U_2 I_2}. \tag{7.60}$$

For practical calculations it can also be convenient to use the expressions resulting from (53)

$$\left.\begin{aligned} e^{-2g} &= \frac{\sqrt{Z_{1o}} - \sqrt{Z_{1s}}}{\sqrt{Z_{1o}} + \sqrt{Z_{1s}}} = \frac{\sqrt{Z_{2o}} - \sqrt{Z_{2s}}}{\sqrt{Z_{2o}} + \sqrt{Z_{2s}}} \\ &= \frac{Z_{1c} - Z_{1s}}{Z_{1c} + Z_{1s}} = \frac{Z_{2c} - Z_{2s}}{Z_{2c} + Z_{2s}} \\ &= \frac{Z_{1o} - Z_{1c}}{Z_{1o} + Z_{1c}} = \frac{Z_{2o} - Z_{2c}}{Z_{2o} + Z_{2c}}. \end{aligned}\right\} \tag{7.61}$$

In the case of a symmetrical four-terminal network ($A_{11} = A_{22}$) the characteristic impedances $Z_{1c}$ and $Z_{2c}$ are equal

$$Z_{1c} = Z_{2c} = Z_c = \sqrt{\left(\frac{A_{12}}{A_{21}}\right)} = \sqrt{(Z_o Z_s)}. \tag{7.62}$$

Consequently, the input impedance of a symmetrical four-terminal network loaded with a characteristic impedance $Z_c$, equals $Z_c$ (Fig. 9). This means that to every symmetrical four-terminal network there corresponds

a certain characteristic impedance $Z_c$ with this property: if the given network is loaded with an impedance $Z_c$, the ratios of voltage to current at the input and output of the network will be identical, i.e.

$$\frac{\dot{U}_1}{\dot{I}_1} = \frac{\dot{U}_2}{\dot{I}_2} = Z_c.$$

On the basis of (50) the equations for a symmetrical four-terminal network with an arbitrary load termination are written in the hyperbolic form

$$\left.\begin{aligned} \dot{U}_1 &= \dot{U}_2 \cosh g + Z_c \dot{I}_2 \sinh g, \\ \dot{I}_1 &= \dot{I}_2 \cosh g + \frac{1}{Z_c} \dot{U}_2 \sinh g; \end{aligned}\right\} \tag{7.63}$$

here

$$\cosh g = A_{11} = A_{22}, \quad Z_c \sinh g = A_{12}, \quad \frac{1}{Z_c} \sinh g = A_{21}.$$

If the load is matched, i.e. $Z_c\dot{I}_2 = \dot{U}_2$ and $1/Z_c\dot{U}_2 = \dot{I}_2$, then

$$e^a = e^b e^{ja} = \frac{\dot{U}_1}{\dot{U}_2} = \frac{\dot{I}_1}{\dot{I}_2}. \tag{7.64}$$

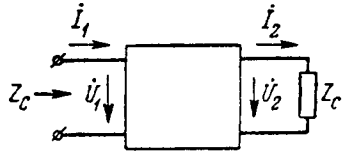



FIG. 9. Image impedance of a symmetrical four-terminal network.

In this case the change in the amplitude of both voltage and current is determined by the factor $e^b$ and the phase difference between the voltages or currents by the angle $a$.

From (64)

$$\left.\begin{aligned} g &= \ln \frac{\dot{U}_1}{\dot{U}_2} = \ln \frac{\dot{I}_1}{\dot{I}_2}, \\ b &= \ln \frac{U_1}{U_2} = \ln \frac{I_1}{I_2}. \end{aligned}\right\} \tag{7.65}$$

The quantities $g$, $b$ and $a$ have no dimensions. Angle $a$ is calculated in radians (rad). The image attenuation constant $b$ in (60) and (65) is calculated

in nepers (nep). An attenuation of one neper corresponds to a decrease in voltage or current of $e = 2{\cdot}718$ times since, when

$$b_{\text{nepers}} = \ln \frac{U_1}{U_2} = 1, \quad \frac{U_1}{U_2} = 2{\cdot}718.$$

In radio engineering attenuation is calculated in decibels (db) from the formulae

$$b = 10 \lg \frac{U_1 I_1}{U_2 I_2} = 20 \lg \frac{U_1}{U_2} = 20 \lg \frac{I_1}{I_2}.$$

It follows from these expressions that an attenuation of 1 db corresponds to a decrease in power of 1·26 times or a decrease in voltage or current of 1·12 times since, when

$$20 \lg \frac{U_1}{U_2} = 1, \quad \frac{U_1}{U_2} = 1{\cdot}12.$$

The following relationships are used for conversion from nepers to decibels and vice versa:

$$1 \text{ neper} = 8{\cdot}636 \text{ db},$$

$$1 \text{ db} = 0{\cdot}115 \text{ nepers}.$$

### 5. Amplitude-phase Characteristic or Transfer Function

The *amplitude-phase characteristic* or *transfer function** of a four-terminal network is the ratio of the complex amplitudes or complex effective values of the electrical quantities at its output to those at its input in given transfer conditions.

The ratios of like electrical quantities (output-to-input voltage ratio and current ratio)

$$K_U = \frac{\dot{U}_2}{\dot{U}_1}, \quad K_I = \frac{\dot{I}_2}{\dot{I}_1} \tag{7.66}$$

are dimensionless quotients usually complex and dependent on frequency. In accordance with amplifying devices, these ratios are called voltage amplification or current amplification. The word 'gain' is sometimes used as a synonym for amplification.

* The transfer function for a complex frequency is discussed in Section 20.

The ratios of unlike electrical quantities, namely transfer impedance and admittance*

$$Z_{21} = \frac{\dot{U}_2}{\dot{I}_1}, \quad Y_{21} = \frac{\dot{I}_2}{\dot{U}_1} \tag{7.67}$$

are also generally complex quantities dependent on frequency.

The moduli of the complex values $K_U$, $K_I$, $Y_{21}$ and $Z_{21}$ are the amplitude-frequency characteristics, and their arguments are the phase-frequency characteristics of the four-terminal network.

Amplitude and phase characteristics in the band of transmittable frequencies are of great significance in radio-engineering and automation equipment.

As (51) shows, for a matched load the amplitude-phase characteristics of an unsymmetrical four-terminal network are determined by the formulae

$$\left.\begin{aligned} K_{Um} &= \sqrt{\left(\frac{Z_{2c}}{Z_{1c}}\right)} e^{-g}, \quad K_{Im} = \sqrt{\left(\frac{Z_{1c}}{Z_{2c}}\right)} e^{-g}, \\ Z_{21} &= \sqrt{(Z_{1c}Z_{2c})}\, e^{-g}, \quad Y_{21} = \frac{1}{\sqrt{(Z_{1c}Z_{2c})}} e^{-g}. \end{aligned}\right\} \tag{7.68}$$

In the case of a symmetrical four-terminal network, the formulae are simplified accordingly:

$$K_{Um} = K_{Im} = e^{-g}, \quad Z_{21} = Z_c e^{-g}, \quad Y_{21} = \frac{1}{Z_c} e^{-g}. \tag{7.69}$$

Transfer functions corresponding to open-circuit and short-circuit conditions are often used in radio and control engineering. The network equations (17) and (26) make it possible to establish a connexion between the factors $K_{Uo}$ and $K_{Is}$ for forward transmission and the factors $K'_{Uo}$ and $K'_{Is}$ for reverse transmission, and also to express these factors by $A$ parameters:

$$K_{Us} = K'_{Is} = \frac{1}{A_{11}}, \quad K_{Io} = K'_{Us} = \frac{1}{A_{22}}. \tag{7.70}$$

Thus, in a reciprocal four-terminal network the voltage transfer function for an open circuit and forward transmission equals the current transfer function for a short circuit and reverse transmission; and in turn the current transfer function for a short circuit and forward transmission equals the voltage transfer function for an open circuit and reverse transmission.

* In present-day literature these concepts are expressed by the general term "transfer immittance".

It follows from (4), (16) and (70) that $K_{Uo} = \Delta_{12}/\Delta_{22}$ and $K_{Is} = \Delta_{12}/\Delta_{11}$, which agrees with the conclusion drawn in Chapter IV, Section 9.

The network equations can be expressed by the transfer functions $K_{Uo}$ and $K_{Is}$ and the short-circuit and open-circuit impedances $Z_{1s}$ and $Z_{1o}$.

Thus, taking the parameters $Z_{1s} = A_{12}/A_{22}$ and $Z_{1o} = A_{11}/A_{21}$, into account, equations (17) take the form

$$\left.\begin{aligned} \dot{U}_1 &= \frac{1}{K_{Uo}}\dot{U}_2 + \frac{Z_{1s}}{K_{Is}}\dot{I}_2, \\ \dot{I}_1 &= \frac{1}{K_{Uo}Z_{1o}}\dot{U}_2 + \frac{1}{K_{Is}}\dot{I}_2. \end{aligned}\right\} \tag{7.71}$$

According to (18), for a reciprocal four-terminal network

$$\frac{Z_{1o}}{Z_{1s}} = \frac{1}{1 - K_{Is}K_{Uo}}. \tag{7.72}$$

On the basis of (71) factors $K_U$ and $K_I$, for an arbitrary load $Z_2$, are expressed by the factors $K_{Uo}$, $K_{Is}$, $Z_{1s}/Z_2$ and $Z_2/Z_{1o}$:

$$\left.\begin{aligned} \frac{1}{K_U} &= \frac{1}{K_{Uo}} + \frac{1}{K_{Is}} \cdot \frac{Z_{1s}}{Z_2}, \\ \frac{1}{K_I} &= \frac{1}{K_{Is}} + \frac{1}{K_{Uo}} \cdot \frac{Z_2}{Z_{1o}}. \end{aligned}\right\} \tag{7.73}$$

The factor $K_U$ can also be expressed by $K_{Uo}$ and the ratio of the input impedance $Z_{2s}$, measured from terminals *2*, with terminals *1* closed, to impedance $Z_2$. In fact, according to the second equation of (5)

$$Y_2\dot{U}_2 = Y_{21}\dot{U}_1 + Y_{22}\dot{U}_2,$$

where

$$Y_2 = \frac{1}{Z_2}.$$

Consequently,

$$K_U = \frac{\dot{U}_2}{\dot{U}_1} = \frac{-Y_{21}}{Y_{22} - Y_2} = \frac{-\frac{Y_{21}}{Y_{22}}}{1 - \frac{Y_2}{Y_{22}}}$$

or

$$K_U = \frac{K_{Uo}}{1 + \frac{Z_{2s}}{Z_2}}. \tag{7.74}$$

Factor $K_I$, with an arbitrary load $Z_2$, is in its turn expressed by the factor $K_{Is}$ which corresponds to short-circuit conditions, and to the ratio $Z_2$ to impedance $Z_{2o}$ measured from terminals *2*, with terminals *1* closed. According to the second equation of (8)

$$Z_2\dot{I}_2 = Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2,$$

from which

$$K_I = \frac{\dot{I}_2}{\dot{I}_1} = \frac{-Z_{21}}{Z_{22} - Z_2} = \frac{-\frac{Z_{21}}{Z_{22}}}{1 - \frac{Z_2}{Z_{22}}}$$

or

$$K_I = \frac{K_{Is}}{1 + \frac{Z_2}{Z_{2o}}}. \tag{7.75}$$

Formulae (74) and (75) can also be easily deduced from the equivalent source theorem.

Many radio-engineering, wire-communication and automation devices consist of several four-terminal networks connected together. Section 9 is devoted to determining the parameters of the over-all four-terminal network from the given parameters of the component networks.

If the $A$ parameters of a composite four-terminal network and the impedance $Z_2$ at its output are known, then the transfer functions can be expressed very easily by these parameters (see Example 1 in Section 12):

$$\left.\begin{aligned} K_U &= \frac{\dot{U}_2}{A_{11}\dot{U}_2 + A_{12}\dot{I}_2} = \frac{1}{A_{11} + A_{12}\frac{1}{Z_2}}, \\ K_I &= \frac{\dot{I}_2}{A_{21}\dot{U}_2 + A_{22}\dot{I}_2} = \frac{1}{A_{22} + A_{21}Z_2}. \end{aligned}\right\} \tag{7.76}$$

Depending on the actual conditions of the problem, the set of the network's parameters most suitable for calculation purposes is used in practice. There is no universal set of parameters suitable for all cases.

## 6. Input Impedance of a Four-terminal Network with Arbitrary Load Conditions

The input impedance of a four-terminal network from terminals *1*, when it is loaded by an impedance $Z_2$ from terminals *2*, is

$$Z_{1\text{in}} = \frac{A_{11}\dot{U}_2 + A_{12}\dot{I}_2}{A_{21}\dot{U}_2 + A_{22}\dot{I}_2} = \frac{A_{11}Z_2 + A_{12}}{A_{21}Z_2 + A_{22}} = \frac{A_{11}}{A_{21}} \cdot \frac{\dfrac{A_{12}}{A_{11}} + Z_2}{\dfrac{A_{22}}{A_{21}} + Z_2}.$$

In accordance with (29)

$$Z_{1\text{in}} = Z_{1o}\frac{Z_{2s} + Z_2}{Z_{2o} + Z_2}. \tag{7.77}$$

The input impedance of a four-terminal network from terminals *2*, when it is loaded by an impedance $Z_1$ from terminals *1*, is

$$Z_{2\text{in}} = \frac{A_{22}Z_1 + A_{12}}{A_{21}Z_1 + A_{11}} = \frac{A_{22}}{A_{21}} \cdot \frac{\dfrac{A_{12}}{A_{22}} + Z_1}{\dfrac{A_{11}}{A_{21}} + Z_1},$$

consequently, in accordance with (29)

$$Z_{2\text{in}} = Z_{2o}\frac{Z_{1s} + Z_1}{Z_{1o} + Z_1}. \tag{7.78}$$

The resulting linear-fractional transformations relating the values of the impedances at the terminals of the four-terminal network illustrate an important property of four-terminal networks – the ability to convert impedances.

It is convenient to use formulae (77) and (78) when the short-circuit and open-circuit parameters are known.

If the image parameters of a four-terminal network are known, the input impedances $Z_{1\text{in}}$ and $Z_{2\text{in}}$ can be determined from formulae (77) and (78), taking (53) into account:

$$Z_{1\text{in}} = Z_{1c}\frac{\dfrac{Z_2}{Z_{2c}} + \tanh g}{1 + \dfrac{Z_2}{Z_{2c}}\tanh g};$$

$$Z_{2\text{in}} = Z_{2c}\frac{\dfrac{Z_1}{Z_{1c}} + \tanh g}{1 + \dfrac{Z_1}{Z_{1c}}\tanh g}.$$

Denoting $Z_2/Z_{2c} = \tanh g_2$ and $Z_1/Z_{1c} = \tanh g_1$, we find

$$\left.\begin{aligned} g_1 &= \frac{1}{2}\ln\frac{Z_{1c} + Z_1}{Z_{1c} - Z_1}, \\ g_2 &= \frac{1}{2}\ln\frac{Z_{2c} + Z_2}{Z_{2c} - Z_2}, \\ Z_{1\text{in}} &= Z_{1c}\tanh(g + g_2), \\ Z_{2\text{in}} &= Z_{2c}\tanh(g + g_1). \end{aligned}\right\} \tag{7.79}$$

With matched loading $g_1 = g_2 = \infty$, $Z_{1\text{in}} = Z_{1c}$, $Z_{2\text{in}} = Z_{2c}$.

With a mismatched load and input impedance $Z_{1\text{in}}$ and $Z_{2\text{in}}$ vary in accordance with the law of change in the hyperbolic tangent of a complex argument (see expressions 79).

Let us call expressions $(Z_{1\text{in}} - Z_{1c})/(Z_{1\text{in}} + Z_{1c})$ the error between the input impedance and the characteristic impedance of the four-terminal network. From (50)

$$Z_{1\text{in}} = \frac{\sqrt{\left(\dfrac{Z_{1c}}{Z_{2c}}\right)}(Z_2\cosh g + Z_{2c}\sinh g)}{\sqrt{\left(\dfrac{Z_{2c}}{Z_{1c}}\right)}\left(\cosh g + \dfrac{Z_2}{Z_{2c}}\sinh g\right)}$$

$$= Z_{1c}\frac{Z_2\cosh g + Z_{2c}\sinh g}{Z_2\sinh g + Z_{2c}\cosh g}$$

$$= Z_{1c}\frac{(Z_2 + Z_{2c})e^{g} + (Z_2 - Z_{2c})e^{-g}}{(Z_2 + Z_{2c})e^{g} - (Z_2 - Z_{2c})e^{-g}}$$

$$= Z_{1c}\frac{1 + \dfrac{Z_2 - Z_{2c}}{Z_2 + Z_{2c}}e^{-2g}}{1 - \dfrac{Z_2 - Z_{2c}}{Z_2 + Z_{2c}}e^{-2g}}.$$

Let us agree to call $n_2 = (Z_2 - Z_{2c})/(Z_2 + Z_{2c})$ the *reflexion factor* at the output of the four-terminal network. Thus,

$$Z_{1\text{in}} = Z_{1c}\frac{1 + n_2 e^{-2g}}{1 - n_2 e^{-2g}}. \tag{7.80}$$

It follows from (80) that

$$\frac{Z_{1\text{in}} - Z_{1c}}{Z_{1\text{in}} + Z_{1c}} = n_2 e^{-2g}. \tag{7.81}$$

The absolute value of the error between the input impedance and the characteristic impedance of the network is

$$\left|\frac{Z_{1\text{in}} - Z_{1c}}{Z_{1\text{in}} + Z_{1c}}\right| = \left|\frac{Z_2 - Z_{2c}}{Z_2 + Z_{2c}}\right| e^{-2b}, \tag{7.82}$$

i.e. it is directly proportional to the reflexion factor and decreases as the image attenuation constant grows.

Similarly, it can be shown that

$$\left.\begin{aligned} Z_{2\text{in}} = Z_{2c}\frac{1 + n_1 e^{-2g}}{1 - n_1 e^{-2g}}; \quad \frac{Z_{2\text{in}} - Z_{2c}}{Z_{2\text{in}} + Z_{2c}} = n_1 e^{-2g}; \\ \left|\frac{Z_{2\text{in}} - Z_{2c}}{Z_{2\text{in}} + Z_{2c}}\right| = \left|\frac{Z_1 - Z_{1c}}{Z_1 + Z_{1c}}\right| e^{-2b}. \end{aligned}\right\} \tag{7.83}$$

Here $n_1 = (Z_1 - Z_{1c})/(Z_1 + Z_{1c})$ is the reflexion factor at the input of the network.

## 7. Effective Attenuation and Insertion Loss of a Four-terminal Network

An international consultative committee on telephony has recommended, for the measuring of transmission, the concepts of effective attenuation and insertion loss (or amplification) as the principal operating parameters to describe the power losses in a four-terminal network.

Let us suppose that a four-terminal network is connected between a voltage source with an internal impedance $Z_1$ and a load with an impedance $Z_2$.

By the *effective attenuation* of the network we understand a half of the natural logarithm or ten times the common logarithm of the ratio of the apparent power $S_{\max}$ which the source would supply direct to impedance

$Z_1$ (Fig. 10*a*), to the apparent power $S_2$ at the output of a four-terminal network loaded with an impedance $Z_2$:

$$b_{\text{eff}} = \frac{1}{2} \ln \frac{S_{\max}}{S_2}, \quad \text{nepers} \tag{7.84}$$

or

$$b_{\text{eff}} = 10 \lg \frac{S_{\max}}{S_2}, \quad \text{db.} \tag{7.85}$$

If the e.m.f. of the source is denoted by $U$ and the current in the impedance $Z_2$ by $I_2$, then

$$S_{\max} = \frac{U_2}{4z_1}$$

and

$$S_2 = z_2 I_2^2.$$

According to (84)

$$b_{\text{eff}} = \frac{1}{2} \ln \frac{U^2}{4z_1 z_2 I_2^2} = \ln \frac{U}{2I_2 \sqrt{(z_1 z_2)}}. \tag{7.86}$$

Thus the effective attenuation serves to measure transmission, estimating the actual conditions of transmission compared with conditions when there

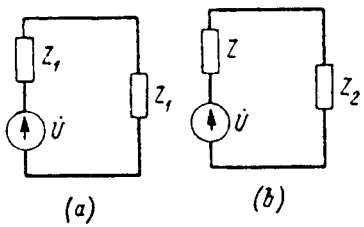



FIG. 10. Circuits for calculating effective attenuation (*a*) and insertion loss (*b*).

s no four-terminal network and the impedance is equal to the internal impedance of the source.

It can be shown (Ref. 1) that

$$b_{\text{eff}} = b + \ln \left| \frac{Z_1 + Z_{1c}}{2\sqrt{(Z_1 Z_{1c})}} \right| + \ln \left| \frac{Z_2 + Z_{2c}}{2\sqrt{(Z_2 Z_{2c})}} \right| + \ln |1 - n_1 n_2 e^{-2a}|. \tag{7.87}$$

Here

$$n_1 = \frac{Z_1 - Z_{1c}}{Z_1 + Z_{1c}}, \quad n_2 = \frac{Z_2 - Z_{2c}}{Z_2 + Z_{2c}}.$$

Consequently, effective attenuation consists of four components. The first component is the image attenuation constant of the network; the second and third are the losses as a result of reflexion at the input and output of the network; and the fourth is the attenuation caused by the interaction of the reflexions at the network's input and output.

If $Z_1 = Z_{1c}$ and $Z_2 = Z_{2c}$, then $b_{\text{eff}} = b$, i.e. when there is perfect matching at the input and output of the network, the effective attenuation equals the network's image attenuation constant.

By the *insertion loss* of a four-terminal network we understand a half of the natural logarithm or ten times the common logarithm (i.e. a logarithm to the base 10) of the ratio of the apparent power $S_1$ which the source would supply direct to impedance $Z_2$ (Fig. 10*b*), to the apparent power $S_2$ at the output of a four-terminal network loaded with an impedance $Z_2$:

$$b_{\text{ins}} = \frac{1}{2} \ln \frac{S_1}{S_2}, \quad \text{nepers} \tag{7.88}$$

or

$$b_{\text{ins}} = 10 \lg \frac{S_1}{S_2}, \quad \text{db.} \tag{7.89}$$

Here

$$S_1 = \frac{U^2 z_2}{|Z_1 + Z_2|^2};$$

consequently, according to (88)

$$b_{\text{ins}} = \frac{1}{2} \ln \frac{U^2 z_2}{|Z_1 + Z_2|^2 z_2 I_2^2} = \ln \frac{U}{|Z_1 + Z_2| I_2}. \tag{7.90}$$

The difference between the values of the effective attenuation and the insertion loss, which is determined in accordance with (86) and (90) as

$$b_{\text{eff}} - b_{\text{ins}} = \ln \frac{U}{2I_2 \sqrt{(z_1 z_2)}} - \ln \frac{U}{I_2 |Z_1 + Z_2|}$$

$$= \ln \frac{|Z_1 + Z_2|}{2 \sqrt{(z_1 z_2)}}, \tag{7.91}$$

is known as the attenuation caused by the mismatching of the internal impedance of the source and the impedance of the load.

The insertion loss is a means of estimating a change in transmission conditions when the network is connected between a source and a load. It is a more general measure than the effective attenuation, which is a particular case of insertion loss: when the impedances of the source and the load are equal ($Z_1 = Z_2$), the attenuation caused by the mismatching of

the internal impedance of the source and the impedance of the load reverts to zero and the insertion loss becomes equal to the effective attenuation.

When the connexion of the impedances at the input and the output of a symmetrical four-terminal network is perfectly matched, i.e. when $Z_1 = Z_2 = Z_{1c} = Z_{2c}$, the insertion loss equals the image attenuation constant of the network.

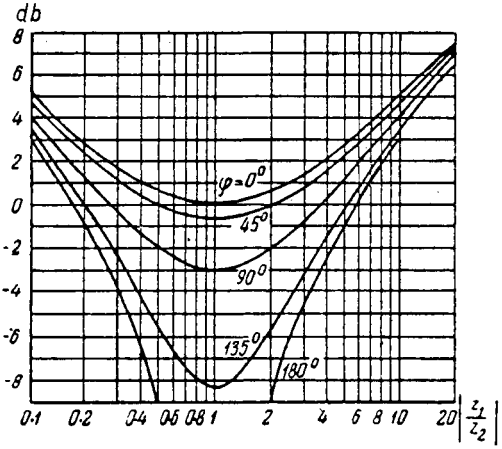



FIG. 11. Graph of attenuation.

Figure 11 shows a graph of attenuation caused by mismatching $Z_1$ and $Z_2$, constructed with respect to $Z_1/Z_2$ for various angles $\varphi$ ($\varphi$ is the argument of the complex value $Z_1/Z_2$) in accordance with the formula

$$20 \lg \left| \frac{Z_1 + Z_2}{2\sqrt{(Z_1 Z_2)}} \right| = 20 \lg \frac{1}{2} \left| \sqrt{\left(\frac{Z_1}{Z_2}\right)} + \sqrt{\left(\frac{Z_2}{Z_1}\right)} \right|, \quad \text{db.}$$

The same graph can be used to find losses as a result of the reflexions at the input and output of the network, as long as the structure of the attenuation expressions in formulae (87) and (91) is the same.

## 8. Cascade of Unsymmetrical Networks on the Image-Impedance Basis

Four-terminal networks connected in such a way that the input terminals of one network are connected to the output terminals of the next are said to be in a *cascade connexion.* This kind of connexion is most widely used since networks through which electrical energy is transmitted (communication channels, etc.) are usually formed of separate sections in a sequence.

The cascade connexion of four-terminal networks, which is based on matching the image impedances (matched cascading), consists in the image impedances being identical at each junction of the networks. Figure 12 shows the cascade connexion of two four-terminal networks.

If the load's impedance is matched with the output image impedance $Z_{3c}$ of the second four-terminal network, the impedance at the input terminals of the second network equals the image impedance ($Z_{2c}$). This im-

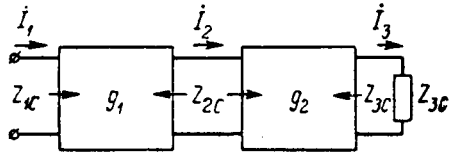



FIG. 12. Cascade of four-terminal networks on the image-impedance basis.

pedance is a matched load for the first network. The impedance at the input of the first network is therefore also equal to the image impedance ($Z_{1c}$).

It follows from this that four-terminal networks joined in matched cascading connexion can be replaced by a composite four-terminal network with image impedances equal to the input image impedance of the first network and the output image impedance of the last (Fig. 13). The propagation function of the composite four-terminal network is determined by the

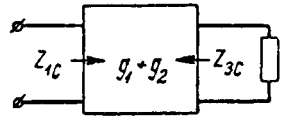



FIG. 13. Over-all four-terminal network (for matched cascading).

algebraic sum of the propagation functions of the component four-terminal networks.

With respect to the network in Fig. 12, according to (51)

$$\dot{I}_2 = \sqrt{\left(\frac{Z_{3c}}{Z_{2c}}\right)}\dot{I}_3 e^{g_2}, \quad \dot{I}_1 = \sqrt{\left(\frac{Z_{2c}}{Z_{1c}}\right)}\dot{I}_2 e^{g_1},$$

from which

$$\dot{I}_1 = \sqrt{\left(\frac{Z_{3c}}{Z_{1c}}\right)}\dot{I}_3 e^{g_1+g_2}. \tag{7.92}$$

The formula obtained confirms what was stated above: the over-all network has image impedances $Z_{1c}$ and $Z_{3c}$ and a propagation function $g = g_1 + g_2$.

Correspondingly, the image attenuation constant of the network is $b = b_1 + b_2$ and the image phase function $a = a_1 + a_2$.

The matched cascading is of great importance in radio engineering, line communication and automation.

Additional losses occur in a network with a mismatched connexion (Section 7).

## 9. Equations of Composite Four-terminal Networks in Matrix Form

In order to obtain the parameters of a four-terminal network composed of a certain combination of simpler four-terminal networks, the parameters of which are given, it is convenient to use a matrix transcription.

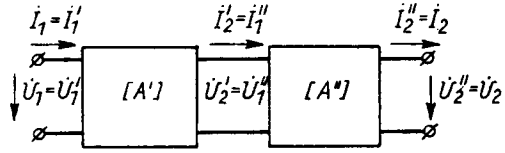



FIG. 14. Cascade connexion.

Depending on how the composite four-terminal network is connected, one or other form of transcribing the equations is used, namely:

(1) the $[A]$ form for cascade connexion (Fig. 14);
(2) the $[Z]$ form for series connexion (Fig. 15);
(3) the $[Y]$ form for parallel connexion (Fig. 16);
(4) the $[H]$ form for series–parallel connexion (Fig. 17);
(5) the $[H]^{-1}$ form for parallel–series connexion (Fig. 18).

### Cascade Connexion of Four-terminal Networks

The equations for the component four-terminal networks in matrix form $[A]$ are

$$\begin{bmatrix} \dot{U}_1' \\ \dot{I}_1' \end{bmatrix} = \begin{bmatrix} A_{11}' & A_{12}' \\ A_{21}' & A_{22}' \end{bmatrix} \begin{bmatrix} \dot{U}_2' \\ \dot{I}_2' \end{bmatrix} = [A'] \begin{bmatrix} \dot{U}_2' \\ \dot{I}_2' \end{bmatrix},$$

$$\begin{bmatrix} \dot{U}_1'' \\ \dot{I}_1'' \end{bmatrix} = \begin{bmatrix} A_{11}'' & A_{12}'' \\ A_{21}'' & A_{22}'' \end{bmatrix} \begin{bmatrix} \dot{U}_2'' \\ \dot{I}_2'' \end{bmatrix} = [A''] \begin{bmatrix} \dot{U}_2'' \\ \dot{I}_2'' \end{bmatrix}.$$

Here, the quantities which relate to the first four-terminal network are primed and those which relate to the second are double-primed.

With a cascade connexion $\dot{U}_1 = \dot{U}_1'$, $\dot{I}_1 = \dot{I}_1'$, $\dot{U}_2 = \dot{U}_2''$, $\dot{I}_2 = \dot{I}_2''$, $\dot{U}_2' = \dot{U}_1''$, $\dot{I}_2' = \dot{I}_1''$; consequently,

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = [A'][A''] \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix} = [A] \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix}. \tag{7.93}$$

Thus, matrix $[A]$ of the composite network equals the product of the matrices of the component four-terminal networks. The product of the matrices is obtained by the so-called 'row-by-column' rule: the element of the resulting matrix lying on the intersection of the $i$-th row and the $k$-th column equals the sum of the products of the elements of the $i$-th row of the first matrix with the elements of the $k$-th column of the second matrix. Consequently,

$$[A] = \begin{bmatrix} A_{11}'A_{11}'' + A_{12}'A_{21}'' & A_{11}'A_{12}'' + A_{12}'A_{22}'' \\ A_{21}'A_{11}'' + A_{22}'A_{21}'' & A_{21}'A_{12}'' + A_{22}'A_{22}'' \end{bmatrix}.$$

The above rule for finding the matrix of the over-all network is also applicable to the cascade connexion of any number of four-terminal networks.

The matrices to be multiplied are then written in the order of the succession of the corresponding four-terminal networks, since the product of matrices is generally not commutative.

**Series Connexion of Four-terminal Networks**

The equations for the component four-terminal networks in matrix form $[Z]$ are

$$\begin{bmatrix} \dot{U}_1' \\ \dot{U}_2' \end{bmatrix} = [Z'] \begin{bmatrix} \dot{I}_1' \\ \dot{I}_2' \end{bmatrix}, \quad \begin{bmatrix} \dot{U}_1'' \\ \dot{U}_2'' \end{bmatrix} = [Z''] \begin{bmatrix} \dot{I}_1'' \\ \dot{I}_2'' \end{bmatrix}.$$

If

$$\dot{I}_1 = \dot{I}_1' = \dot{I}_1'', \qquad \dot{I}_2 = \dot{I}_2' = \dot{I}_2'',$$

$$\dot{U}_1 = \dot{U}_1' + \dot{U}_1'', \quad \dot{U}_2 = \dot{U}_2' + \dot{U}_2'',$$

then

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \Big\{ [Z'] + [Z''] \Big\} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = [Z] \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix}. \tag{7.94}$$

Thus, matrix $[Z]$ of the composite four-terminal network is equal to the sum of the matrices of the component four-terminal networks. The sum

of the matrices is obtained by adding the like elements

$$[Z] = \begin{bmatrix} Z'_{11} + Z''_{11} & Z'_{12} + Z''_{12} \\ Z'_{21} + Z''_{21} & Z'_{22} + Z''_{22} \end{bmatrix}.$$

For the matrix addition to be valid, the current entering one terminal of a part must be the same as that leaving the other terminal. A network

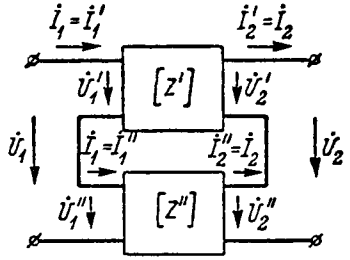



FIG. 15. Series connexion.

interconnexion which satisfies this condition is called a *regular connexion* (see Section 10).

**Parallel Connexion of Four-terminal Networks**

The equations for the component four-terminal network in matrix form $[Y]$ are

$$\begin{bmatrix} \dot{I}'_1 \\ \dot{I}'_2 \end{bmatrix} = [Y'] \begin{bmatrix} \dot{U}'_1 \\ \dot{U}'_2 \end{bmatrix}, \quad \begin{bmatrix} \dot{I}''_1 \\ \dot{I}''_2 \end{bmatrix} = [Y''] \begin{bmatrix} \dot{U}''_1 \\ \dot{U}''_2 \end{bmatrix}.$$

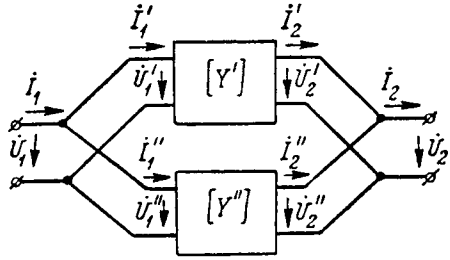



FIG. 16. Parallel connexion.

When four-terminal networks are connected in parallel

$$\dot{U}_1 = \dot{U}'_1 = \dot{U}''_1, \quad \dot{U}_2 = \dot{U}'_2 = \dot{U}''_2,$$
$$\dot{I}_1 = \dot{I}'_1 + \dot{I}''_1, \quad \dot{I}_2 = \dot{I}'_2 + \dot{I}''_2.$$

Consequently,

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \{[Y'] + [Y'']\} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = [Y] \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix}. \tag{7.95}$$

Matrix $[Y]$ of the over-all network is thus equal to the sum of the matrices of the component four-terminal networks

$$[Y] = \begin{bmatrix} Y'_{11} + Y''_{11} & Y'_{12} + Y''_{12} \\ Y'_{21} + Y''_{21} & Y'_{22} + Y''_{22} \end{bmatrix}.$$

As in the previous case it is then supposed that the condition of the regularity of the networks' connexion is observed, and hence the matrix method is valid.

### Series–parallel Connexion of Four-terminal Networks

The equations for the component four-terminal networks in matrix form $[H]$ are

$$\begin{bmatrix} \dot{U}'_1 \\ \dot{I}'_2 \end{bmatrix} = [H'] \begin{bmatrix} \dot{I}'_1 \\ \dot{U}'_2 \end{bmatrix}, \quad \begin{bmatrix} \dot{U}''_1 \\ \dot{I}''_2 \end{bmatrix} = [H''] \begin{bmatrix} \dot{I}''_1 \\ \dot{U}''_2 \end{bmatrix}.$$

If

$$I_1 = \dot{I}'_1 = \dot{I}''_1, \qquad I_2 = \dot{I}'_2 + \dot{I}''_2,$$

$$\dot{U}_1 = \dot{U}'_1 + \dot{U}''_1, \quad \dot{U}_2 = \dot{U}'_2 = \dot{U}''_2,$$

then

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_2 \end{bmatrix} \{[H'] + [H'']\} \begin{bmatrix} \dot{I}_1 \\ \dot{U}_2 \end{bmatrix} = [H] \begin{bmatrix} \dot{I}_1 \\ \dot{U}_2 \end{bmatrix}. \tag{7.96}$$

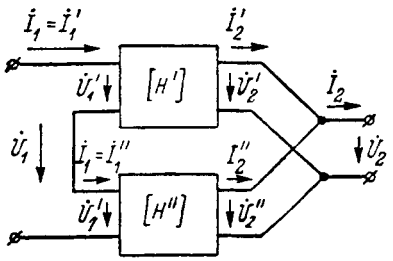



FIG. 17. Series–parallel connexion.

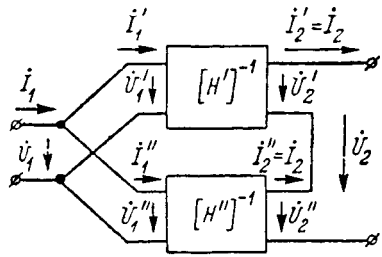



FIG. 18. Parallel–series connexion.

Consequently, matrix $[H]$ of the over-all network is equal to the sum of the matrices of the component four-terminal networks

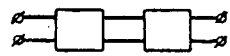
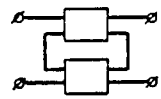
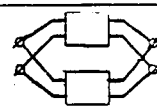
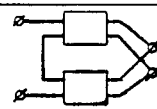
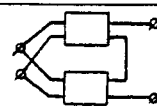
$$[H] = \begin{bmatrix} H'_{11} + H''_{11} & H'_{12} + H''_{12} \\ H'_{21} + H''_{21} & H'_{22} + H''_{22} \end{bmatrix}.$$

### Parallel–Series Connexion of Four-terminal Networks

By analogy with the above, matrix $[H]^{-1}$ of the composite four-terminal network is determined as the sum of the matrices of the component four-terminal networks.

The rules for finding the matrices of composite four-terminal networks are given in Table 3.

TABLE 3. MATRIX EQUATIONS FOR INTERCONNECTED FOUR-TERMINAL NETWORKS

| Connexion | Circuit diagram | Matrix equation |
|---|---|---|
| Cascade |  | $[A] = [A'] \cdot [A'']$ |
| Series |  | $[Z] = [Z'] + [Z'']$ |
| Parallel |  | $[Y] = [Y'] + [Y'']$ |
| Series–parallel |  | $[H] = [H'] + [H'']$ |
| Parallel–series |  | $[H]^{-1} = [H']^{-1} + [H'']^{-1}$ |

## 10. Regular Interconnexions of Four-terminal Networks

Attention has been devoted to the question of *regularity* in the connexion of four-terminal networks in a number of Soviet and foreign works (Refs. 6, 15, 24, 65 and 78).

For the series, parallel, series–parallel and parallel–series connexion of four-terminal networks, the condition of the equality of the currents entering

and leaving at each pair of terminals of the component four-terminal networks may be violated as a result of the electrical interconnexions formed between the networks. Before adding the matrices in accordance with Table 3, it is therefore necessary to ascertain whether the prescribed interconnexion of the networks is regular.

When four-terminal networks are connected in series the Fig. 19 circuits are used for checking.

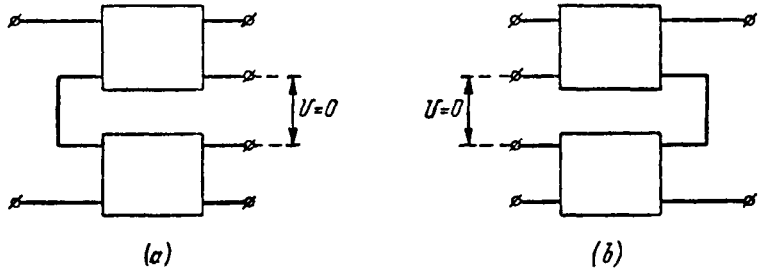



FIG. 19. Diagrams illustrating the validity tests for matrix analysis of series connexion.

The first test corresponds to *forward transmission*, when the left terminals of the four-terminal network are connected in series and the right are open (Fig. 19*a*). The second test corresponds to *reverse transmission*, when the right terminals are connected in series and the left are open (Fig. 19*b*). The voltage $U$ between the lower open terminals of one network and the upper open terminals of the other should be zero. If not, the connexion is not regular.

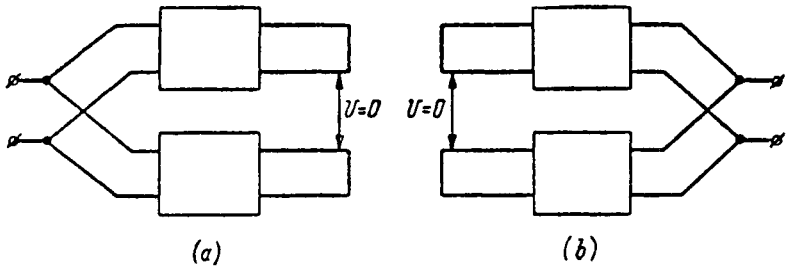



FIG. 20. Diagrams for parallel connexion.

The Fig. 20 circuits are used to check four-terminal networks connected in parallel.

For forward transmission the networks' left terminals are first of all connected in parallel and the right terminals are short-circuited (Fig. 20*a*). Then, for reverse transmission, the right terminals are connected in parallel and the left terminals are short-circuited (Fig. 20*b*). In both cases the vol-

tage $U$ between the short-circuited terminals of the two networks should be zero.

When four-terminal networks have a series–parallel connexion they are checked by the tests shown in Fig. 21.

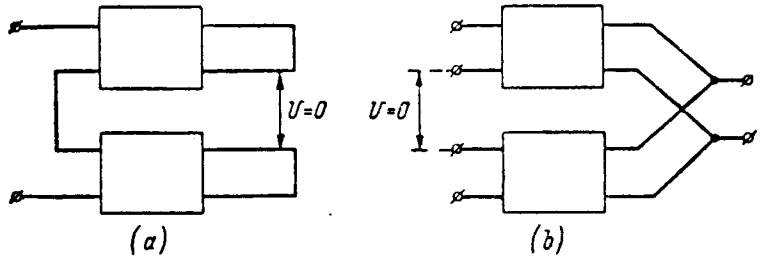



FIG. 21. Diagrams for series-parallel connexion.

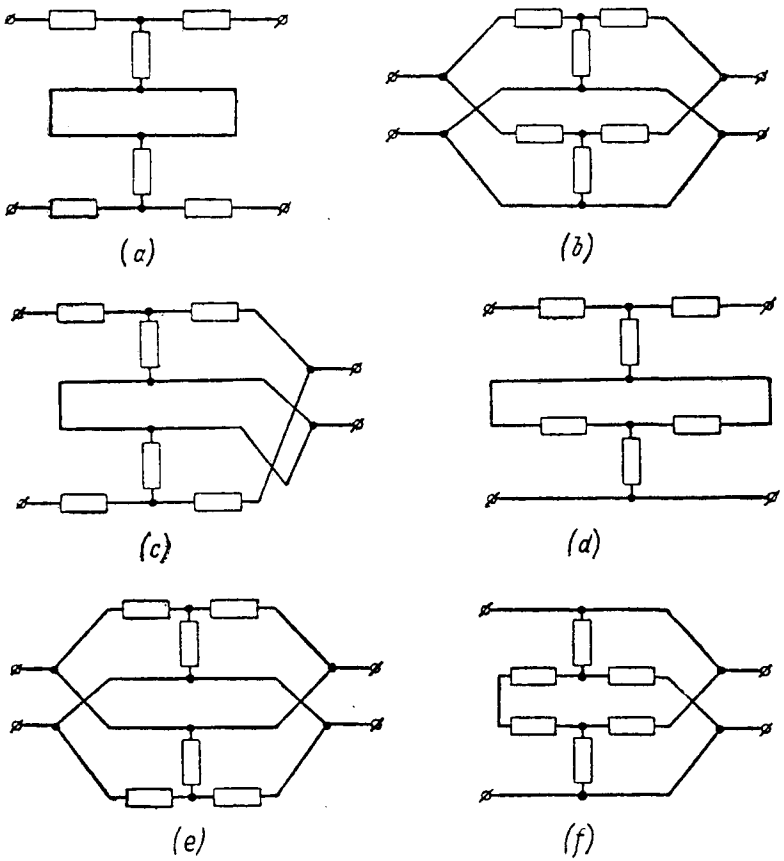



FIG. 22. Examples of connexions for T-networks ($a,b,c$) regular; ($d,e,f$) non-regular.

For forward transmission the networks' left terminals are connected in series and the right terminals are short-circuited (Fig. 21*a*). For reverse transmission the right terminals are connected in parallel and the left terminals are open (Fig. 21*b*). In both cases voltage $U$ is zero if the condition of regularity is observed.

Figure 22*a*, *b* and *c* shows examples of regular connexions of T-networks, and Fig. 22*d*, *e* and *f* gives examples of non-regular connexions.

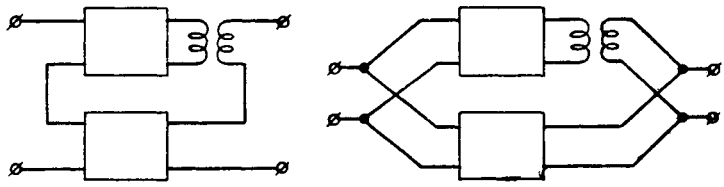


FIG. 23. Examples of regular connexions with intermediate transformers.

The regularity of network interconnections is ensured by intermediate transformers (Fig. 23), and in a number of cases ideal transformers with a transformer ratio of 1:1 can be considered. In Section 17 a transformer, together with its matrix, is examined as a four-terminal network.

## 11. Single-element Four-terminal Networks

The simplest four-terminal networks are single-element four-terminal networks consisting of a series (Fig. 24*a*) or parallel impedance (Fig. 24*b*).

The equations in the $[A]$ form for the first of these are written as follows:

$$\dot{U}_1 = \dot{U}_2 + Z\dot{I}_2,$$

$$\dot{I}_1 = \dot{I}_2,$$

or, the same thing,

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix}. \tag{7.97}$$

Using Table 1, the matrix equation of the same network in the $[Y]$ form can be given as

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{Z} & -\frac{1}{Z} \\ \frac{1}{Z} & -\frac{1}{Z} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix}. \tag{7.98}$$

In the given case a transcription of the equations in the [Z] form is not applicable since the elements of the [Z] matrix equal infinity.

The equation in the [A] form for a single-element four-terminal network with a parallel impedance is written as follows:

$$\dot{U}_1 = \dot{U}_2,$$

$$\dot{I}_1 = \frac{\dot{U}_2}{Z} + \dot{I}_2,$$

or, the same thing,

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z} & 1 \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix}. \tag{7.99}$$

Using Table 1, the matrix equation of the same four-terminal network in the [Z] form can be given as

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} Z & -Z \\ Z & -Z \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix}. \tag{7.100}$$

In this case a transcription of the equations in the [Y] form is not applicable since the elements of the [Y] matrix equal infinity.

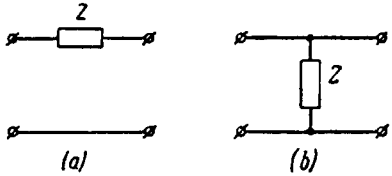



FIG. 24. Single-element four-terminal networks: (*a*) series connexion; (*b*) parallel connexion.

If we suppose that $Z = 0$ in the first network (Fig. 24*a*) or, the same thing, that $Z = \infty$ in the second network (Fig. 24*b*), an equation in the [A] form is obtained:

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix}, \tag{7.101}$$

which corresponds to the *direct (non-crossed) connexion* shown in Fig. 25*a*.

It is easy to confirm that the equation in the [$A$] form corresponds to the *crossed connexion* (Fig. 25$b$).

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} -\dot{U}_2 \\ -\dot{I}_2 \end{bmatrix}. \tag{7.102}$$

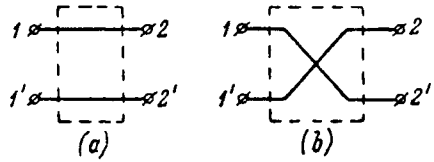



FIG. 25. Connexions: (*a*) direct; (*b*) crossed.

Therefore, when wires are crossed at the input or the output of any four-terminal network, its [$A$] matrix is multiplied by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ which is equivalent to reversing the signs of all the elements of the [$A$] matrix. The signs for $Z_{12}$, $Z_{21}$, $Y_{12}$, $Y_{21}$, $H_{12}$ and $H_{21}$ are then reversed.

## 12. L-Network

The $A$, $Z$, $Y$ and $H$ parameters of an L-network (Fig. 26) can be obtained directly from the formulae given in Section 2.

For example, for the Fig. 26$a$ circuit

$$A_{11} = \left(\frac{\dot{U}_1}{\dot{U}_2}\right)_{I_2=0} = 1,$$

$$A_{22} = \left(\frac{\dot{I}_1}{\dot{I}_2}\right)_{U_2=0} = \frac{\dot{I}_1}{\dot{I}_1 \dfrac{Z_a}{Z_a + Z_c}} = 1 + \frac{Z_c}{Z_a},$$

$$A_{12} = \left(\frac{\dot{U}_1}{\dot{I}_2}\right)_{U_2=0} = \frac{U_1}{\dfrac{U_1}{Z_c}} = Z_c, \quad A_{21} = \left(\frac{\dot{I}_1}{\dot{U}_2}\right)_{I_2=0} = \frac{1}{Z_a}.$$

The remaining parameters can be calculated analogously.

Another method of finding the parameters of an L-network consists in representing the latter in the form of a cascade of the two simplest single-element two-terminal networks — a parallel and a series impedance (Fig. 26$a$) or a series and a parallel impedance (Fig. 26$b$).

In accordance with (93), (97) and (99) the $[A]$ matrix of the L-network in Fig. 26*a* equals

$$[A] = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_a} & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & Z_c \\ \frac{1}{Z_a} & 1 + \frac{Z_c}{Z_a} \end{bmatrix}, \tag{7.103}$$

and in 26*b*

$$[A] = \begin{bmatrix} 1 & Z_b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_a} & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{Z_b}{Z_a} & Z_b \\ \frac{1}{Z_a} & 1 \end{bmatrix}. \tag{7.104}$$

The $[Z]$, $[Y]$, $[H]$ and $[H]^{-1}$ matrices can be obtained from (103) and (104) with the help of Table 1.

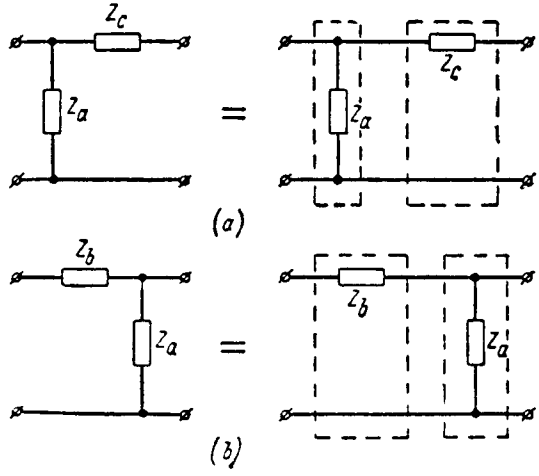



FIG. 26. L-network.

For example, for the Fig. 26*a* circuit

$$[Z] = \begin{bmatrix} Z_a & -Z_a \\ Z_a & -(Z_a + Z_c) \end{bmatrix};$$

$$[Y] = \begin{bmatrix} \frac{1}{Z_a} + \frac{1}{Z_c} & -\frac{1}{Z_c} \\ \frac{1}{Z_c} & -\frac{1}{Z_c} \end{bmatrix} \tag{7.105}$$

and for the Fig. 26*b* circuit

$$[Z] = \begin{bmatrix} Z_a + Z_b & -Z_a \\ Z_a & -Z_a \end{bmatrix};$$

$$[Y] = \begin{bmatrix} \frac{1}{Z_b} & -\frac{1}{Z_b} \\ \frac{1}{Z_b} & -\left(\frac{1}{Z_a} + \frac{1}{Z_b}\right) \end{bmatrix}. \tag{7.106}$$

The image parameters of an L-network can be obtained from formulae (37) and (38) or (40).

For the Fig. 26*a* circuit

$$\left.\begin{aligned} Z_{1c} = Z_a\sqrt{\left(\frac{Z_c}{Z_a + Z_c}\right)}, \quad Z_{2c} = \sqrt{[Z_c(Z_a + Z_c)]}, \\ \cosh g = \sqrt{\left(1 + \frac{Z_c}{Z_a}\right)} \end{aligned}\right\} \tag{7.107}$$

and for the Fig. 26*b* circuit

$$\left.\begin{aligned} Z_{1c} = \sqrt{[Z_b(Z_a + Z_b)]}, \quad Z_{2c} = Z_a\sqrt{\left(\frac{Z_b}{Z_a + Z_b}\right)}, \\ \cosh g = \sqrt{\left(1 + \frac{Z_b}{Z_a}\right)}. \end{aligned}\right\} \tag{7.108}$$

For the calculation of electrical filters and in a number of other cases the Fig. 27*a* and *b* circuits are taken as the initial L-network circuits, the

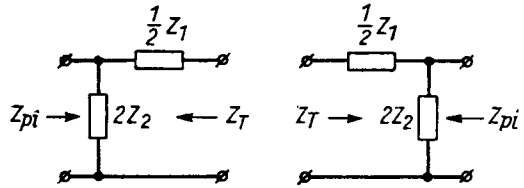



FIG. 27. L-network.

propagation function of the L-network being denoted by $g/2$ so that when two such four-terminal networks are matched in cascade connexion a T- or pi-network with a propagation function $g$ is obtained (see Sections 13 and 14).

The image impedance seen from the parallel-arm side is then denoted by $Z_{pi}$ and that from the series-arm side by $Z_T$.

From (107) and (108)

$$\left.\begin{aligned} Z_T &= \sqrt{\left[Z_1 Z_2\left(1+\frac{Z_1}{4Z_2}\right)\right]}, \quad Z_{p1} = \sqrt{\left(\frac{Z_1 Z_2}{1+\frac{Z_1}{4Z_2}}\right)}, \\ \cosh\frac{g}{2} &= \sqrt{\left(1+\frac{Z_1}{4Z_2}\right)}, \quad \sinh\frac{g}{2} = \sqrt{\left(\frac{Z_1}{4Z_2}\right)}. \end{aligned}\right\} \tag{7.109}$$

These formulae are used in filter theory (Chapter VIII).

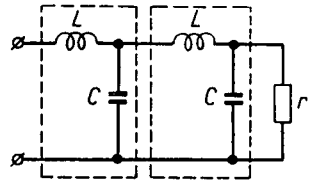



FIG. 28. Example 1.

EXAMPLE 1. To determine the transfer function $K_U$ of the Fig. 28 circuit from (76). A load $r$ is connected to the output of a four-terminal network which can be regarded as the cascade connexion of two L-networks. The $[A]$ matrix of the composite network is equal to

$$[A] = \begin{bmatrix} 1-\omega^2 LC & j\omega L \\ j\omega C & 1 \end{bmatrix}^2 = \begin{bmatrix} 1-3\omega^2 LC+\omega^4 L^2C^2 & j(2\omega L-\omega^3 L^2 C) \\ j(2\omega C-\omega^3 LC^2) & 1-\omega^2 LC \end{bmatrix}.$$

Consequently,

$$A_{11} = 1-3\omega^2 LC+\omega^4 L^2C^2; \quad A_{12} = j(2\omega C-\omega^3 LC^2).$$

Since the impedance of the load $Z_2 = r$, then according to (76)

$$K_U = \frac{1}{A_{11}+A_{12}\frac{1}{Z_2}} = \frac{1}{1-3\omega^2 LC+\omega^4 L^2C^2+j\frac{2\omega C-\omega^3 LC^2}{r}}.$$

## 13. T-Network

The $A$ parameters of a T-network (Fig. 29*a*) are either found directly from the formulae given in Section 2 or by multiplying the $[A]$ matrices of two four-terminal networks, which, connected in cascade, form a T-network

$$\begin{aligned} [A] &= \begin{bmatrix} 1 & Z_b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & Z_c \\ \frac{1}{Z_a} & 1+\frac{Z_c}{Z_a} \end{bmatrix} \\ &= \begin{bmatrix} 1+\frac{Z_b}{Z_a} & Z_b+Z_c+\frac{Z_bZ_c}{Z_a} \\ \frac{1}{Z_a} & 1+\frac{Z_c}{Z_a} \end{bmatrix} \end{aligned} \tag{7.110}$$

or, the same thing,

$$[A] = \begin{bmatrix} 1 + \dfrac{Z_b}{Z_a} & Z_b \\ \dfrac{1}{Z_a} & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_c \\ 0 & 1 \end{bmatrix}.$$

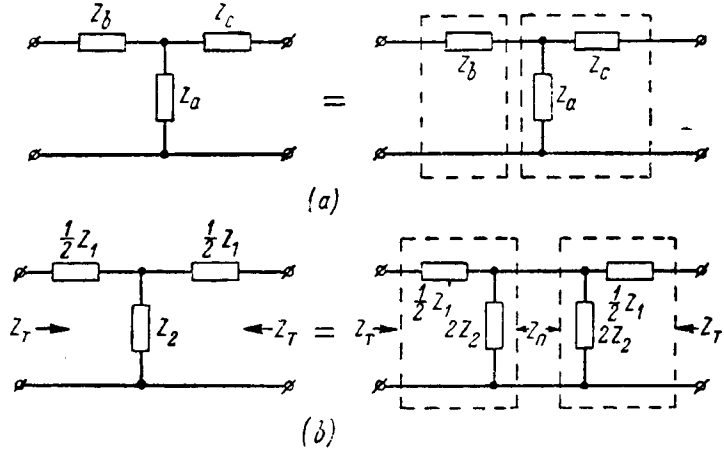



FIG. 29. T-network: (*a*) unsymmetrical; (*b*) symmetrical (composed of two L-networks).

The image parameters of an unsymmetrical T-network are found from formulae (37)–(40) or (53):

$$\left.\begin{aligned} Z_{1c} &= \sqrt{\left[\frac{(Z_a + Z_b)(Z_aZ_b + Z_bZ_c + Z_cZ_a)}{Z_a + Z_c}\right]}, \\ Z_{2c} &= \sqrt{\left[\frac{(Z_a + Z_c)(Z_aZ_b + Z_bZ_c + Z_cZ_a)}{Z_a + Z_b}\right]}, \\ \cosh g &= \frac{\sqrt{[(Z_a + Z_b)(Z_a + Z_c)]}}{Z_a}. \end{aligned}\right\} \qquad (7.111)$$

The symmetrical T-network shown in Fig. 29*b* is composed of two L-networks connected in cascade with image matching. Therefore, in accordance with (109), the resulting T-network has an image impedance,

$$Z_T = \sqrt{\left[Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)\right]}$$

and a propagation function $g$ twice that of an L-network.

Taking (109) and the formula

$$\cosh g = 1 + 2\sinh^2 \frac{g}{2},$$

into account, we find

$$\cosh g = A_{11} = 1 + \frac{Z_1}{2Z_2}. \tag{7.112}$$

The same result is obtained from (111) when

$$Z_a = Z_2, \quad Z_b = Z_c = \tfrac{1}{2}Z_1.$$

## 14. Pi-Network

The $A$ parameters of a pi-network (Fig. 30*a*) are found as above, either directly from the formulae given in Section 2, or by multiplying the $[A]$ matrices of two four-terminal networks which, connected in cascade, form a pi-network, e.g.

$$[A] = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_{ab}} & 1 \end{bmatrix} \begin{bmatrix} 1 + \dfrac{Z_{bc}}{Z_{ac}} & Z_{bc} \\ \dfrac{1}{Z_{ac}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \dfrac{Z_{bc}}{Z_{ac}} & Z_{bc} \\ \dfrac{1}{Z_{ab}} + \dfrac{1}{Z_{ac}} + \dfrac{Z_{bc}}{Z_{ab}Z_{ac}} & 1 + \dfrac{Z_{bc}}{Z_{ab}} \end{bmatrix}. \tag{7.113}$$

The image parameters of an unsymmetrical pi-network are found from formulae (37)–(40) or (53).

$$\left.\begin{aligned} Z_{1c} &= Z_{ab}\sqrt{\left[\frac{(Z_{bc} + Z_{ac})Z_{bc}}{(Z_{bc} + Z_{ab})(Z_{bc} + Z_{ac} + Z_{ab})}\right]}; \\ Z_{2c} &= Z_{ac}\sqrt{\left[\frac{(Z_{bc} + Z_{ab})Z_{bc}}{(Z_{bc} + Z_{ac})(Z_{bc} + Z_{ac} + Z_{ab})}\right]}; \\ \cosh g &= \sqrt{\left[\left(1 + \frac{Z_{bc}}{Z_{ac}}\right)\left(1 + \frac{Z_{bc}}{Z_{ab}}\right)\right]}. \end{aligned}\right\} \tag{7.114}$$

The symmetrical pi-network shown in Fig. 30*b* consists of two cascaded L-networks matched on an image-impedance basis.

Therefore, according to (109), the resulting pi-network has an image impedance $Z_{pi}$ and a propagation function $g$ twice that of an L-network; i.e. in accordance with (109) and (112) $Z_{pi} = \sqrt{[Z_1 Z_2/(1 + Z_1/4Z_2)]}$ and

$$\cosh g = A_{11} = 1 + \frac{Z_1}{2Z_2}.$$

The same result is obtained from (114) when

$$Z_{bc} = Z_1, \quad Z_{ab} = Z_{ac} = 2Z_2.$$

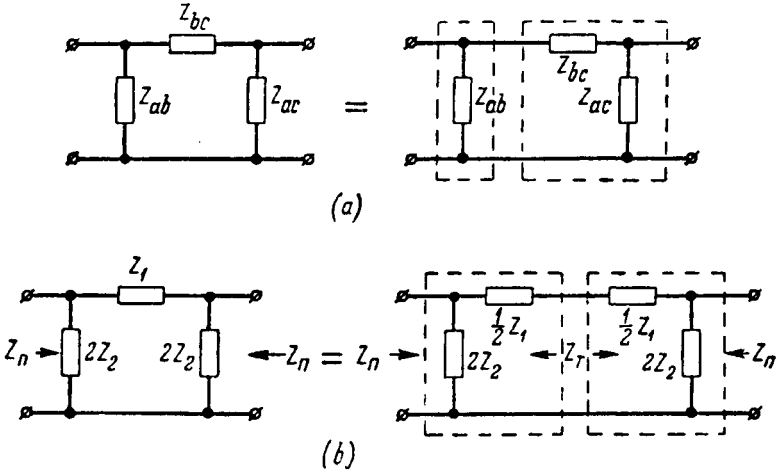



FIG. 30. Pi-network: (*a*) unsymmetrical; (*b*) symmetrical (composed of two L-networks).

## 15. Lattice (Bridge) Network

The lattice diagram in Fig. 31 *a* can be represented as the series–parallel connexion of two L-networks, the terminals of one of them being crossed.

In practice, *symmetrical* lattice networks in which $Z_1 = Z_3$ and $Z_2 = Z_4$ are most widely used as filters, correctors, etc.

A symmetrical lattice network can be represented in the form of the parallel connexion of two four-terminal networks, the terminals of one of them being crossed (Fig. 31 *b*). The $[Y]$ matrix of such a symmetrical lattice network is found by taking Section 11 into account:

$$[Y] = \begin{bmatrix} \frac{1}{2Z_1} & -\frac{1}{2Z_1} \\ \frac{1}{2Z_1} & -\frac{1}{2Z_1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2Z_2} & \frac{1}{2Z_2} \\ -\frac{1}{2Z_2} & -\frac{1}{2Z_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) & \frac{1}{2}\left(\frac{1}{Z_2} - \frac{1}{Z_1}\right) \\ \frac{1}{2}\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) & -\frac{1}{2}\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) \end{bmatrix}. \tag{7.115}$$

According to Table 1, the $Z$ and $A$ matrices of the same four-terminal network have the form

$$[Z] = \begin{bmatrix} \frac{1}{2}(Z_1 + Z_2) & \frac{1}{2}(Z_1 - Z_2) \\ \frac{1}{2}(Z_2 - Z_1) & -\frac{1}{2}(Z_1 + Z_2) \end{bmatrix}; \tag{7.116}$$

$$[A] = \begin{bmatrix} \frac{Z_1 + Z_2}{Z_2 - Z_1} & \frac{2Z_1Z_2}{Z_2 - Z_1} \\ \frac{2}{Z_2 - Z_1} & \frac{Z_1 + Z_2}{Z_2 - Z_1} \end{bmatrix}. \tag{7.117}$$

The open-circuit and short-circuit parameters of a symmetrical lattice network are

$$Z_o = \frac{Z_1 + Z_2}{2}, \quad Z_s = \frac{2Z_1Z_2}{Z_1 + Z_2}. \tag{7.118}$$

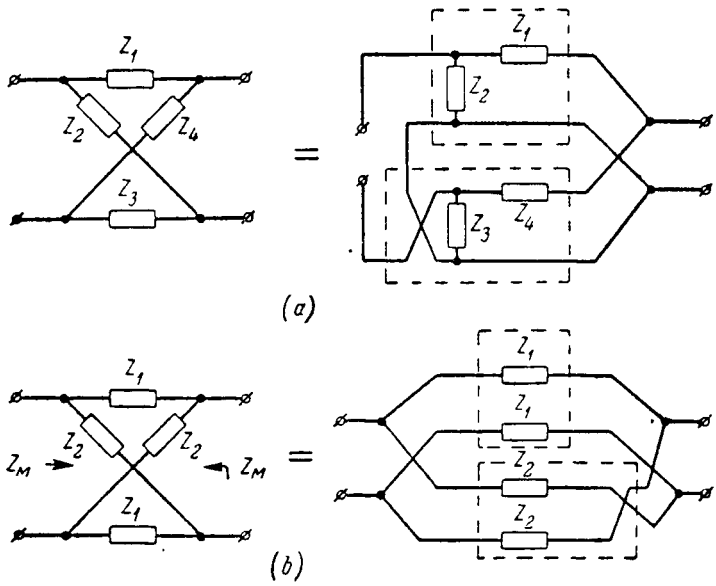



FIG. 31. Lattice networks: (*a*) unsymmetrical; (*b*) symmetrical.

Consequently, its image parameters in accordance with (53) equal, respectively,

$$Z_{\text{M}} = \sqrt{(Z_1 Z_2)}, \quad \tanh g = \frac{2\sqrt{(Z_1 Z_2)}}{Z_1 + Z_2}. \tag{7.119}$$

On the basis of (40) $\cosh g = (Z_1 + Z_2)/(Z_2 - Z_1)$, from which $\sinh g = 2\sqrt{(Z_1 Z_2)}/(Z_2 - Z_1)$ and

$$\left.\begin{aligned} \sinh \frac{g}{2} &= \sqrt{\left(\frac{\cosh g - 1}{2}\right)} = \sqrt{\left(\frac{Z_1}{Z_2 - Z_1}\right)}; \\ \cosh \frac{g}{2} &= \sqrt{\left(1 + \sinh^2 \frac{g}{2}\right)} = \sqrt{\left(\frac{Z_2}{Z_2 - Z_1}\right)}; \\ \tanh \frac{g}{2} &= \sqrt{\left(\frac{Z_1}{Z_2}\right)}. \end{aligned}\right\} \tag{7.120}$$

## 16. Bridged T-Network

A bridged T-network can be represented as the parallel connexion of a T-network with a four-terminal network consisting of one series impedance or as the series connexion of a pi-network with a four-terminal network consisting of one parallel impedance (Fig. 32*a*).

The $A$ matrix of the resulting four-terminal network is

$$[A] = \begin{bmatrix} \dfrac{M + (Z_a + Z_b)Z_d}{M + Z_a Z_d} & \dfrac{MZ_d}{M + Z_a Z_d} \\ \dfrac{Z_b + Z_c + Z_d}{M + Z_a Z_d} & \dfrac{M + (Z_a + Z_c)Z_d}{M + Z_a Z_d} \end{bmatrix}, \tag{7.121}$$

where $M = Z_a Z_b + Z_b Z_c + Z_a Z_c$.

In the case of a symmetrical bridged T-network where $Z_b = Z_c = \frac{1}{2}Z_1$, $Z_a = Z_2$ and $Z_d = Z_3$ (Fig. 32*b*), the image parameters equal, respectively,

$$Z_{\text{TM}} = \sqrt{\left(\frac{Z_1 Z_3 (Z_1 + 4Z_2)}{4(Z_1 + Z_3)}\right)},$$

$$\cosh g = 1 + \frac{2Z_1 Z_3}{Z_1^2 + 4Z_1 Z_2 + 4Z_2 Z_3}. \tag{7.122}$$

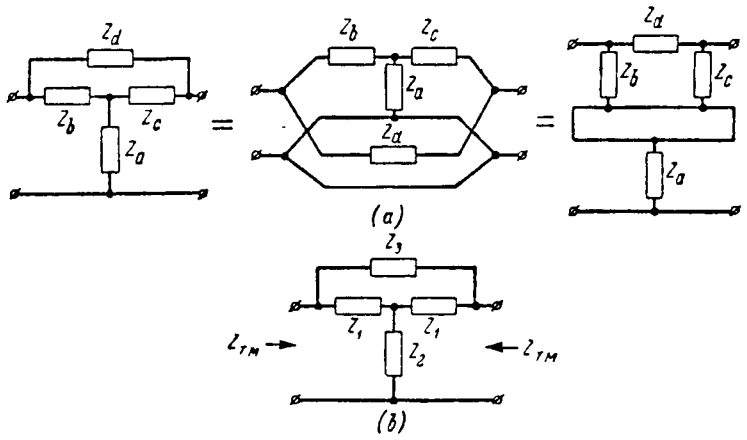



FIG. 32. Bridged T-network: (*a*) unsymmetrical; (*b*) symmetrical.

## 17. Ideal Transformer as a Four-terminal Network

We have already encountered the concept of an "ideal transformer" in Chapter IV, Section 12.

The $[A]$ matrix of the transformer shown in Fig. 33*a* has the form

$$[A] = \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}, \tag{7.123}$$

where $n$ is the transformer turns ratio.

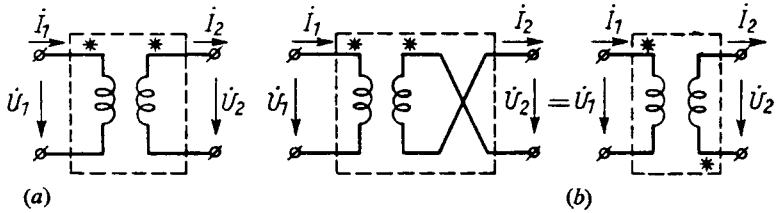



FIG. 33. Transformer as a four-terminal network.

Consequently, connecting an ideal transformer to the input or output of a given four-terminal network transforms the matrix in accordance with the formula

$$\begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} nA_{11} & nA_{12} \\ \frac{1}{n}A_{21} & \frac{1}{n}A_{22} \end{bmatrix} \tag{7.124}$$

or, correspondingly,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} nA_{11} & \frac{1}{n}A_{12} \\ nA_{21} & \frac{1}{n}A_{22} \end{bmatrix}. \tag{7.125}$$

According to Table 1 the [Z] and [Y] matrices of the resulting four-terminal network will have the following form:

when the transformer is connected to the input

$$[Z] = \begin{bmatrix} n^2 Z_{11} & nZ_{12} \\ nZ_{21} & Z_{22} \end{bmatrix};$$

$$[Y] = \begin{bmatrix} \frac{1}{n^2} Y_{11} & \frac{1}{n} Y_{12} \\ nY_{21} & Y_{22} \end{bmatrix}, \tag{7.126}$$

when the transformer is connected to the output

$$[Z] = \begin{bmatrix} Z_{11} & \frac{1}{n} Z_{12} \\ \frac{1}{n} Z_{21} & \frac{1}{n^2} Z_{22} \end{bmatrix};$$

$$[Y] = \begin{bmatrix} Y_{11} & nY_{12} \\ nY_{21} & n^2 Y_{22} \end{bmatrix}. \tag{7.127}$$

Crossing one pair of the transformer's terminals is equivalent to changing the polarity of the windings (Fig. 33*b*). The sign of the elements in the matrix of (123) then changes and the [*A*] matrix takes the form

$$[A] = \begin{bmatrix} -n & 0 \\ 0 & -\frac{1}{n} \end{bmatrix}. \tag{7.128}$$

Correspondingly, in the matrices of (124)–(127) the sign of the elements containing the factor $n$ or $1/n$ changes.

An ideal transformer with a transformation ratio of unity has a unit matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Connecting such a transformer at the input or output

of a given four-terminal network and, correspondingly, multiplying the unit matrix in any sequence by the $[A]$ matrix of the given network does not transform the latter matrix.

When $n = -1$, $\quad [A] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

## 18. Lattice Equivalent of a General Symmetrical Four-terminal Network

Equivalent four-terminal networks, i.e. those with unidentical circuits but with identical parameters, have already been dealt with. Thus, for example, it was shown in Section 3 that any reciprocal four-terminal network can be replaced by an equivalent T- or pi-network (Fig. 7). But such a network is not always physically realizable.

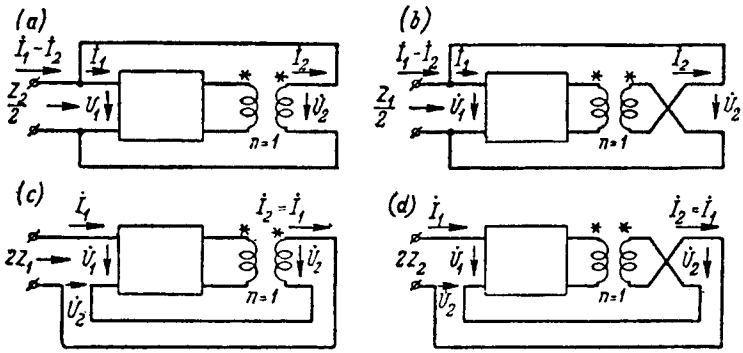



FIG. 34. Determining the impedances of the arms of a lattice equivalent to a given symmetrical four-terminal network.

It will be shown later that a lattice network is a physically realizable equivalent of any practicable symmetrical passive four-terminal network (Refs. 15, 24 and 97).

Let us connect an ideal transformer with a transformation ratio $n = 1$ at the output of a given symmetrical passive four-terminal network (Fig. 34). If the secondary winding of the transformer is connected in parallel to the input of the network, then with a direct connexion (Fig. 34*a*) we get

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix},$$

from which, in view of the equality $\dot{U}_1 = \dot{U}_2$,

$$\dot{I}_1 = (Y_{11} + Y_{12})\dot{U}_1, \quad \dot{I}_2 = (Y_{21} + Y_{22})\dot{U}_1.$$

Regarding the entire Fig. 34*a* circuit from the input terminals as a two-terminal network, we can find the input admittance of this two-terminal network:

$$Y_{in} = \frac{\dot{I}_1 - \dot{I}_2}{\dot{U}_1} = Y_{11} + Y_{12} - Y_{21} - Y_{22}$$

$$= 2(Y_{11} + Y_{12}).$$

When the connexion is crossed (Fig. 34*b*)

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & -Y_{12} \\ -Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix},$$

from which, taking the equality $\dot{U}_1 = \dot{U}_2$ into account,

$$\dot{I}_1 = (Y_{11} - Y_{12})\dot{U}_1, \quad \dot{I}_2 = (-Y_{21} + Y_{22})\dot{U}_1,$$

$$Y''_{in} = \frac{\dot{I}_1 - \dot{I}_2}{\dot{U}_1} = Y_{11} - Y_{12} + Y_{21} - Y_{22} = 2(Y_{11} - Y_{12}).$$

Consequently,

$$Y_{11} = \frac{Y'_{in} + Y''_{in}}{4}, \quad Y_{12} = \frac{Y'_{in} - Y''_{in}}{4}. \tag{7.129}$$

It is evident from comparing (129) with (115), i.e. with the elements of the $[Y]$ matrix of a symmetrical lattice network, that

$$Z_1 = \frac{2}{Y''_{in}}, \quad Z_2 = \frac{2}{Y'_{in}}. \tag{7.130}$$

Consequently, the input impedance of the Fig. 34*a* circuit is $Z_2/2$ and the input impedance of the Fig. 34*b* circuit is $Z_1/2$. Here $Z_1$ and $Z_2$ are the impedances of the arms of a symmetrical lattice equivalent to the given four-terminal network.

Thus, impedances $Z_1$ and $Z_2$ of a symmetrical lattice equivalent to the given network can be found from two measurements of the input impedances in the Fig. 34*a* and *b* circuits.

Instead of connecting the secondary winding of the ideal transformer to the input of the four-terminal network, it is possible to use a series connexion (Fig. 34*c* and *d*).

In this case, for the Fig. 34*c* circuit,

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix},$$

from which, on the strength of the equality $\dot{I}_1 = \dot{I}_2$

$$\dot{U}_1 = (Z_{11} + Z_{12})\dot{I}_1, \quad \dot{U}_2 = (Z_{21} + Z_{22})\dot{I}_1,$$

$$Z'_{in} = \frac{\dot{U}_1 - \dot{U}_2}{\dot{I}_1} = Z_{11} + Z_{12} - Z_{21} - Z_{22} = 2(Z_{11} + Z_{12}).$$

For the Fig. 34*d* circuit

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix},$$

from which, when $\dot{I}_1 = \dot{I}_2$,

$$\dot{U}_1 = (Z_{11} - Z_{12})\dot{I}_1, \quad \dot{U}_2 = (-Z_{21} + Z_{22})\dot{I}_1,$$

$$Z''_{in} = \frac{\dot{U}_1 - \dot{U}_2}{\dot{I}_1} = Z_{11} - Z_{12} + Z_{21} - Z_{22} = 2(Z_{11} - Z_{12}).$$

Consequently,

$$Z_{11} = \frac{Z'_{in} + Z''_{in}}{4}, \quad Z_{12} = \frac{Z'_{in} - Z''_{in}}{4}. \tag{7.131}$$

It is evident from comparing (131) with (116), i.e. with the elements of the [*Z*] matrix of the symmetrical lattice network, that the input impedances in the Fig. 34*c* and *d* circuits equal, respectively,

$$Z'_{in} = 2Z_1 \quad \text{and} \quad Z''_{in} = 2Z_2.$$

If the given symmetrical four-terminal network is a three-pole network (Fig. 35), then from the condition of the equality of the currents of the primary and secondary windings of an ideal transformer with a transformation ratio of $n = 1$, the Fig. 35*a* and *b* circuits, serving to determine $Z_2/2$ and $2Z_1$, can easily be converted into simpler circuits (Fig. 35*c* and *d*) without ideal transformers.

The conversion of a symmetrical lattice network into a T equivalent takes place according to the formulae

$$Z_a = \frac{Z_2 - Z_1}{2},$$

$$Z_b = Z_c = Z_1. \tag{7.132}$$

The expression for $Z_a$ shows that the T equivalent to the given lattice is not always physically practicable, e.g. the real part of $Z_2 - Z_1$ is negative.

On the other hand, a lattice network equivalent to a physically existing symmetrical T-network is always realizable, as can be seen from the formulae for the conversion of a T-network into a lattice:

$$Z_1 = Z_b, \quad Z_2 = Z_b + 2Z_a. \tag{7.133}$$

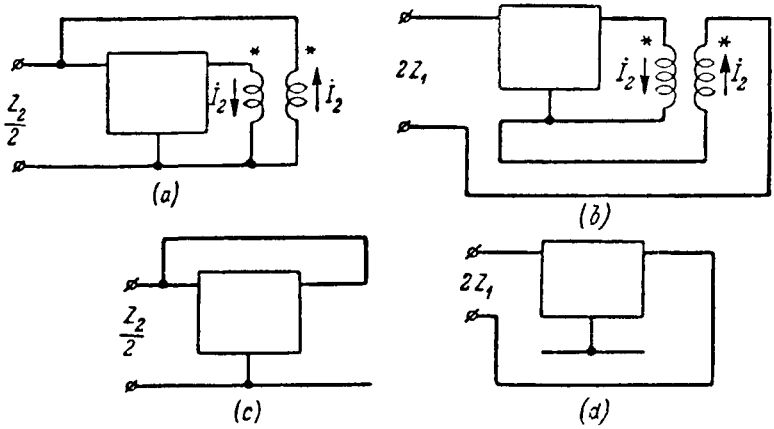



FIG. 35. Determining the impedances of the arms of a lattice equivalent to a given symmetrical four-terminal network.

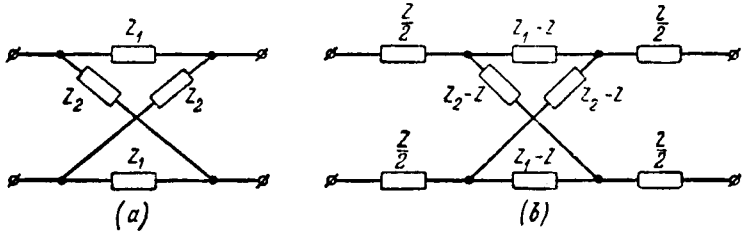



FIG. 36. Removing impedances from lattice network.

If $Z_a$ and $Z_b$ are realizable, then the sum $Z_b - 2Z_a$ is also physically practicable.

In conclusion it should be noted that if the arm impedances of the lattice $Z_1$ and $Z_2$ have a common additive term $Z$, the latter may be removed outside the limits of the lattice: the common impedance may be regarded as being in series with both the input and output terminals, as Fig. 36 shows. An analogous procedure is possible with respect to the admittance $Y$ which is a common additive term of admittances $Y_1$ and $Y_2$. The common admittance may be regarded as being in parallel with both the input and output

terminals (Fig. 37). The equivalence of the Fig. 36*a* and *b* circuits or of the Fig. 37*a* and *b* circuits is easily established by comparing their matrices.

Extending the impedances of the series arms or the admittances of the crossed arms of lattice networks to the terminals makes it possible to convert lattice structures into ladder equivalents in many cases.

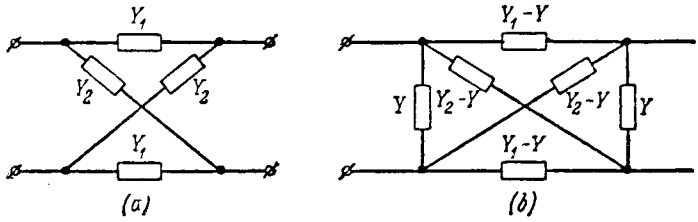



FIG. 37. Removing admittances from lattice network.

In order to determine the parameters of a lattice network equivalent to a given symmetrical four-terminal network consisting of four-terminal networks connected in cascade, the *bisection theorem* is used. By bisection we mean the division of a circuit into two parts along the axis of symmetry, each part being a mirror image of the other. The number of connecting wires joining the two parts can then be arbitrary (Fig. 38*a*).

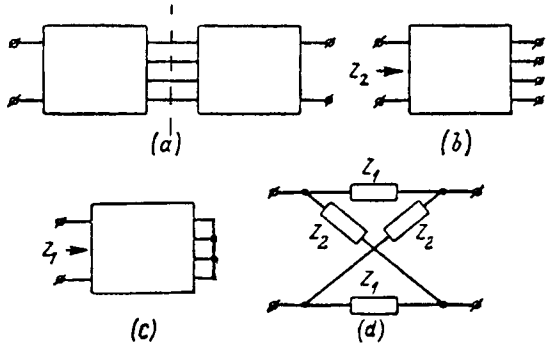



FIG. 38. Bisection theorem.

According to this theorem the input impedances of the half-section under open-circuit (Fig. 38*b*) and short-circuit (Fig. 38*c*) conditions on the side corresponding to the bisection, are equal to the crossed-arm and non-crossed-arm impedances ($Z_2$ and $Z_1$) repectively of the equivalent lattice network (Fig. 38*d*).

If some of the connecting wires are crossed on the side of the bisection, the impedance $Z_2$ is determined by the input impedance of the half-section with the non-crossing wires open and with the crossing wires short-circuited

together and $Z_1$ is determined by the input impedance of the half-section with the non-crossing wires short-circuited and with the crossing wires open.

## 19. Conversion of an Ideal Transformer with a Series or Parallel Arm into an Equivalent Four-terminal Network

A four-terminal network consisting of an ideal transformer at the input of which an admittance $Y$ is connected, can be converted into an equivalent T-network (Fig. 39).

Regarding the given network as a single-element four-terminal network and an ideal transformer connected in cascade, we get a resulting $A$ matrix

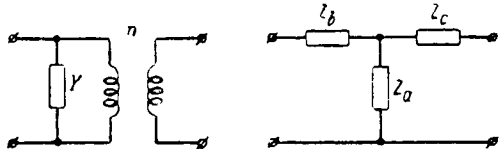



FIG. 39. Replacing an ideal transformer with an admittance at the input by a T equivalent.

by multiplying the matrices of the component four-terminal networks. Taking (99) and (123) into account

$$\begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} n & 0 \\ nY & \frac{1}{n} \end{bmatrix}.$$

A comparison of the resulting matrix with matrix (110) of the T-network

$$[A] = \begin{bmatrix} 1 + \frac{Z_b}{Z_a} & Z_b + Z_c + \frac{Z_b Z_c}{Z_a} \\ \frac{1}{Z_a} & 1 + \frac{Z_c}{Z_a} \end{bmatrix}$$

results in a set of equations

$$\left.\begin{aligned} n = 1 + \frac{Z_b}{Z_a}, \quad Z_b + Z_c + \frac{Z_b Z_c}{Z_a} = 0, \\ nY = \frac{1}{Z_a}, \quad \frac{1}{n} = 1 + \frac{Z_c}{Z_a}. \end{aligned}\right\}$$

Any three equations of this set are independent and the fourth dependent. A solution with respect to the unknowns gives:

$$Z_a = \frac{1}{nY}, \quad Z_b = \frac{n-1}{nY},$$

$$Z_c = \frac{1-n}{n^2 Y} = -\frac{1}{n} Z_b. \tag{7.134}$$

When $n > 1$ the sign of $Z_b$ agrees with the sign of $Y^{-1}$, and when $n < 1$ it is opposite. As (134) shows, the impedances $Z_b$ and $Z_c$ have different signs and, consequently, depending on the numerical value of $n$, one of the series arms of the equivalent T-network is not physically realizable. Thus for example, with an inductive susceptance $Y$ and $n > 1$, $Z_c$ is an inductive reactance with a negative sign. If an inductance is connected in series with $Z_c$, then compensation of $Z_c$ is possible. In that case the whole circuit is physically realizable without the use of an ideal transformer.

Similarly, it can be proved that a four-terminal network consisting of an ideal transformer at the output of which an impedance $Z$ is connected, can be converted into an equivalent pi-network (Fig. 40).

On the basis of (97), (113) and (123)

$$\begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} n\frac{Z}{n} \\ 0\ \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{Z_{bc}}{Z_{ac}} & Z_{bc} \\ \frac{1}{Z_{ab}} + \frac{1}{Z_{ac}} + \frac{Z_{bc}}{Z_{ab}Z_{ac}} & 1 + \frac{Z_{bc}}{Z_{ab}} \end{bmatrix},$$

from which we obtain a set of equations

$$n = 1 + \frac{Z_{bc}}{Z_{ac}}, \quad \frac{Z}{n} = Z_{bc},$$

$$0 = \frac{1}{Z_{ab}} + \frac{1}{Z_{ac}} + \frac{Z_{bc}}{Z_{ab}Z_{ac}}, \quad \frac{1}{n} = 1 + \frac{Z_{bc}}{Z_{ab}},$$

whence

$$Z_{ab} = \frac{Z}{1-n}, \quad Z_{bc} = \frac{Z}{n},$$

$$Z_{ac} = \frac{Z}{n(n-1)} = -\frac{Z_{ab}}{n}. \tag{7.135}$$

When $n > 1$, the sign of $Z_{ac}$ agrees with the sign of $Z$, and when $n < 1$, it is opposite. As (135) shows, the impedances $Z_{ab}$ and $Z_{ac}$ have different

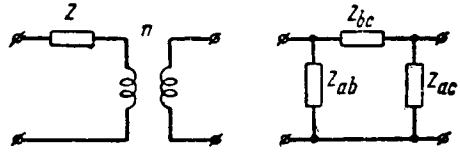



FIG. 40. Replacing an ideal transformer with an admittance at the input by a $\Pi$ equivalent.

signs and, consequently, depending on the numerical value of $n$, one of the parallel arms of the equivalent pi-network is not physically realizable. It can be compensated by an external element connected in parallel.

The conversions shown in Figs. 39 and 40 are used in filters.

## 20. Poles and Zeros of Transfer Functions

As in the case of two-terminal networks (Chapter VI, Section 13), the introduction of the complex frequency $p = c + j\omega$ increases the possibility of studying the general properties of four-terminal networks (Refs. 81, 107 and 114).

If the voltages and currents are regarded as functions of $p$, the transfer function of a four-terminal network (Section 5) can express one of the following ratios:

(1) the ratio of the output voltage to the input voltage (voltage transfer function)

$$K_U(p) = \frac{U_2(p)}{U_1(p)};$$

(2) the ratio of the output current to the input current (current transfer function) $K_I(p) = I_2(p)/I_1(p)$;

(3) the ratio of the output current to the input voltage (the transfer admittance) $Y_{21}(p) = I_2(p)/U_1(p)$ or the ratio of the output voltage to the input current (the transfer impedance)

$$Z_{21}(p) = \frac{U_2(p)}{I_1(p)}.$$

The transfer function can be expressed by the ratio of the polynomials:

$$K(p) = \frac{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0}{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_0}. \tag{7.136}$$

In contrast to the impedance or admittance of a two-terminal network, the powers of the polynomials in the numerator and the denominator of a four-terminal network's transfer function may differ by more than unity.

A transfer function representing the ratio of the electrical quantities in two different loops or nodes of an electrical circuit is free from certain limitations inherent in the impedance and admittance of a two-terminal network. Thus, for example, the transfer impedance and transfer admittance of a four-terminal network can have a negative real part.

As in the case of a two-terminal network, the poles of a transfer function representing the ratio of the electrical values at the output and input of a linear four-terminal network cannot have a positive real part: they are located in the left halfplane since they coincide with the poles of the input impedance or the output admittance of the four-terminal network.

But this limitation does not extend to the zeros of a network's transfer function: they can lie either in the left or the right halfplane.

Transfer functions without zeros in the right halfplane are called *minimum phase functions*, and transfer functions whose zeros are located in the

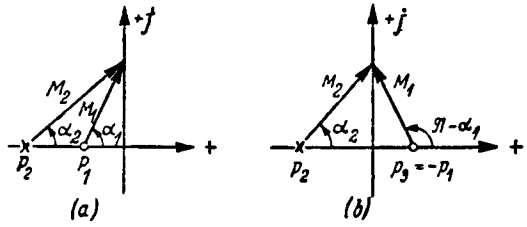


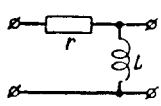
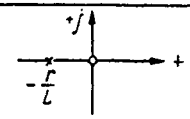
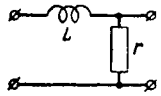
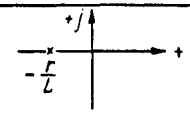
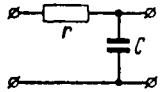
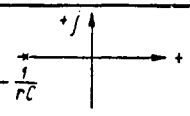
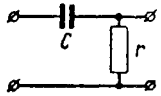
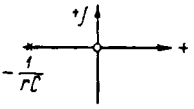
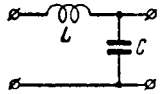
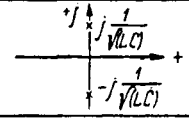
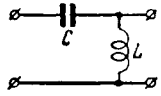
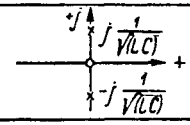
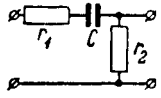
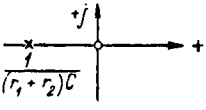
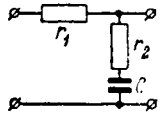
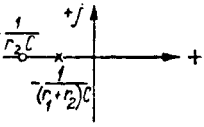
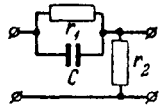
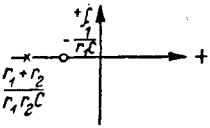

FIG. 41. Zero and pole of transfer function: (*a*) minimum phase; (*b*) non-minimum phase.

right halfplane are known as *non-minimum phase functions*. Correspondingly, we distinguish between minimum phase and non-minimum phase networks or electrical systems.

As an example, Fig. 41 shows the distribution of the zeros and poles of minimum (Fig. 41*a*) and non-minimum (Fig. 41*b*) phase transfer functions. The amplitude characteristic and the phase response can be constructed on the basis of formula (92) given in Chapter VI.

In the case in question, the relationships $M_1/M_2$ are equal and therefore, when the $H$ factors are equal, the amplitude characteristics coincide with each other. But the phase responses are not identical: the minimum phase transfer function is always characterized by a phase angle smaller, in the

TABLE 4. FOUR-TERMINAL NETWORKS, THEIR TRANSFER FUNCTIONS AND POLE-ZERO CONFIGURATION

| N° | Circuit | Transfer function | Zeros and poles |
|---|---|---|---|
| 1 |  | $\frac{p}{p+\frac{r}{L}}$ |  |
| 2 |  | $\frac{\frac{r}{L}}{p+\frac{r}{L}}$ |  |
| 3 |  | $\frac{\frac{1}{rC}}{p+\frac{1}{rC}}$ |  |
| 4 |  | $\frac{p}{p+\frac{1}{rC}}$ |  |
| 5 |  | $\frac{\frac{1}{LC}}{p^2+\frac{1}{LC}}$ |  |
| 6 |  | $\frac{p^2}{p^2+\frac{1}{LC}}$ |  |
| 7 |  | $\frac{r_2}{r_1+r_2}\cdot\frac{p}{p+\frac{1}{(r_1+r_2)C}}$ |  |
| 8 |  | $\frac{r_2}{r_1+r_2}\cdot\frac{p+\frac{1}{r_2C}}{p+\frac{1}{(r_1+r_2)C}}$ |  |
| 9 |  | $\frac{p+\frac{1}{r_1C}}{p+\frac{r_1+r_2}{r_1r_2C}}$ |  |

algebraic sense, than the non-minimum phase function for the same amplitude characteristic.

A pictorial representation of the above is given in Fig. 41, where angle $\alpha_1$ (Fig. 41*a*) contained in the numerator of formula (92) of Chapter VI is smaller than the angle which complements it to 180° (Fig. 41*b*).

An essential feature of minimum phase functions is that the amplitude characteristic and the phase characteristic determine each other simply: if we have one of these characteristics we can find the other (Section 21).

Minimum phase transfer functions corresponding to certain four-terminal networks and the arrangement of the zeros and poles in a complex plane are given in Table 4.

In ladder networks, the current in the load can equal zero in the case where any impedance in the series arms is infinitely great or any impedance in the parallel arms equals zero.

Therefore, the transfer function for a ladder network is always a minimum phase function: its zeros depend on the poles of the series impedances or the zeros of the parallel impedances which form the given ladder network. They are always located in the left halfplane (Chapter VI, Section 15).

An interesting example of a non-minimum phase transfer function is the function whose zeros are located in the right halfplane and are negatives of the poles. A four-terminal network with such a transfer function is characterized by the property of passing all frequencies with the same attenuation. Such a network is termed an "all-pass" network. In fact, when the frequency varies from zero to infinity the modulus of formula (92) remains constant, i.e. the amplitude characteristic is a straight line parallel to the axis; and the argument of formula (92) in Chapter VI, i.e. the phase difference between the electrical values at the input and the output of the network, does not remain constant but alters with the frequency.

Such a change in phase with a constant amplitude is, for example, the property of a symmetrical lattice network terminated in a matched impedance $r = \sqrt{(Z_1 Z_2)}$. In this case the transfer function, in accordance with (76) and (117), has the form

$$K_U = \frac{r(Z_2 - Z_1)}{2Z_1 Z_2 + r(Z_1 + Z_2)}.$$

Multiplying the numerator and denominator by $\sqrt{(Z_1/Z_2)}$ we get

$$K_U = \frac{r^2 - Z_1^2}{r^2 + Z_1^2 + 2rZ_1},$$

from which

$$K_U = \frac{r - Z_1}{r + Z_1}. \tag{7.137}$$

It follows from the condition $r^2 = Z_1 Z_2$ that $Z_1$ and $Z_2$ are mutually inverse two-terminal networks. Let $Z_1$ and $Z_2$ be constructed in accordance with Fig. 42*b* and *c*.

In this case

$$Z_1(p) = \frac{1}{C_1} \frac{p}{p^2 + \frac{1}{L_1 C_1}}, \quad Z_2(p) = L_2 \frac{p^2 + \frac{1}{L_2 C_2}}{p}.$$

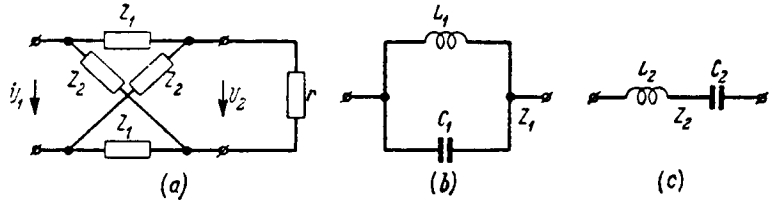



FIG. 42. All-pass lattice network: (*a*) circuit diagram; (*b*) two-terminal network $Z_1$; (*c*) two-terminal network $Z_2$.

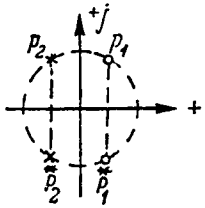



FIG. 43. Distribution of zeros and poles.

In order to simplify the calculations let us suppose $L_1 = L_2 = 1\text{H}$, $C_1 = C_2 = 1\text{F}$, and $r = 1\Omega$. In this case

$$Z_1 = \frac{p}{p^2 + 1}, \quad Z_2 = \frac{p^2 + 1}{p},$$

$$K_U = \frac{1 - \frac{p}{p^2 + 1}}{1 + \frac{p}{p^2 + 1}} = \frac{p^2 - p + 1}{p^2 + p + 1} = \frac{(p - p_1)(p - \overset{*}{p}_1)}{(p - p_2)(p - \overset{*}{p}_2)},$$

where

$$p_1, \overset{*}{p}_1 = \frac{1}{2} \pm j \frac{\sqrt{3}}{2} = e^{\pm j60°};$$

$$p_2, \overset{*}{p}_2 = -\frac{1}{2} \pm j \frac{\sqrt{3}}{2} = e^{\pm j120°}.$$

The zeros and poles are arranged around a circumference of radius 1 (Fig. 43).

In this case, the modulus of the transfer function is equal to unity. The characteristic of the transfer function in a complex plane and the amplitude and phase characteristics in a rectangular system of coordinates are shown in Fig. 44*a* and *b*. The all-pass network is used as a phase-shifting network in communication engineering.

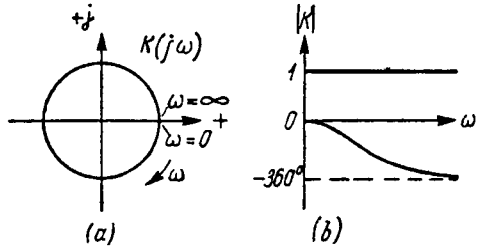



FIG. 44. Transfer function of an all-pass network: (*a*) characteristic in complex plane; (*b*) amplitude and phase characteristics.

## 21. The Relation Between the Amplitude Characteristic and the Phase Characteristic of a Four-terminal Network

Let us suppose that a four-terminal network representing a minimum phase system has a transfer function $K(j\omega) = k(\omega)e^{-j\varphi(\omega)}$. In order to establish the relationship between the amplitude characteristic $k(\omega)$ and the phase characteristic $\varphi(\omega)$ it is convenient in this case to start with the expression for the natural logarithm of this function

$$\ln K(j\omega) = \ln k(\omega) - j\varphi(\omega).$$

The functional transformation relating to the frequency characteristics of the resistance and reactance of a two-terminal network (Chapter VI, Section 18) is applicable to the frequency characteristics $\ln k(\omega)$ and $\varphi(\omega)$.

For example, if the amplitude characteristic $k(\omega)$ is given, the phase characteristic can be found by analogy with (95)

$$\varphi(\omega_0) = -\frac{2\omega_0}{\pi}\int_0^\infty \frac{\ln k(\omega)}{\omega^2 - \omega_0^2}\,d\omega.$$

Here the minus sign is introduced in front of the integral in view of the fact that a negative sign appears in the index of the transfer function $K(j\omega)$.

## 22. Construction of Four-terminal Networks with Prescribed Frequency Characteristics

The construction of four-terminal networks with prescribed frequency characteristics, which belongs to the synthesis of electrical circuits, is generally a more complex problem than the construction of two-terminal networks. Depending on the conditions of the problem, the construction of four-terminal networks can, in many cases, be based on the common methods of synthesizing two-terminal networks (Refs. 81, 107 and 108).

Let us suppose that we have to construct a symmetrical reciprocal four-terminal network with any two parameters given in a rational-fractional form, e.g.: $Z_{11} = -Z_{22}$ and $Z_{12} = -Z_{21}$; $Y_{11} = -Y_{22}$ and $Y_{12} = -Y_2$; $A_{11} = A_{22}$ and $A_{12}$, etc.

The required network can be realized in the form of a lattice network, the impedances of which, $Z_1$ and $Z_2$, are determined according to Section 15:

$$Z_1 = Z_{11} + Z_{12} = \frac{1}{Y_{11} - Y_{12}} = \frac{A_{12}}{A_{11} + 1},$$

$$Z_2 = Z_{11} - Z_{12} = \frac{1}{Y_{11} + Y_{12}} = \frac{A_{12}}{A_{11} - 1}.$$

The problem of constructing a symmetrical four-terminal network from two parameters given in the $[Z]$, $[Y]$ or $[A]$ form can thus be reduced to constructing the two-terminal networks $Z_1$ and $Z_2$. The methods described in Chapter VI, Section 16 are used for finding these two-terminal networks.

EXAMPLE 2. To construct a symmetrical four-terminal network with a prescribed parameter

$$Z_{21} = \frac{1}{2p^3 + 4p^2 + 4p + 2}.$$

Let us convert the given expression

$$Z_{21} = \frac{1}{2}\left(\frac{1}{p+1} - \frac{p}{p^2+p+1}\right) = \frac{1}{2}(Z_2 - Z_1).$$

Taking

$$Z_1 = \frac{p}{p^2+p+1}, \quad Z_2 = \frac{1}{p+1},$$

we get the lattice shown in Fig. 45*a* (here $r = 1, L = 1$ and $C = 1$). In accordance with the rule mentioned in Section 18 (Fig. 37), the lattice can be converted into a pi-network (Fig. 45*b*).

The one essential condition for the realizability of the transfer function $K_U(p)$ of a lattice network with matched loading generally follows from formula (137).

In view of the fact that $Z_1(p)$ is a rational positive-real function, $K_U(p)$ is rational and real for real values of $p$.

It follows from the condition for the positive character of function $Z_1(p)$, which denotes that $\operatorname{Re} Z_1(p) \geqq 0$ when $\operatorname{Re} p \geqq 0$, that:

$$|K_U(p)|^2 = \left|\frac{r - \operatorname{Re} Z_1(p)}{r + \operatorname{Re} Z_1(p)}\right|^2 = \frac{[r - \operatorname{Re} Z_1(p)]^2 + [\operatorname{Im} Z_1(p)]^2}{[r + \operatorname{Re} Z_1(p)]^2 + [\operatorname{Im} Z_1(p)]^2} \leqq 1,$$

i.e. modulus $K_U(p)$ does not exceed unity in the right halfplane and on the imaginary axis (function $K_U(p)$ has no poles in the right halfplane $p$, including the imaginary axis). This is the essential condition for the realizability of $K_U(p)$.

In turn, if the function $Z_1(p)$ resulting from (137)

$$Z_1(p) = r\,\frac{1 - K_U(p)}{1 + K_U(p)}$$

is positive and real, that is a sufficient condition for $K_U(p)$ to be realizable.

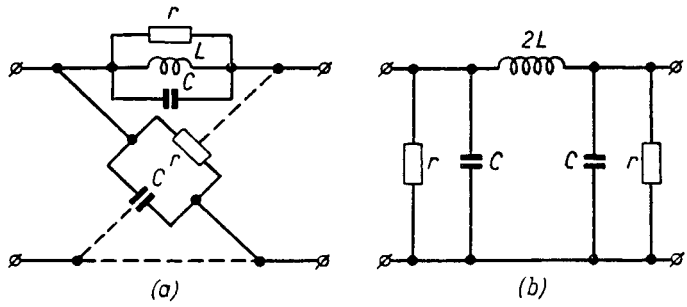



FIG. 45. Example 2.

If $K_U(p)$ is real for real values of $p$, and $|K_U(p)| \leqq 1$ when $\operatorname{Re} p \geqq 0$, then $Z_1(p)$ is a positive-real function and, correspondingly, $Z_1(p)$ is realizable.

The one essential condition for the realizability of the transfer function $Z_U(p)$ of a lattice network with a matched load can thus be reduced to the following:

(1) $K_U(p)$ is a rational function, real for real values of $p$;

(2) $|K_U(p)| \leqq 1$ when $\operatorname{Re} p \geqq 0$.

Let us now suppose that we have to construct a reactive four-terminal network, loaded with a resistance, with a prescribed transfer function.

The equality of the active (average) powers at the input and output of a reactive four-terminal network makes it possible to obtain the relation between the given transfer function and the real part of the input impedance (or output admittance) of a four-terminal network. Having determined the

real part, it is then possible to find the input impedance with the aid of which the design of the four-terminal network itself can also be established.

Let us examine the above-mentioned types of network supply.

1. *Feed from a current source* (Fig. 46*a*).

Multiplying the given transfer function (transfer impedance) $Z_{21}(j\omega) = \dot{U}_2/\dot{I}_1$ by the conjugate function $Z_{21}(-j\omega) = \overset{*}{U}_2/\overset{*}{I}_1$,

$$Z_{21}(j\omega) \cdot Z_{21}(-j\omega) = \left(\frac{U_2}{I_1}\right)^2.$$

The power $P_2 = U_2^2/r_2$ at the output of the network is equal to the power $P_1 = I_1^2 r_{1\text{in}}$ at its input. Consequently,

$$\left(\frac{U_2}{I_1}\right)^2 = r_2 r_{1\text{in}},$$

from which

$$r_{1\text{in}} = \operatorname{Re} Z_{1\text{in}}(j\omega) = \frac{Z_2(j\omega)Z_{21}(-j\omega)}{r_2}.$$

Let us represent the unknown function $Z_1(p)$ in the form of the ratio of the polynomials $A(p) = A_e + A_o$ and $B(p) = B_e + B_o$, when the subscripts "$e$" and "$o$" denote the parts of the polynomials with even and odd powers of $p$.

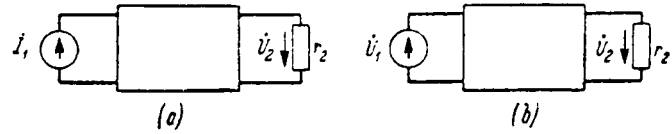



FIG. 46. Reactive four-terminal networks supplied by: (*a*) current source; (*b*) e.m.f. source.

By analogy with (109) (Chapter VI, Section 18)

$$\frac{Z_{21}(p)Z_{21}(-p)}{r_2} = \frac{A_eB_e - A_oB_o}{B(p) \cdot B(-p)}. \tag{7.138}$$

By equating the factors for the same powers of $p$ in the right and left parts of equation (138) we find the unknown factors and thus obtain expressions for $Z_{1\text{in}}(p)$ and $Y_{1\text{in}}(p) = 1/Z_{1\text{in}}(p)$. By expanding one of these functions into a continued fraction we can obtain the rquired four-terminal network in the form of a ladder network.

2. *Feed from an e.m.f. source* (Fig. 46*b*).

Multiplying the given transfer function $K(j\omega) = \dot{U}_2/\dot{U}_1$ by the conjugate function $K(-j\omega) = \overset{*}{U}_2/\overset{*}{U}_1$, we get

$$K(j\omega)K(-j\omega) = \left(\frac{U_2}{U_1}\right)^2.$$

Since the network is dissipationless, the power $P_2 = U_2^2/r_2$ at the output of the four-terminal network is equal to the active power $P_1 = U_2^1 g_{1\text{in}}$ at its input. Consequently,

$$\left(\frac{U_2}{U_1}\right)^2 = r_2 g_{1\text{in}},$$

from which

$$g_{1\text{in}} = \operatorname{Re} Y_{1\text{in}}(j\omega) = \frac{K(j\omega)K(-j\omega)}{r_2}.$$

The same procedure as above can be used for finding functions $Y_{1\text{in}}(p)$ and $Z_{1\text{in}}(p)$ and constructing the network.

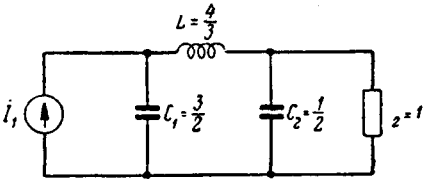



FIG. 47. Example 3.

EXAMPLE 3 (Ref. 107). To construct a reactive four-terminal network supplied by a current source (Fig. 46*a*) if the transfer function

$$Z_{21}(p) = \frac{1}{p^3 + 2p^2 + 2p + 1},$$

is given, and $r_2 = 1$.

In the given case

$$\frac{Z_{21}(p)Z_{21}(-p)}{r_2} = \frac{1}{(p^3 + 2p^2 + 2p + 1)(-p^3 + 2p^2 - 2p + 1)}.$$

Denoting

$$Z_{1\text{in}}(p) = \frac{A(p)}{B(p)} = \frac{a_3p^3 + a_2p^2 + a_1p + a_0}{p^3 + 2p^2 + 2p + 1},$$

we get

$$\frac{A_eB_e - A_eB_e}{B(p)\,.\,B(-p)} = \frac{(a_2p^2 + a_0)(2p^2 + 1) - (a_3p^3 + a_1p)(p^3 + 2p)}{(p^3 + 2p^2 + 2p + 1)(-p^3 + 2p^2 - 2p + 1)}$$

$$= \frac{-a_3p^6 + (-2a_3 + 2a_2 - a_1)p^4 + (a_2 - 2a_1 + 2a_0)p^2 + a_0}{(p^3 + 2p^2 + 2p + 1)(-p^3 + 2p^2 - 2p + 1)}.$$

Consequently, from (138) $-a_3 = 0$, $-2a_3 + 2a_2 - a_1 = 0$, $a_2 - 2a_1 + 2a_0 = 0$, $a_0 = 1$.

From this, $a_2 = 2/3$ and $a_1 = 4/3$.

Thus

$$Z_{1\text{in}}(p) = \frac{\frac{2}{3}p^2 + \frac{4}{3}p + 1}{p^3 + 2p^2 + 2p + 1}$$

or

$$Y_{1\text{in}}(p) = \frac{p^3 + 2p^2 + 2p + 1}{\frac{2}{3}p^2 + \frac{4}{3}p + 1} = \frac{3}{2}p + \cfrac{1}{\frac{4}{3}p + \cfrac{1}{\frac{1}{2}p + 1}}.$$

The circuit diagram given in Fig. 47 corresponds to the resulting input admittance.

CHAPTER VIII

# RESONANT CIRCUITS. FILTERS

## 1. Resonant Circuits and their Characteristics

Electrical circuits containing inductances and capacitances, in which the phenomena of *series resonance* or *parallel resonance* may occur are called *resonant circuits.* These circuits are an integral part of all radio-engineering systems and are used extensively in electrical engineering.

Several features of resonant circuits have been dealt with in the previous chapters devoted to the basic relationships of the electrical quantities in circuits with $r$, $L$ and $C$ elements connected in series (Chapter II, Section 3) and parallel (Chapter II, Section 4), loci (Chapter V), and the frequency characteristics of two-terminal networks (Chapter VI).

In this section we shall examine the resonance curves and phase characteristics of resonant circuits which are the simplest two-terminal networks consisting of $r$, $L$ and $C$ elements. The following sections are devoted to electrical filters.

### Series Resonance

As we have said above, *series resonance occurs in an electrical circuit when the inductive reactance of one part of the circuit is compensated by the capacitive reactance of another part, connected in series to the first.* During series resonance the total reactance of the circuit is zero and the power factor across the terminals of the circuit is unity. Consequently, the reactive power at the terminals of the circuit equals zero.

When $r$, $L$ and $C$ are connected in series (Fig. 1 *a*) the *resonant frequency* is

$$\omega_0 = \frac{1}{\sqrt{(LC)}} \tag{8.1}$$

and the *Q-factor* (quality factor) of the circuit for the resonant frequency is

$$Q = \frac{U_L}{U} = \frac{U_C}{U} = \frac{\omega_0 L}{r} = \frac{1}{\omega_0 C r} = \frac{\sqrt{\left(\frac{L}{C}\right)}}{r}. \tag{8.2}$$

The value

$$\varrho = \sqrt{\left(\frac{L\,\mathrm{H}}{C\,\mathrm{F}}\right)} = 30\sqrt{\left(\frac{L\,\mathrm{cm}}{C\,\mathrm{cm}}\right)} \tag{8.3}$$

is known as the *characteristic impedance* of the circuit.

Consequently,

$$Q = \frac{\varrho}{r}. \tag{8.4}$$

Multiplying the numerator and denominator of the expression $\omega_0 L/r$ in (2) by $\frac{1}{2}I_m^2$ we get

$$Q = \omega_0 \frac{\frac{1}{2}LI_m^2}{\frac{1}{2}I_m^2 r} = 2\pi \frac{\frac{1}{2}LI_m^2}{\frac{1}{2}I_m^2 \frac{r}{f_0}},$$

or, in a more general form,

$$Q = \omega_0 \frac{W_{\max}}{P}. \tag{8.5}$$

Here $W_{\max}$ is the maximum energy stored by the inductance or capacitance during resonance; and $P$ is the average power consumed in the impedance.

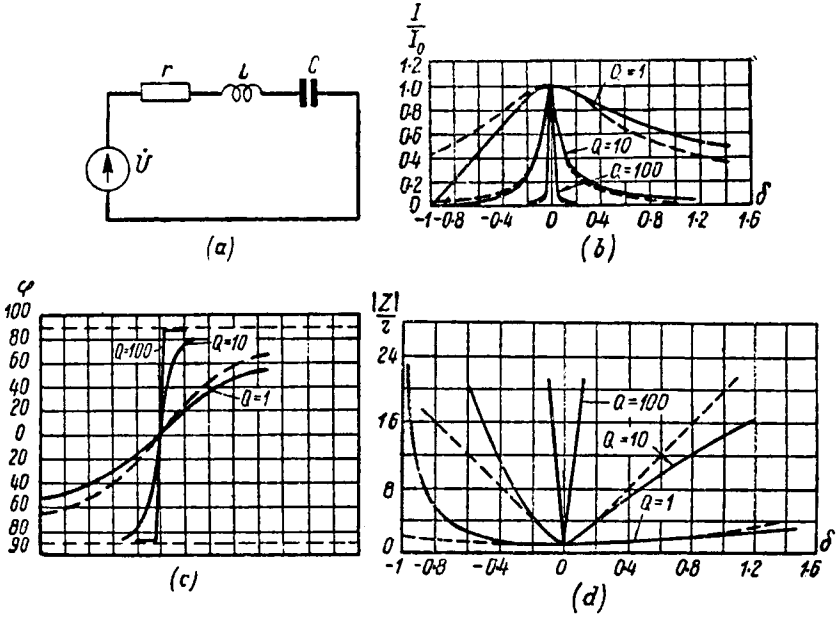



FIG. 1. Series resonance: (*a*) resonance circuit; (*b*) resonance curve $I/I_0$; (*c*) phase characteristic; (*d*) resonance curve $z/R$.

Thus, *the Q-factor of the circuit is determined from the ratio of the maximum energy stored in the magnetic or electrical field to the energy dissipated per cycle.*

Let us agree to call the value

$$\delta = \frac{\omega - \omega_0}{\omega_0} = \frac{\omega}{\omega_0} - 1 \tag{8.6}$$

the *relative fractional deviation* from the resonant frequency, or the *relative fractional detuning.*

The impedance of the circuit, taking (2) and (6) into account, is

$$Z = r + j\left(\omega L - \frac{1}{\omega C}\right) = r\left[1 + j\frac{\omega_0 L}{r}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

$$= r\left[1 + jQ\frac{(\omega - \omega_0)(\omega + \omega_0)}{\omega_0\omega}\right] = r\left(1 + jQ\delta\frac{\delta + 2}{\delta + 1}\right),$$

from which

$$\left.\begin{aligned} z &= r\sqrt{\left[1 + \left(Q\delta\frac{\delta + 2}{\delta + 1}\right)^2\right]}, \\ \varphi &= \arctan\frac{Q\delta(\delta + 2)}{\delta + 1}. \end{aligned}\right\} \tag{8.7}$$

At a frequency close to the resonant frequency, the magnitude of $\delta$ is considerably less than unity and therefore, approximately,

$$\left.\begin{aligned} Z &\cong r(1 + j2Q\delta) = r\sqrt{[1 + (2Q\delta)^2]}\, e^{j\varphi}, \\ \varphi &\cong \arctan 2Q\delta, \\ \dot{I} &\cong \frac{\dot{U}}{r}\frac{1}{1 + j2Q\delta}. \end{aligned}\right\} \tag{8.8}$$

In practice, formulae (8) are sufficiently accurate when $|\delta| < 0{\cdot}1$. When $|\delta| = 0{\cdot}2$ the error in the impedance is less than 10%.

Figure 1 gives the curves on a per unit basis: in Fig. 1*b* the ratios of the currents to the maximum current at resonance

$$\frac{I}{I_0} = \frac{U}{z} : \frac{U}{r} = \frac{r}{z} = \frac{1}{\sqrt{\left[1 + \left(Q\delta\frac{\delta + 2}{\delta + 1}\right)^2\right]}}$$

$$\approx \frac{1}{\sqrt{[1 + (2Q\delta)^2]}} \tag{8.9}$$

are plotted on the ordinates axis.

In Fig. 1*d* the reciprocals, i.e. the ratios of the circuit impedance $z$ to the resistance $r$, are plotted on the ordinates axis.

The curves in Fig. 1*b*, *c* and *d* shown by continuous lines correspond to accurate formulae; the dotted curves are constructed from approximate formulae (Ref. 109).

Thus the value $Q$ describes the sharpness of the resonance by a curve ("tuning sharpness"); in accordance with (5), the greater the ratio of the peak energy stored in the magnetic or electrical field to the energy dissipated in the resonant circuit during one cycle, the sharper the resonance curve.

The range of frequencies near resonance, at the edges of which the current decreases to $1/\sqrt{2} = 0{\cdot}707$ of the maximum (resonance) value, is usually called the *bandwidth of the circuit*. When the current $I = I_0/\sqrt{2}$ the power dissipated in resistance $r$ is a half of the power dissipated at resonance (if the voltage across the circuit is of constant magnitude). The bandwidth is therefore described as the band whose limits correspond to half the maximum power. At the edges of the bandwidth of a resonant circuit the resistance and the reactance are equal in absolute value. Correspondingly, the phase angle between the voltage applied to the circuit and the current is 45°: at the lower limit the current leads the applied voltage and at the upper limit lags behind the applied voltage by 45°.

On the basis of (9) the conditions for the limits of the bandwidth can be written in the following form:

$$\frac{1}{\sqrt{\left[1 + \left(Q\delta \frac{\delta + 2}{\delta + 1}\right)^2\right]}} = \frac{1}{\sqrt{2}}$$

or

$$Q\delta \frac{\delta + 2}{\delta + 1} = \pm 1,$$

from which

$$\delta_{1,2} = -1 \mp \frac{1}{2Q} + \sqrt{\left(1 + \frac{1}{4Q^2}\right)}. \tag{8.10}$$

The subscripts 1 and 2 and, correspondingly, the plus and minus signs in expression (10) relate to the limits below and above resonance.

By definition, the bandwidth of the resonant circuit can be found from the condition

$$\delta_2 - \delta_1 = \frac{1}{Q}$$

or

$$\frac{\omega_2 - \omega_1}{\omega_0} = \frac{1}{Q} = d. \tag{8.11}$$

The value $d$, which is the reciprocal of the $Q$-factor of the circuit, is sometimes called the *dissipation factor.*

With a sufficiently high $Q$-factor for the resonant circuit ($Q \gg 1$), the radicand expression in (10) can be equated to unity, from which $\delta_{1,2} = \mp 1/2Q$, i.e. the bandwidth is, in practice, symmetrical with respect to the resonant frequency. The same result is obtained from the approximate formula (8).

The resonance properties of the circuit in question depend on the distribution of the zeros of function $Z(p)$. In Chapter VI, Section 17 it was

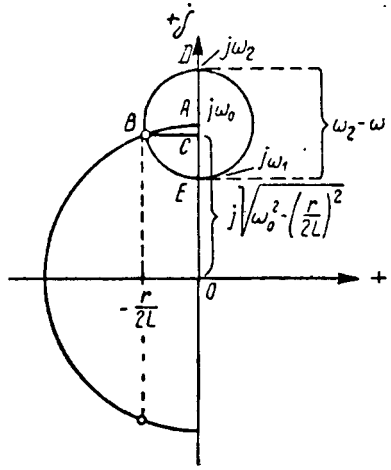



FIG. 2. Dependence of bandwidth on distribution of zeros of function $Z(p)$.

shown that when $r$, $L$ and $c$ are connected in series, function $Z(p)$ has a pair of conjugate-complex zeros:

$$p, \overset{*}{p} = -\frac{r}{2L} \pm j\sqrt{\left[\frac{1}{LC} - \left(\frac{r}{2L}\right)^2\right]},$$

the resonant frequency $\omega_0$ being determined by the radius of the circle passing through zero, i.e. by the distance from the origin to the zero of function $Z(p)$ (Fig. 49 in Chapter VI and Fig. 2 below).

From (11)

$$\omega_2 - \omega_1 = \frac{\omega_0}{Q} = \frac{\omega_0}{\frac{\omega_0 L}{r}} = 2\frac{r}{2L}. \tag{8.12}$$

By comparing (12) with the real part of the zero of the function it is easy to confirm that the bandwidth of the circuit under examination is determined as twice the length of the section $BC$. The higher the $Q$-factor,

the nearer point $B$ lies to the imaginary axis, i.e. the shorter the section $BC$ and, correspondingly, the sharper the resonant frequency. As point $B$ approaches the imaginary axis, the length of the section $BC$ approaches the distance between points $A$ and $B$. Therefore when the $Q$-factor of the resonant circuit is sufficiently high its bandwidth can be determined

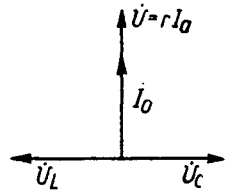



FIG. 3. Vector diagram of voltages for series resonance.

approximately by the diameter of a circle drawn through point $B$ with its centre at $A$ (Fig. 2).

*In conditions close to resonance the voltages across the inductance and capacitance can be very high,* which must be taken into account in averting an insulation fault.

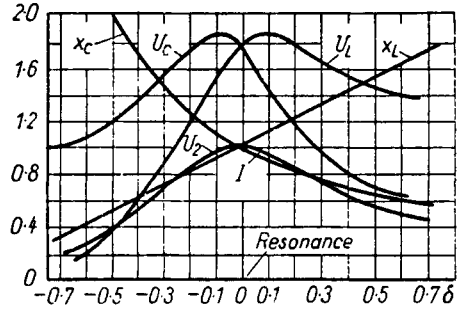



FIG. 4. Curves of $U_L$, $U_C$ and $U_r$ per unit.

Figure 3 shows a vector diagram of voltages at resonance. At resonance the voltages on the inductance and capacitance are determined from the formulae

$$\dot{U}_L = -\dot{U}_C = \frac{\dot{U}}{r} j\omega_0 L = j\dot{U}Q. \tag{8.13}$$

When $Q > 1$ these voltages exceed the voltages $U$ applied to the resonant circuit. But the values obtained from (13) are not extreme: the maximum of the voltage $U_L$ lies somewhat above, and the maximum $U_C$ somewhat below, the resonant frequency (Fig. 4).

This follows from analysing the formulae

$$\dot{U}_L = \dot{U}\frac{j\omega L}{r + j\left(\omega L - \dfrac{1}{\omega C}\right)} = \dot{U}\frac{jQ(\delta + 1)}{1 + jQ\delta\dfrac{\delta + 2}{\delta + 1}}$$

and

$$\dot{U}_C = \dot{U}\frac{-j\dfrac{1}{\omega C}}{r + j\left(\omega L - \dfrac{1}{\omega C}\right)} = \dot{U}\frac{-j\dfrac{Q}{\delta + 1}}{1 + jQ\delta\dfrac{\delta + 2}{\delta + 1}}.$$

Voltage $U_L$ reaches a maximum when

$$\delta = \frac{1}{\sqrt{\left(1 - \dfrac{1}{2Q^2}\right)}} - 1,$$

and voltage $U_C$ reaches a maximum when

$$\delta = \sqrt{\left(1 - \frac{1}{2Q^2}\right)} - 1.$$

Disregarding $\delta$ with respect to unity, we get the approximate formula

$$\dot{U}_L \cong -\dot{U}_C \cong \dot{U}\frac{jQ}{1 + j2Q\delta}. \tag{8.14}$$

At resonance the maximum values of the energy of the electrical and magnetic fields is:

$$W_{L\max} = \frac{LI_m^2}{2}, \quad W_{C\max} = \frac{C}{2}\left(\frac{I_m}{\omega_0 C}\right)^2 = \frac{LI_m^2}{2}.$$

The instantaneous values $W_L$ and $W_C$ oscillate with a doubled frequency about the average value $LI_m^2/4$, and a continuous redistribution of the energy of the electrical and magnetic fields takes place, its total value remaining constant:

$$W_L + W_C = \frac{LI_m^2}{2}\left[\sin^2(\omega t + \alpha) + \cos^2(\omega t + \alpha)\right] = \frac{LI_m^2}{2}.$$

Energy supplied to the circuit from the source is dissipated in resistance $r$.

The simplest single-loop network has been examined above. As well as the analytic calculation method it is also convenient to use the locus method

(Chapter V, Section 4) to examine resonance phenomena in more complex multi-loop circuits where resonance can occur at one frequency or at several frequencies.

**Parallel Resonance**

*Parallel resonance occurs in an electrical circuit when the inductive susceptance of one part of the circuit is compensated by the capacitive susceptance of another part, connected in parallel to the first.* At parallel resonance the susceptance of the circuit is zero and the power factor at the terminals of the circuit equals unity, as for series resonance. Correspondingly, the reactive power at the terminals of the circuit is zero.

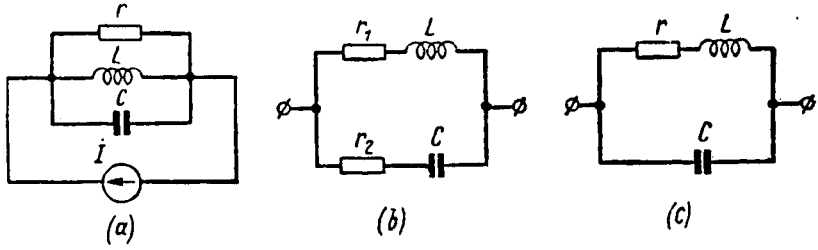



FIG. 5. Parallel resonance circuits.

Figure 5 shows the different types of the simplest circuits in which parallel resonance is possible.

If a circuit composed of $r$, $L$ and $C$ elements connected in parallel is supplied from a current source (Fig. 5*a*), then regarding it as a dual circuit with respect to that shown in Fig. 1*a*, we can use the relationships educed earlier. Substituting current and voltage for voltage and current, and parameters $g$, $C$, $L$ and $Y$ for $r$, $L$, $C$ and $Z$, respectively, we get the calculation formulae for the circuit under examination.

Thus, the admittance

$$Y = g - j\left(\frac{1}{\omega L} - \omega C\right) = g\left(1 + jQ\delta\,\frac{\delta + 2}{\delta + 1}\right),$$

the node voltage

$$\dot{U} = \frac{\dot{I}}{Y} = \frac{\dot{I}}{g\left(1 + jQ\delta\,\dfrac{\delta + 2}{\delta + 1}\right)},$$

etc.

Dual circuits possess the same $Q$-factor, formula (5) giving, for the Fig. 5*a* circuit,

$$Q = \frac{\omega_0 C}{g} = \frac{1}{\omega_0 L g} = \frac{1}{g}\sqrt{\left(\frac{C}{L}\right)}.$$

In this case the bandwidth is determined from (11)

$$\omega_2 - \omega_1 = \frac{\omega_0}{Q} = \frac{g}{C} = 2\frac{g}{2C}.$$

The resonant frequency $\omega_0$ and the bandwidth can easily be found graphically from the distribution of the poles of the function

$$Z(p) = \frac{p}{C\left(p^2 + p\frac{g}{C} + \frac{1}{LC}\right)},$$

$$p, \overset{*}{p} = -\frac{g}{2C} \pm j\sqrt{\left[\frac{1}{LC} - \left(\frac{g}{2C}\right)^2\right]}.$$

The resonant frequency is determined by the magnitude of the radius of the circle passing through a pole and the bandwidth is determined as being twice the abscissa of a pole (Fig. 6). The nearer the pole is to the imaginary axis, the higher the $Q$-factor and the sharper the resonance curve.

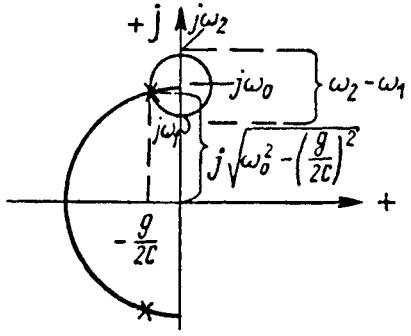



FIG. 6. Dependence of bandwidth on distribution of poles of function $Z(p)$.

Losses in the capacitor can be taken into account by the admittance $g$ in Fig. 5*a*. But these losses are negligible in comparison with the losses in the impedance of the induction coil. Therefore we generally use an equivalent circuit in which the inductance $L$ is connected in series to the resistance $r$ of the coil (Fig. 5*b* and *c*).

The condition for parallel resonance in the Fig. 5*b* circuit is written in the form of the equality of the susceptances

$$\frac{\omega_0 L}{r_1^2 + (\omega_0 L)^2} = \frac{\frac{1}{\omega_0 C}}{r_2^2 + \left(\frac{1}{\omega_0 C}\right)^2},$$

from which

$$\omega_0 = \sqrt{\left(\frac{1}{LC} \cdot \frac{\frac{L}{C} - r_1^2}{\frac{L}{C} - r_2^2}\right)}. \tag{8.15}$$

Parallel resonance is only possible when the radicand of formula (15) has a positive sign.

A vector diagram for parallel resonance is given in Fig. 7. The currents in the inductive and capacitive arms are added from the resistive ($\dot{I}_{La}$, $\dot{I}_{Ca}$) and reactive ($\dot{I}_{Lp}$, $\dot{I}_{Cp}$) components, and then

$$\dot{I}_{Lp} = -\dot{I}_{Cp}, \quad \text{and} \quad \dot{I}_{La} + \dot{I}_{Ca} = \dot{I}.$$

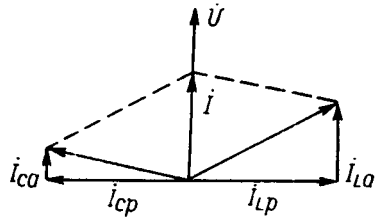



FIG. 7. Vector diagram of currents for parallel resonance.

At resonance the circuit has a conductance

$$Y = g = \frac{r_1}{r_1^2 + (\omega_0 L)^2} + \frac{r_2}{r_2^2 + \left(\frac{1}{\omega_0 C}\right)^2} = \frac{r_1 + r_2 \omega_0^2 LC}{r_1^2 + (\omega_0 L)^2}.$$

For a circuit with small losses it is possible to disregard the terms $r_1^2$ and $r_2^2$ with respect to $(\omega_0 L)^2$, $(1/\omega_0 C)^2$ and $L/C$.

The admittance of the resonant circuit is then approximately expressed by a formula which is widely used in practical calculations in radio engineering:

$$Y \approx \frac{r_1 + r_2}{(\omega_0 L)^2} \approx \frac{(r_1 + r_2)\, C}{L}.$$

If the inductance and capacitance are expressed in centimetric units, then

$$Y \approx \frac{(r_1 + r_2)\, C}{900 L}.$$

When $r_1 = r_2 = \sqrt{(L/C)}$ formula (15) becomes indeterminate. The admittance of the circuit is then a pure conductive one for any value of $\omega$, i.e. the circuit resonates at any frequency.

The given circuit, which has a fixed resistance in the whole spectrum of frequencies, may be matched with a long line which has an image impedance $\varrho = \sqrt{(L/C)}$ (Chapter 9). It is used in communication channels and in systems designed for transmitting a broad spectrum of frequencies, e.g. in wide-band amplifiers.

Disregarding the losses in the branch with the capacitance we get the Fig. 5*c* circuit, which is a particular case of the previous circuit ($r_2 = 0$). The resonant frequency of this circuit, in accordance with (15),

$$\omega_0 = \frac{1}{\sqrt{(LC)}} \sqrt{\left[1 - \left(\frac{r}{\varrho}\right)^2\right]}, \tag{8.16}$$

$Q$ is determined from formula (5): $Q = \omega_0 L/r$.

It is evident from comparing (16) with (1) that for the same $r$, $L$ and $C$ parameters, the resonant frequencies of the circuits in Figs. 1*a* and 5*c* differ by $\sqrt{[1 - (\varrho/r)^2]}$ times. When $\varrho/r > 10$ the difference between the resonant frequencies of the two circuits does not exceed 1 per cent. In addition, formula (16) shows that parallel resonance is only possible when $\varrho/r > 1$.

The impedance of the resonant circuit in Fig. 5*c*

$$Z = \frac{(r + j\omega L)\left(-j\dfrac{1}{\omega C}\right)}{r + j\left(\omega L - \dfrac{1}{\omega C}\right)} = \frac{L\left(1 - j\dfrac{r}{\omega L}\right)}{rC\left[1 + j\dfrac{\omega L}{r}\left(1 - \dfrac{1}{\omega^2 LC}\right)\right]}.$$

Taking into account the correlations

$$\left(\frac{\varrho}{r}\right)^2 = \frac{L}{r^2 C} = 1 + Q^2,$$

$$\frac{r}{\omega L} = \frac{\omega_0}{\omega}\frac{r}{\omega_0 L} = \frac{1}{Q(1 + \delta)},$$

$$\frac{1}{\omega^2 LC} = \frac{\omega_0^2}{\omega^2 \omega_0^2 LC} = \frac{1}{\left(1 - \dfrac{r^2}{\varrho^2}\right)(1 + \delta)^2} = \frac{1 + Q^2}{Q^2(1 + \delta)^2},$$

we get an expression for the impedance of the resonant circuit

$$Z = \frac{r(1 + Q^2)\left[1 - j\dfrac{1}{Q(1 + \delta)}\right]}{1 + jQ(1 + \delta)\left[1 - \dfrac{1 + Q^2}{Q^2(1 + \delta)^2}\right]}.$$

At the resonant frequency ($\delta = 0$)

$$Z = r(1 + Q^2).$$

In those cases where the value $Q^2$ is very large in comparison with unity, the expression for the impedance of the resonant circuit is simplified thus:

$$Z \approx rQ^2 \frac{1 - j\dfrac{1}{Q(\delta + 1)}}{1 + jQ\delta \cdot \dfrac{\delta + 2}{\delta + 1}}.$$

In conditions close to resonance, when $\delta$ is incommensurably smaller than unity, the given formula can be replaced by the approximate formula

$$Z \approx rQ^2 \frac{1 - j\dfrac{1}{Q}}{1 + j2Q\delta}.$$

When the resonant circuit has a high $Q$-factor

$$Z \approx \frac{rQ^2}{1 + j2Q\delta}.$$

The currents in the branches are then

$$\left.\begin{aligned} \dot{I}_L &= \frac{Z}{r + j\omega L}\dot{I} \approx -\dot{I}\frac{jQ}{1 + j2Q\delta}, \\ \dot{I}_C &= j\omega CZ\dot{I} \approx \dot{I}\frac{jQ}{1 + j2Q\delta}. \end{aligned}\right\} \tag{8.17}$$

The node voltage

$$\dot{U} = Z\dot{I} \approx \dot{I}\frac{rQ^2}{1 + j2Q\delta}. \tag{8.18}$$

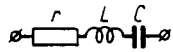
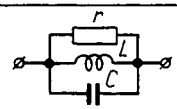
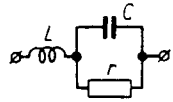
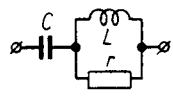
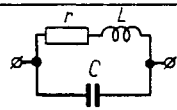
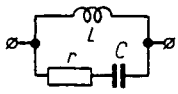
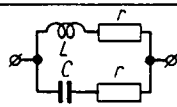
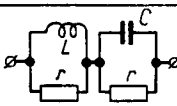
The approximate formulae (17) and (18) are analogous to formulae (14) and (8) which were educed for the Fig. 1 circuit, on the condition of replacing the voltages by currents and vice versa.

Therefore the curves of the impedance, currents and voltages which correspond to the circuit in Fig. 1 *a*, approximately express the admittance, voltages and currents of the Fig. 5 *c* circuit on a certain scale.

The power relationships at parallel resonance do not differ essentially from those examined in the case of series resonance.

Formulae for the resonant frequencies and resistances at frequencies of resonance for the simplest resonant circuits are given in Table 1.

TABLE 1. RESONANT FREQUENCIES AND RESISTANCES AT RESONANCE

| N° | Circuit | Resonant frequency | Resistance of circuit at resonance |
|---|---|---|---|
| 1 |  | $\frac{1}{\sqrt{(LC)}}$ | $r$ |
| 2 |  | $\frac{1}{\sqrt{(LC)}}$ | $r$ |
| 3 |  | $\sqrt{\left(\frac{1}{LC}-\frac{1}{(rC)^2}\right)}$ | $\frac{L}{rC}$ |
| 4 |  | $\frac{1}{\sqrt{\left[LC-\left(\frac{L}{r}\right)^2\right]}}$ | $\frac{L}{rC}$ |
| 5 |  | $\sqrt{\left[\frac{1}{LC}-\left(\frac{r}{L}\right)^2\right]}$ | $\frac{L}{rC}$ |
| 6 |  | $\frac{1}{\sqrt{(LC-(rC)^2)}}$ | $\frac{L}{rC}$ |
| 7 |  | $\frac{1}{\sqrt{(LC)}}$ | $\frac{r}{2}+\frac{L}{2rC}$ |
| 8 |  | $\frac{1}{\sqrt{(LC)}}$ | $\frac{2r\frac{L}{C}}{r^2+\frac{L}{C}}$ |

Let us consider the conditions in which the admittance of the resonant circuit shown in Fig. 5*c* reaches a minimum. As will be shown later (Ref.99), when the frequency or inductance $L$ alter, the conditions for the minimum admittance of the resonant circuit do not agree with condition (16), which characterizes parallel resonance. In the case where capacitance $C$ is varied, the admittance of the resonant circuit reaches a minimum at parallel resonance.

The admittance of the circuit

$$Y = \frac{r}{r^2 + \omega^2 L^2} - j\left(\frac{\omega L}{r^2 + \omega^2 L^2} - \omega C\right)$$

$$= \frac{r - j[\omega L - \omega C(r^2 + \omega^2 L^2)]}{r^2 + \omega^2 L^2}.$$

The square of the absolute value of the admittance is expressed by the formula

$$y^2 = \frac{1 - 2\omega^2 LC + \omega^2 C^2(r^2 + \omega^2 L^2)}{r^2 + \omega^2 L^2}.$$

For a variable

$$\frac{\partial y^2}{\partial \omega} = \frac{-4\omega r^2 LC + 2\omega r^4 C^2 + 4\omega^3 r^2 L^2 C^2 + 2\omega^5 L^4 C^2 - 2\omega L^2}{(r^2 + \omega^2 L^2)^2} = 0,$$

from which

$$\omega = \sqrt{\left[\frac{1}{LC}\sqrt{\left(1 + \frac{2r^2 C}{L}\right) - \left(\frac{r}{L}\right)^2}\right]}.$$

Only when $2r^2C/L$ is negligibly small in comparison with unity does the resulting formula coincide with (16).

When $L$ is varied

$$\frac{\partial y^2}{\partial L}$$

$$= \frac{(-2\omega^2 C + 2\omega^4 LC^2)(r^2 + \omega^2 L^2) - [1 - 2\omega^2 LC + \omega^2 C^2(r^2 + \omega^2 L^2)]2\omega^2 L}{(r^2 + \omega^2 L^2)^2}$$

$$= 0,$$

from which

$$\omega = \sqrt{\left[\frac{1}{LC} + \left(\frac{r}{L}\right)^2\right]}$$

and

$$\omega^2 LC = \frac{1 \pm \sqrt{(1 + 4\omega^2 r^2 C^2)}}{2}.$$

The resulting equations do not agree with (16).

When $C$ is varied

$$\frac{\partial y^2}{\partial C} = \frac{-\omega^2 L + \omega^2 r^2 C + \omega^4 L^2 C}{(r^2 + \omega^2 L^2)^2} = 0,$$

from which

$$\omega = \sqrt{\left[\frac{1}{LC} - \left(\frac{r}{L}\right)^2\right]}.$$

Consequently, when capacitance $C$ varies the minimum admittance, or, the same thing, the maximum impedance of the circuit occurs at parallel resonance.

In practice this phenomenon is used for tuning radio equipment.

## 2. General Information about Electrical Filters

A device which is used to segregate electrical oscillations of different frequencies is called an *electrical filter*. It is a passive four-terminal network which passes a desired band of frequencies with little attenuation. The attenuation outside this band of frequencies is great.

The band of frequencies in which the attenuation is small is known as the *pass band* or the transmission band of the filter. The remaining range of frequencies is called the *attenuation band* or stop band of the filter.

Electrical filters have a very wide and varied practical application. They are used in radio engineering, wire communication, automation, techniques of measurement and many other fields of present-day engineering which make use of the principle of segregating frequencies.

A simultaneous telegraph and telephone system over the same wire, using inductances and capacitances to prevent one form of communication interfering with the other was first used in Russia in 1880 by G. G. Ignat'ev, a captain in the Engineers, who succeeded in separating the two channels of communication (telephone and telegraph) by using induction coils and capacitors.

These elements, used for isolating a band of lower frequencies from a band of higher frequencies, made the simplest form of electrical filter.

The perfecting of electrical filters is inseparably linked with the subsequent development of high-frequency engineering and electronics. Making multi-channel wire and radio communication entailed the development of filter theory, methods of calculating filters, and the continuous improvement of the filters themselves.

Filter theory has been developed by many foreign and Soviet scientists.

This theory is based on the general theory of four-terminal networks. When this theory and methods of designing filters were first being developed,

about 1915–1923, actual circuits of ideal reactive filters composed of L-, T- and pi-structures were examined.

During the next period (1930–1935) a general theory of ladder and lattice filters was developed, which made it possible to design filters with prescribed frequency characteristics and containing a minimum number of elements. The mathematical methods developed by P. L. Chebyshev (Section 8) were then used to approximate to these given characteristics.

The need to increase the steepness of the attenuation characteristic and to reduce losses in the filter elements necessitated the use of new magnetic materials, piezoelectric resonators, etc.

Inductionless filters consisting of resistances and capacitors, and possessing a number of advantages over other types of filter (they are simple to produce, have small dimensions, etc.) have been widely used in recent years.

Electrical filters can be classified in various ways:

*Classification according to the frequencies passed.* Depending on the spectrum of frequencies which they pass, filters can be divided into *low-pass*, *high-pass*, *band-pass* and *band-elimination* (rejector) filters.

*Classification according to sections.* Filters can consist of L-, T-, pi- or lattice sections, etc. Depending on the number of its sections a filter can be single-section or composite (ladder-type).

*Classification according to characteristics.* Filters of higher classes are distinguished from the simplest *k*-filters: *m*-derived filters, *mm'* filters, etc. (the meaning of the letters *k* and *m* will be explained later).

*Classification according to the types of element.* The filters are classified as: *reactive* (consisting of *L* and *C* elements); *piezoelectric* (made basically of quartz crystals); *inductionless* (consisting of *r* and *C* elements), etc.

### 3. Conditions for T- and Pi-Networks as Filters

Two is the least number of elements which a filter can contain (L-shaped section). Filters containing T- and pi-sections may be regarded as combinations of L-sections.

In the case of a T- or pi-filter (Fig. 8*a* and *b*) parameters $A_{11}$ and $g$ are calculated from formula (112) in Chapter VII,

$$A_{11} = \cosh g = 1 + \frac{Z_1}{2Z_2}. \tag{8.19}$$

The propagation function $g$ of a symmetrical T- or pi-filter can be determined as twice the propagation function $g/2$ of an L-filter. According to (109) in Chapter VII

$$\sinh \frac{g}{2} = \sqrt{\left(\frac{Z_1}{4Z_2}\right)}. \tag{8.20}$$

Let us suppose that an L- or symmetrical T- or pi-network consists of purely reactive elements, i.e.

$$Z_1 = \pm jx_1, \quad Z_2 = \pm jx_2. \tag{8.21}$$

Here $x_1$ and $x_2$ are absolute values of the reactances; the positive signs relate to the inductive reactances and the negative to the capacitive reactances.

The pass band of a reactive filter is the region of frequencies in which the attenuation of the filter is zero ($b = 0$). It is easy to show that a four-terminal network consisting of inductive reactances or capacitive reactances only (signs $Z_1$ and $Z_2$ being identical) cannot have attenuation $b$

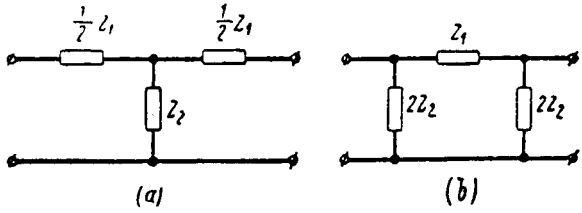



FIG. 8. Symmetrical T- and pi-filters.

equal to zero in the band of frequencies, i.e. cannot have a pass band. In other words, such a network is not a filter.

In fact, according to (20)

$$\sinh \frac{g}{2} = \sinh\left(\frac{b}{2} + j\frac{a}{2}\right) = \sinh \frac{b}{2} \cos \frac{a}{2} + j \cosh \frac{b}{2} \sin \frac{a}{2}$$

$$= \pm \sqrt{\left(\frac{x_1}{4x_2}\right)},$$

from which

$$\sinh \frac{b}{2} \cos \frac{a}{2} = \pm \sqrt{\left(\frac{x_1}{4x_2}\right)}, \tag{8.22}$$

$$\cosh \frac{b}{2} \sin \frac{a}{2} = 0. \tag{8.23}$$

In view of the fact that the hyperbolic cosine of the real argument cannot equal zero $\cosh b/2 \geqq 1$, then on the basis of (23) $\sin a/2 = 0$ and $\cos a/2 = \pm 1$.

Thus when the impedances $Z_1$ and $Z_2$ are the same type of reactance the attenuation of the filter is determined from the formula

$$\cosh \frac{b}{2} = \sqrt{\left(\frac{x_1}{4x_2}\right)}, \tag{8.24}$$

i.e. $b > 0$ (the network has no pass band).

The network only acquires the properties of a filter when the impedances $Z_1$ and $Z_2$ are opposite types of reactance, i.e. they have different signs:

$$Z_1 = \pm jx_1, \quad Z_2 = \mp jx_2. \tag{8.25}$$

Then

$$\sinh \frac{g}{2} = \pm j \sqrt{\left(\frac{x_1}{4x_2}\right)}$$

and, consequently

$$\sinh \frac{b}{2} \cos \frac{a}{2} = 0, \tag{8.26}$$

$$\cosh \frac{b}{2} \sin \frac{a}{2} = \pm \sqrt{\left(\frac{x_1}{4x_2}\right)}. \tag{8.27}$$

Equation (26) is satisfied when

$$\sinh \frac{b}{2} = 0 \tag{8.28}$$

or

$$\cos \frac{a}{2} = 0. \tag{8.29}$$

Condition (28) corresponds to the pass band ($b = 0$). In this case $\cosh b/2 = 1$ and, according to (27),

$$\sin \frac{a}{2} = \pm \sqrt{\left(\frac{x_1}{4x_2}\right)}. \tag{8.30}$$

The phase function of the filter varies in the pass band in accordance with this law ($x_1$ and $x_2$ depend on frequency).

In its turn, condition (29) corresponds to the attenuation band. In this case $a = \pm\pi$, $\sin a/2 = \pm 1$ and, by (27),

$$\cosh \frac{b}{2} = \sqrt{\left(\frac{x_1}{4x_2}\right)}. \tag{8.31}$$

The filter's attenuation varies in the stop band in accordance with the same law. Formula (31) is true for L- and symmetrical T- and pi-filters.

Then $b/2$ stands for the attenuation of each L-section. A T- or pi-filter consists of two L-sections and its attenuation is therefore equal to $b$.

It is easy to determine the width of the filter's pass band with the aid of formula (19). In fact, when $b = 0$

$$\cosh g = \cosh (b + ja) = \cosh ja = \cos a.$$

The cosine varies from $-1$ to $+1$:

$$-1 \leqq \cos a \leqq 1;$$

consequently

$$-1 \leqq 1 + \frac{Z_1}{2Z_2} \leqq 1$$

or

$$-2 \leqq \frac{Z_1}{2Z_2} \leqq 0.$$

Dividing both parts of the inequality by two we get

$$-1 \leqq \frac{Z_1}{4Z_2} \leqq 0. \tag{8.32}$$

Thus the limit frequencies lying on the edges of the pass band (the so-called *cut-off frequencies*) satisfy the conditions

$$Z_1 = -4Z_2, \tag{8.33}$$

$$Z_1 = 0. \tag{8.34}$$

The inequality (32) is known as the filter's *pass condition.*

The cut-off frequencies can be found either analytically (if the functional expressions of $Z_1$ and $Z_2$, with a variable frequency, are given) or graphically (if the frequency characteristics of $Z_1$ and $Z_2$ are given) from equations (33) and (34).

As an example, Fig. 9*b* shows the frequency characteristics of $Z_1$ and $Z_2$ of the filter given in Fig. 9*a*.

Cut-off frequencies can also be obtained by examining the frequency characteristic of the input impedance, i.e. by using the image impedance of the filter.

In the case of a T-filter the image impedance is

$$Z_T = \sqrt{\left[Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)\right]}; \tag{8.35}$$

in the case of a pi-filter it is

$$Z_{pi} = \sqrt{\left(\frac{Z_1 Z_2}{1 + \frac{Z_1}{4Z_2}}\right)}. \tag{8.36}$$

As we proved above, the condition $b = 0$ can only obtain when $Z_1$ and $Z_2$ have different signs. In this case

$$\frac{Z_1}{4Z_2} \leqq 0, \tag{8.37}$$

and $Z_1 Z_2$ is real.

Expressions (35) and (36) are real when

$$1 + \frac{Z_1}{4Z_2} \geqq 0,$$

i.e. when

$$-1 \leqq \frac{Z_1}{4Z_2}. \tag{8.38}$$

A comparison of (37) and (38) with (32) shows that the condition for which the filter's image impedance has a real value coincides with the condition for which the filter passes. The limit frequencies of the band which

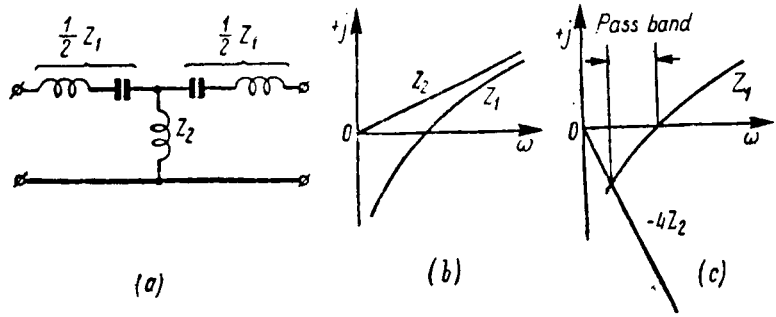



FIG. 9. Determining the pass band graphically from formulae (33) and (34).

corresponds to the real values of the characteristic impedance, are therefore the cut-off frequencies.

This can be explained physically as follows. In the band of frequencies for which the filter's characteristic impedance is real, the filter is loaded with a resistance $r$ (equal to the characteristic impedance as a result of the conditions of matched connexion). The power delivered to this load is also active and equals

$$P_2 = I_2^2 r = \frac{U_2^2}{r}, \tag{8.39}$$

where $I_2$ and $U_2$ are effective values of current and voltage at the output of the filter.

The input impedance of a filter with a matched load is also equal to $r$ and therefore the power at the filter's input is

$$P_1 = I_1^2 r = \frac{U_1^2}{r}. \tag{8.40}$$

Here $I_1$ and $U_1$ are effective values of current and voltage at the input of the filter.

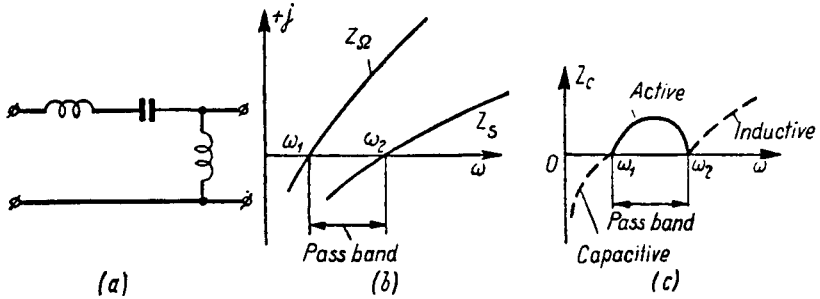



FIG. 10. Determining the pass band graphically from formula (43).

Since we are considering a lossless filter, it is clear that all power entering the network is dissipated in the load and therefore

$$P_1 = P_2. \tag{8.41}$$

It follows from (39) and (40) that

$$I_1 = I_2 \quad \text{and} \quad U_1 = U_2, \tag{8.42}$$

i.e. the attenuation is zero.

So, *the characteristic impedance is a pure resistance in the pass band, and a pure reactance in the attenuation band.*

A simple method of finding cut-off frequencies from the open-circuit ($Z_o$) and short-circuit ($Z_s$) frequency characteristics of the input impedances follows from this.

The characteristic impedance $Z_c$ of the filter is related to $Z_o$ and $Z_s$ by the formula

$$Z_c = \sqrt{(Z_o Z_s)}. \tag{8.43}$$

It is evident that $Z_c$ is real when the signs of $Z_o$ and $Z_s$ are different.

Figure 10*b* gives the curves of $Z_o$ and $Z_s$ for the filter shown in Fig. 10*a*. After multiplying $Z_o$ by $Z_s$ and extracting the root we get the curve shown in Fig. 10*c*, the real part of which corresponds to the filter's pass band.

It is clear that his band is bounded by the frequencies $\omega_1$ and $\omega_2$ (Fig. 10*c*); within the limits of this band, $Z_o$ and $Z_s$ have different signs. Outside this band $Z_o$ and $Z_s$ have the same sign and, consequently, in accordance with (43) the characteristic impedance is imaginary (a pure reactance) indicated by the broken line in Fig. 10*c*.

This condition is common to every type of filter.

### 4. Constant-*k* Filters

Let the component elements of an L-section be inverse two-terminal networks, i.e. the product of their impedances in the whole range of frequencies is constant (does not depend on frequency)

$$\tfrac{1}{2}Z_1 \cdot 2Z_2 = Z_1 Z_2 = k^2, \tag{8.44}$$

where $k$ is a real number.

It can be shown that the product of the characteristic impedances $Z_T$ and $Z_{pi}$ of the same L-section also equals $k^2$.

In fact, multiplying (35) and (36) we get

$$Z_T Z_{pi} = Z_1 Z_2 = k^2. \tag{8.45}$$

Filters which satisfy condition (44) are called *constant-k filters.*

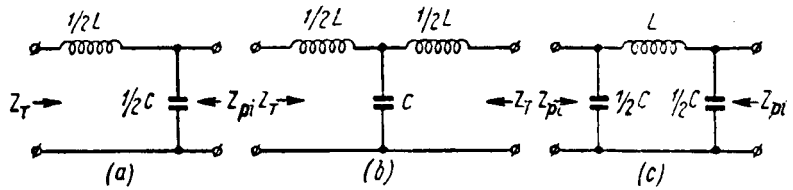



FIG. 11. Low-pass constant-*k* filter.

Taking (25) and (44) into account, formulae (30)–(32), (35) and (36) take the form:

$$\sin\frac{a}{2} = \pm\frac{k}{2x_2} = \pm\frac{x_1}{2k}, \tag{8.30a'}$$

$$\cosh\frac{b}{2} = \frac{k}{2x_2} = \frac{x_1}{2k}, \tag{8.31a'}$$

$$1 \geqq \frac{k}{2x_2} \geqq 0, \quad 1 \geqq \frac{x_1}{2k} \geqq 0, \tag{8.32a'}$$

$$Z_T = k\sqrt{\left[1 - \left(\frac{k}{2x_2}\right)^2\right]} = k\sqrt{\left[1 - \left(\frac{x_1}{2k}\right)^2\right]}, \quad (8.35a')$$

$$Z_{pi} = \frac{k}{\sqrt{\left[1 - \left(\frac{k}{2x_2}\right)^2\right]}} = \frac{k}{\sqrt{\left[1 - \left(\frac{x_1}{2k}\right)^2\right]}}. \quad (8.36a')$$

It is thus only necessary to use one of the frequency functions $x_1$ or $x_2$ (either analytically or graphically) to determine all the other characteristics of a constant-$k$ filter, if the value of $k$ is given.

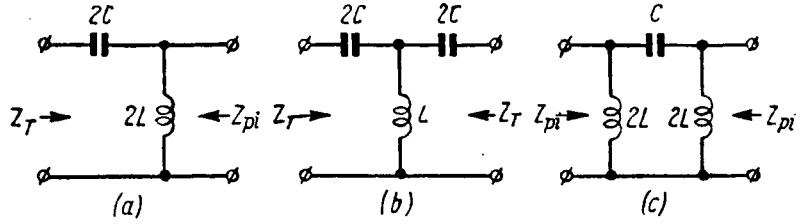



FIG. 12. High-pass constant-$k$ filter.

The formulae given above are true for L-, T- and pi-sections. In other words, for identical values of $k$ and $x_1$ (or $x_2$) the width of the pass band and the characteristic impedances of an L-filter are the same as in the case of T- and pi-filters.

The only difference is in the attenuation and the phase constant which, in the case of a T- or pi-section, are double those of an L-section.

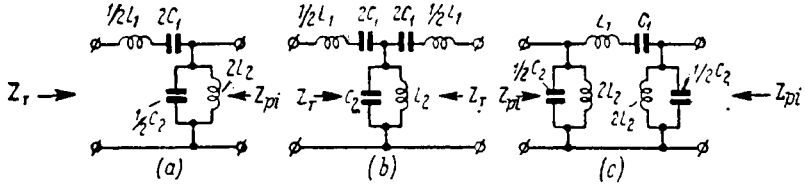



FIG. 13. Band-pass constant-$k$ filter.

Figures 11–13 show the circuits of low-pass, high-pass and band-pass filters (for L-, T- and pi-structures).

The frequency characteristics $b$, $a$, $Z_T$ and $Z_{pi}$ have the form shown in Figs. 14–16.

The calculation formulae for the principal characteristics and parameters of constant-$k$ filters are given in Tables 2 and 3.

As the calculation formulae and characteristics of Figs. 14 and 15 show, within the pass band the voltage at the input of a symmetrical filter with a matched load either leads the voltage at the output ($a > 0$—low-pass filter) or lags behind it ($a < 0$—high-pass filter). In the case of a band-pass filter the sign of $a$ varies from minus to plus within the pass band.

TABLE 2. CHARACTERISTICS OF CONSTANT-$k$ FILTERS

| Characteristics | Low-pass filter | High-pass filter | Band-pass filter |
|---|---|---|---|
| Pass band<br>$\sin\frac{a}{2}$<br>when $b = 0$ | $\frac{f}{f_c}$ | $-\frac{f_c}{f}$ | $F$ |
| Attenuation band<br>$\cosh\frac{b}{2}$<br>$a$ | $\frac{f}{f_c}$<br>$\pi$ | $\frac{f_c}{f}$<br>$-\pi$ | $\lvert F\rvert$<br>$\mp\pi$ |
| $Z_T$ | $k\sqrt{\left[1-\left(\frac{f}{f_c}\right)^2\right]}$ | $k\sqrt{\left[1-\left(\frac{f_c}{f}\right)^2\right]}$ | $k\sqrt{(1-F^2)}$ |
| $Z_{pi}$ | $\frac{k}{\sqrt{\left[1-\left(\frac{f}{f_c}\right)^2\right]}}$ | $\frac{k}{\sqrt{\left[1-\left(\frac{f_c}{f}\right)^2\right]}}$ | $\frac{k}{\sqrt{(1-F^2)}}$ |
| $k$ | $\sqrt{\left(\frac{L}{C}\right)}$ | $\sqrt{\left(\frac{L}{C}\right)}$ | $\sqrt{\left(\frac{L_1}{C_2}\right)} = \sqrt{\left(\frac{L_2}{C_1}\right)}$ |

N.B. $F = \dfrac{\frac{f}{f_m} - \frac{f_m}{f}}{\frac{f_2}{f_m} - \frac{f_m}{f_2}}$; $f_m = \sqrt{(f_1 f_2)}$.

It should be noted that if, for instance, a low-pass L-filter is loaded with the characteristic impedance $Z_{pi}$ then the load is a pure resistance in the pass band.

It can be shown that as a result of connecting the resistance ($Z_{pi}$) in parallel to the capacitive reactance of the shunt arm of the L-section ($2Z_2$), we get, in the case of a low-pass filter, an active-capacitive impedance the capacitive component of which compensates the inductive reactance of the

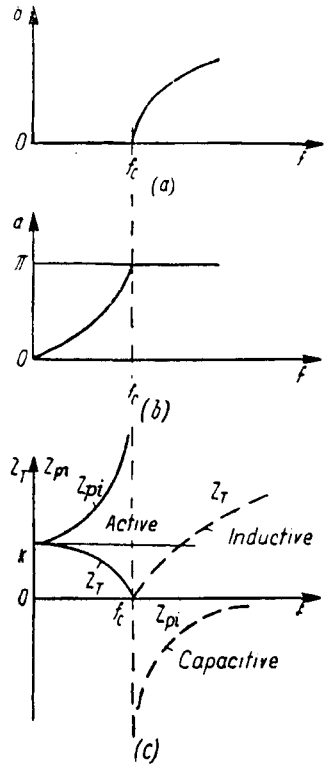



FIG. 14. Frequency characteristics of low-pass constant-$k$ filters.

TABLE 3. PARAMETERS OF CONSTANT-$k$ FILTERS

| Low-pass filter | High-pass filter | Band-pass filter |
|---|---|---|
| $L = \frac{k}{\pi f_c}$ | $L = \frac{k}{4\pi f_c}$ | $L_1 = \frac{k}{\pi(f_2 - f_1)}$ $L_2 = \frac{4\pi f_1 f_2}{k(f_2 - f_1)}$ |
| $C = \frac{1}{\pi f_c k}$ | $C = \frac{1}{4\pi f_c k}$ | $C_1 = \frac{f_2 - f_1}{4\pi k f_1 f_2}$ $C_2 = \frac{1}{\pi k (f_2 - f_1)}$ |
| $f_c = \frac{1}{\pi\sqrt{(LC)}}$ | $f_c = \frac{1}{4\pi\sqrt{(LC)}}$ | $f_1 = \frac{1}{2\pi}\left[\sqrt{\left(\frac{1}{L_1C_2} + \frac{1}{L_1C_1}\right)} - \frac{1}{\sqrt{(L_1C_2)}}\right]$ $f_2 = \frac{1}{2\pi}\left[\sqrt{\left(\frac{1}{L_1C_2} + \frac{1}{L_1C_1}\right)} + \frac{1}{\sqrt{(L_1C_2)}}\right]$ |

series arm ($\frac{1}{2}Z_1$). Consequently, for a given frequency $f$ only the resistive component equal to the characteristic impedance $Z_T$ remains.

Constant-$k$ filters possess the following shortcomings:

1. The characteristic impedances $Z_T$ and $Z_{pi}$ vary widely over the pass band, so that a satisfactory impedance match is not possible.

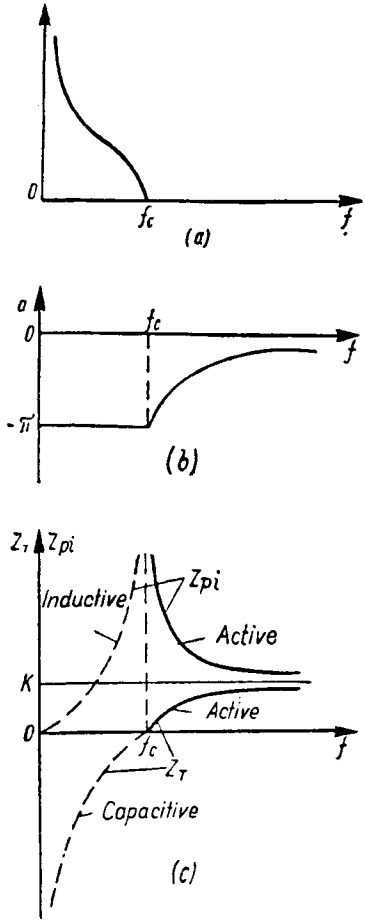



FIG. 15. Frequency characteristics of high-pass constant-$k$ filters.

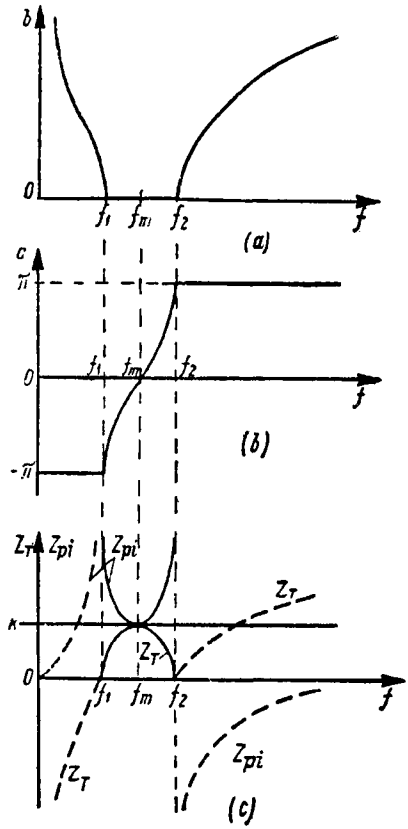



FIG. 16. Frequency characteristics of constant-$k$ band-pass filters.

2. The attenuation does not rise very rapidly at cut-off, so that frequencies just outside the pass band are not appreciably attenuated.

This may be improved by cascading a number of sections, but such methods are expensive and do not improv the characteristic impedance.

## 5. *m*-derived Filters

In order to match the load and the filter as well as possible, the characteristic impedance of the filter should be constant in the pass band. In conjunction with this we try to change the series or parallel arm of the constant-*k* L-section so that we get a new L-section with a characteristic impedance which varies less according to the frequency in the pass band.

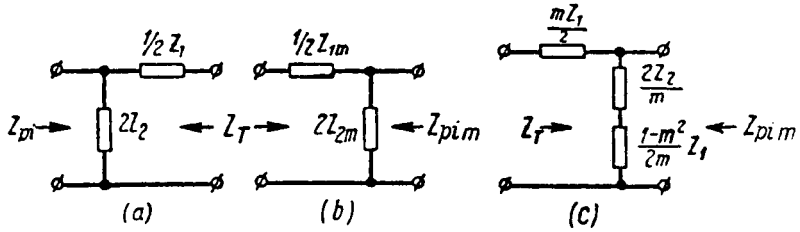



FIG. 17. Formation of *m*-derived series section.

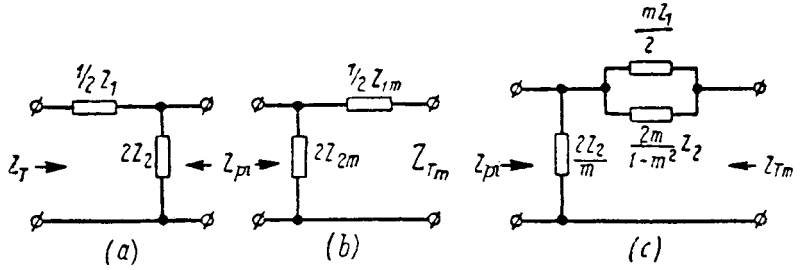



FIG. 18. Formation of *m*-derived parallel section.

The second characteristic impedance of this section should equal the characteristic impedance of the original *k*-section (called the "prototype").

The equality of the characteristics of the new filter (a so-called *m-derived filter*) and the prototype makes it possible to match them, thus forming composite filters combining the advantages of both types of filter. In view of the fact that the L-prototype has two characteristic impedances, there are two possible variants in the given case.

1. The impedances $Z_T$ remain identical; the resulting *m*-section is then called a *series-derived section* (Fig. 17).

2. The impedances $Z_{pi}$ remain identical; in this case the *m*-section is called a *parallel-derived section* (Fig. 18).

**First Variant**

It follows from the condition for the equality of the characteristic impedances $Z_T$ of the sections shown in Fig. 17*a* and *b* that

$$Z_T = \sqrt{\left[Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)\right]} = \sqrt{\left[Z_{1m} Z_{2m} \left(1 + \frac{Z_{1m}}{4Z_{2m}}\right)\right]}. \quad (8.46)$$

Let

$$Z_{1m} = mZ_1, \quad (8.47)$$

then

$$1 \geqq m \geqq 0. \quad (8.48)$$

Substituting (47) in (46) and solving the resulting equation with respect to $Z_{2m}$ we get:

$$Z_{2m} = \frac{Z_2}{m} + \frac{Z_1(1 - m^2)}{4m}. \quad (8.49)$$

As (49) shows, the parallel arm of the series-derived L-section in Fig. 17*c* consists of two impedances connected in series:

$$\frac{2Z_2}{m} \quad \text{and} \quad \frac{Z_1}{2} \cdot \frac{1 - m^2}{m}.$$

**Second Variant**

Starting with the condition for the equality of the characteristic impedances $Z_{pi}$ with respect to Fig. 18*a* and *b* we get:

$$Z_{pi} = \sqrt{\left(\frac{Z_1 Z_2}{1 + \frac{Z_1}{4Z_2}}\right)} = \sqrt{\left(\frac{Z_{1m} Z_{2m}}{1 + \frac{Z_{1m}}{4Z_{2m}}}\right)}. \quad (8.50)$$

Let

$$Z_{2m} = \frac{Z_2}{m}, \quad (8.51)$$

for which condition (48) is observed.

After substituting (51) in (50), the solution of the equation with respect to $Z_{1m}$ gives:

$$\frac{1}{Z_{1m}} = \frac{1}{mZ_1} + \frac{1}{Z_2} \frac{1 - m^2}{4m}. \quad (8.52)$$

Consequently, the series arm of the parallel-derived L-section (Fig. 18*c*) consists of the impedances $mZ_1/2$ and $2Z_2 m/(1 - m^2)$ connected in parallel.

In accordance with (49) and (52) we can find expressions for the characteristic impedances $Z_{\mathrm{pi}\,m}$ (Fig. 17) and $Z_{\mathrm{T}\,m}$ (Fig. 18):

$$Z_{\mathrm{pi}\,m} = Z_{\mathrm{pi}}\left[1 + (1 - m^2)\frac{Z_1}{4Z_2}\right], \tag{8.53}$$

$$Z_{\mathrm{T}\,m} = Z_{\mathrm{T}}\frac{1}{1 + (1 - m^2)\dfrac{Z_1}{4Z_2}}. \tag{8.54}$$

The relationship

$$Z_{\mathrm{T}\,m}Z_{\mathrm{pi}\,m} = Z_{\mathrm{T}}Z_{\mathrm{pi}} = k^2. \tag{8.55}$$

is obvious.

The pass bands of constant-$k$ filters and the $m$-derived filters obtained from them coincide. In fact, from (47) and (49)

$$\frac{Z_{1\,m}}{4Z_{2\,m}} = \frac{mZ_1}{4\left[\dfrac{Z_2}{m} + \dfrac{Z_1(1 - m^2)}{4m}\right]} = \frac{m^2}{\dfrac{4Z_2}{Z_1} + 1 - m^2}. \tag{8.56}$$

The same formulae can be obtained from (51) and (52).

When conditions (33) and (34), which correspond to the cut-off frequencies of a constant-$k$ filter, are observed, according to (56) the following conditions which correspond to the cut-off frequencies of an $m$-derived filter (with a series- or parallel-derived section) are satisfied simultaneously:

$$\frac{Z_{1\,m}}{4Z_{2\,m}} = 0, \quad \frac{Z_{1\,m}}{4Z_{2\,m}} = -1. \tag{8.57}$$

We can also estimate the extent to which the pass band of the prototype and the sections derived from it agree from the equality of the characteristic impedances (which are pure resistance in the pass band).

In the attenuation band, according to the general formula (31),

$$\cosh\frac{b}{2} = \sqrt{\left|\frac{Z_{1\,m}}{4Z_{2\,m}}\right|} = \frac{m}{\sqrt{\left|\dfrac{4Z_2}{Z_1} + 1 - m^2\right|}}. \tag{8.58}$$

The attenuation becomes infinite at the frequency $f_\infty$, when

$$4Z_2/Z_1 + 1 - m^2 = 0$$

or, the same thing,

$$\frac{Z_1}{4Z_2} = -\frac{1}{1 - m^2}. \tag{8.59}$$

Frequency $f_\infty$ corresponds to the resonance in the parallel arm—in the case of a series-derived section, or in the series arm—in the case of a

parallel-derived. At frequencies $f < f_\infty < f_c$ and $f > f_\infty > f_c$ there should be $\sinh \frac{1}{2}b$ instead of $\cosh \frac{1}{2}b$ in the left-hand side of (58), since impedances $Z_{1m}$ and $Z_{2m}$ have the same sign (Section 3). With $Z_1/4Z_2$ tending to infinity,

$$\sinh \frac{b}{2} \to \frac{m}{\sqrt{(1 - m^2)}}. \tag{8.60}$$

Figure 19 shows the family of curves for the attenuation of an $m$-derived filter (for different values of $m$) in the function of $\sqrt{|Z_1/4Z_2|}$. When $m = 1$ the attenuation curve reverts to the characteristic of a constant-$k$ filter.

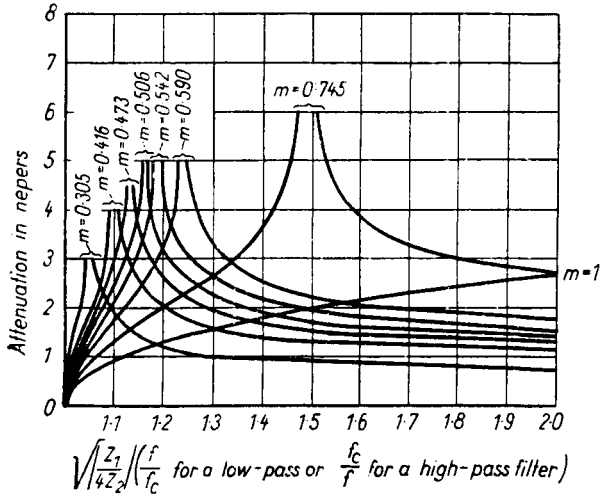



FIG. 19. Attenuation curves of $m$-derived filter.

The smaller the factor $m$, the smaller the value of $f_\infty$, i.e. the steeper the attenuation curve $b$. But according to (60), the limit value of $b$ decreases when $Z_1/4Z_2 \to \infty$ as $m$ decreases.

In addition to the great steepness of their attenuation curve, $m$-derived filters differ from constant-$k$ filters with respect to their relatively constant characteristic impedances $Z_{Tm}$ or $Z_{\pi m}$ (compared with the frequency characteristics $Z_T$ and $Z_\pi$).

Figure 20 shows the curves $Z_{Tm}/k$ and, correspondingly, $k/Z_{\pi m}$, for various values of $\sqrt{|Z_1/4Z_2|}$ which equals $f/f_c$ for low-pass and $f_c/f$ for high-pass filters.

In the spectrum of frequencies corresponding to $\sqrt{|Z_1/4Z_2|} = 0$ to 0·9 the characteristic impedances $Z_{Tm}$ and $Z_{\pi m}$ deviate from the value of $k$ by $\pm 5\%$ when $m = 0{\cdot}59$ and by $-10\%$ when $m = 0{\cdot}54$.

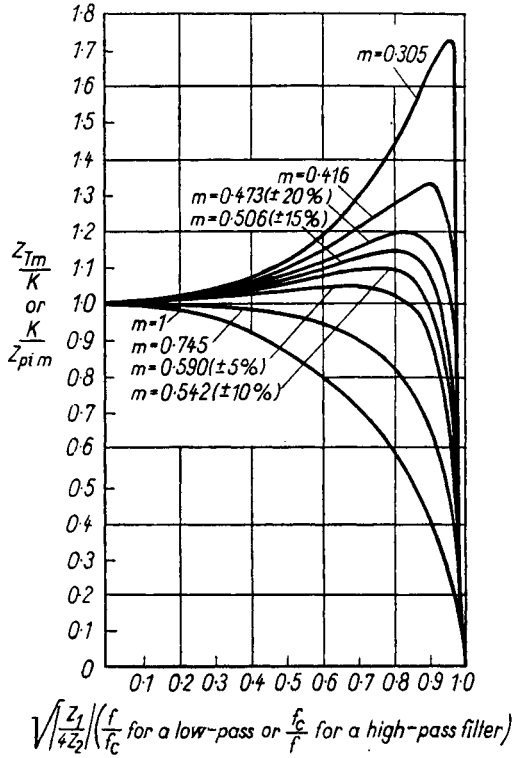



FIG. 20. Characteristic impedance of *m*-derived filter.

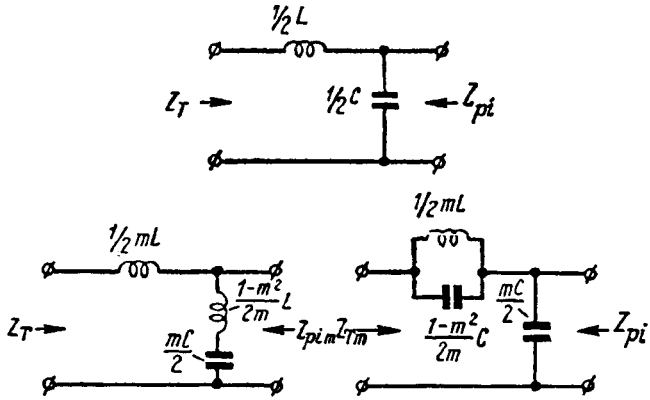



FIG. 21. Low-pass *m*-derived filter.

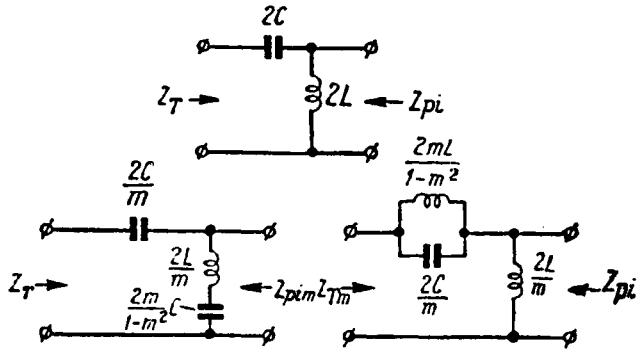



FIG. 22. High-pass *m*-derived filter.

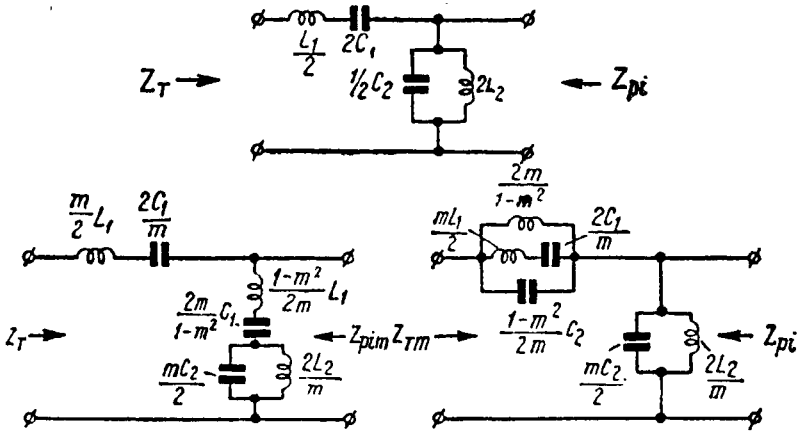



FIG. 23. *m*-derived band-pass filter.

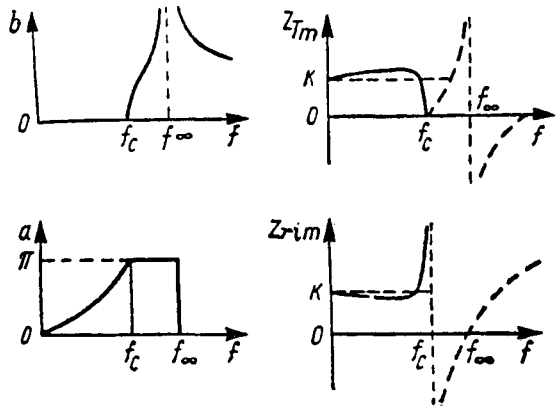



FIG. 24. Frequency characteristics of low-pass *m*-derived filter.

Circuits of the L-shaped sections of *m*-derived low-pass, high-pass and band-pass filters are given in Figs. 21–23; and the frequency characteristics of *m*-derived filters are shown in Figs. 24–26.

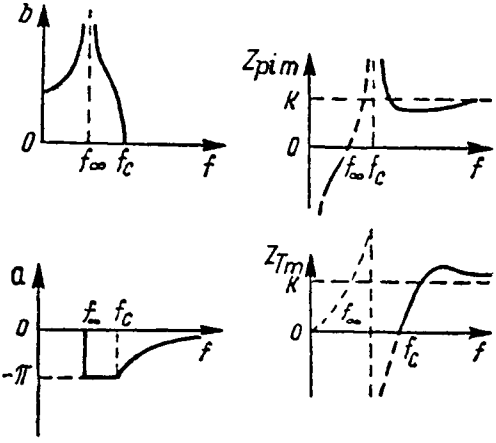



FIG. 25. Frequency characteristics of high-pass *m*-derived filter.

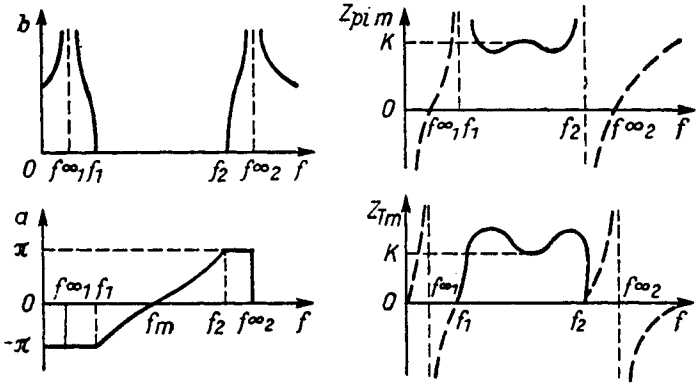



FIG. 26. Frequency characteristics of *m*-derived band-pass filter.

As the frequency characteristics show, in the pass band the characteristic impedance of *m*-derived filters is more constant than that of constant-*k* filters. In addition they possess a steeper attenuation curve.

The drawback of *m*-derived filters is that attenuation decreases when $f > f_\infty > f_c$ (for low-pass and band-pass) and when $f < f_\infty < f_c$ (for high-pass and band-pass).

By connecting *k*- and *m*-sections in series it is possible to ensure a certain degree of constancy in the characteristic impedances and the steepness of the attenuation curve, at the same time maintaining the necessary

amount of attenuation before or after a frequency of infinitely great attenuation. As an example Fig. 27*a* shows a band filter consisting of a constant-*k* T-filter (in the middle) and two L-type *m*-sections (at the ends). Figure 27*b* shows the attenuation characteristic of this filter.

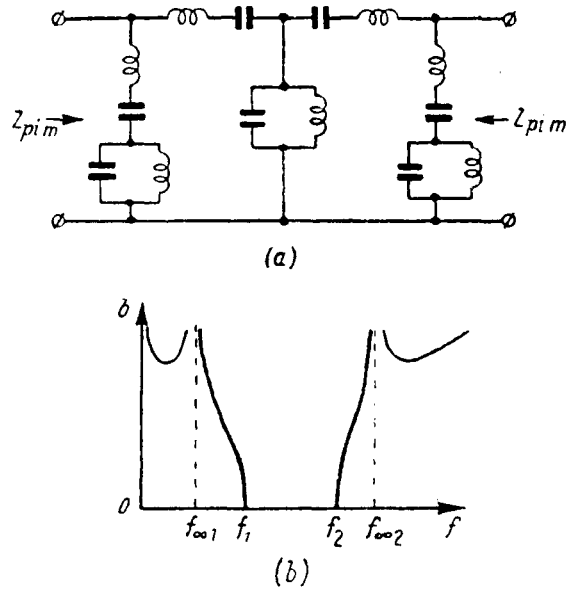



FIG. 27. Composite filter: (*a*) circuit; (*b*) attenuation characteristic.

EXAMPLE 1 (Ref. 99). It is required to calculate a composite low-pass filter consisting of *k*- and *m*-sections. Given: $k = 500\ \Omega$, cut-off frequency $f_c = 1000$ c/s, and frequencies of infinite attenuation $f_\infty = 1065$ and 1250 c/s.

The parameters of the prototype (constant-*k* filter):

$$L = \frac{k}{\pi f_c} = \frac{500}{\pi . 1000} = 0{\cdot}159\ \text{H},$$

$$C = \frac{1}{\pi f_c k} = \frac{1}{\pi . 1000 . 500} = 0{\cdot}636\ \mu\text{F}.$$

Figure 28*a* shows a constant-*k* T-section with an inductance $L/2$ in each series arm. The constant $m$ of a filter with an infinite attenuation frequency of 1065 c/s is found from (59). Taking into account the fact that

$$\frac{Z_1}{4Z_2} = -\left(\frac{f_\infty}{f_c}\right)^2,$$

equation (59) assumes the form

$$\left(\frac{f_\infty}{f_c}\right)^2 = \frac{1}{1 - m^2},$$

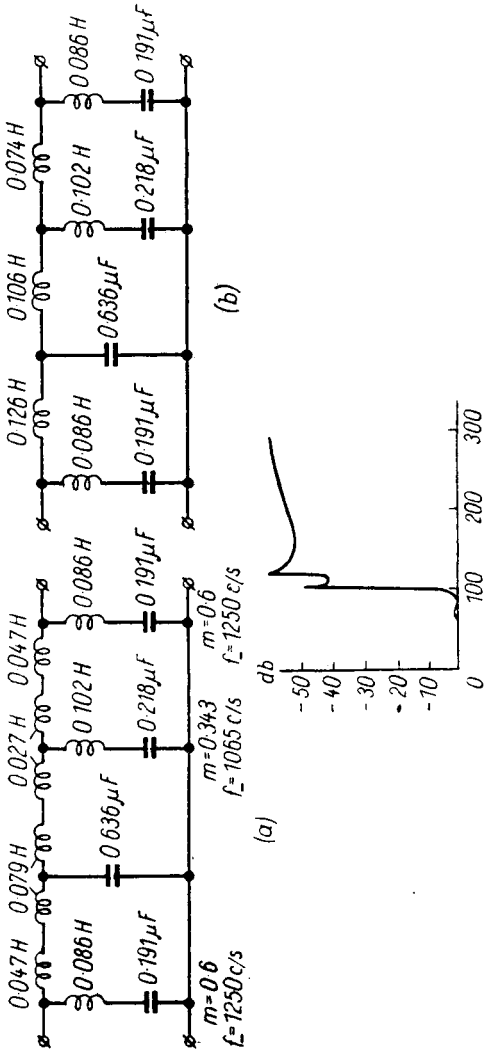



FIG. 28. Example 1: (*a*, *b*) filter network; (*c*) attenuation curve (experimental).

from which

$$m = \sqrt{\left[1 - \left(\frac{f_c}{f_\infty}\right)^2\right]} = \sqrt{\left[1 - \left(\frac{1000}{1065}\right)^2\right]} = \sqrt{(1 - 0{\cdot}882)} = 0{\cdot}343.$$

The parameters of an $m$-derived T-filter

$$\frac{mL}{2} = \frac{0{\cdot}343 \,.\, 0{\cdot}159}{2} = 0{\cdot}0273 \text{ H};$$

$$\frac{1 - m^2}{4m} L = \frac{1 - 0{\cdot}118}{1{\cdot}372} \,.\, 0{\cdot}159 = 0{\cdot}102 \text{ H},$$

$$mC = 0{\cdot}343 \,.\, 0{\cdot}636 = 0{\cdot}218 \,\mu\text{F}.$$

The $m$-derived L-sections connected to the ends of the filter and with an infinite attenuation frequency of 1250 c/s have the following parameters:

$$m = \sqrt{\left[1 - \left(\frac{1000}{1250}\right)^2\right]} = \sqrt{(1 - 0{\cdot}64)} = 0{\cdot}6;$$

$$\frac{1 - m^2}{2m} L = \frac{1 - 0{\cdot}36}{1{\cdot}2} 0{\cdot}159 = 0{\cdot}086 \text{ H};$$

$$\frac{mC}{2} = \frac{0{\cdot}6 \,.\, 0{\cdot}636}{2} = 0{\cdot}191 \,\mu\text{F}.$$

Adding the inductances connected in series in Fig. 28*a* we arrive at the final circuit which is shown in Fig. 28*b*.

Figure 28*c* shows the frequency characteristic of the given filter obtained experimentally (the quality factor of the induction coils being in the order of $Q = 40$).

## 6. Coupled Circuits as Selective Networks

When their parameters are matched coupled circuits can be regarded as a band-pass filter passing a given band of frequencies. Let us consider the circuit shown in Fig. 29 (the inductance of the winding $L/2$ and the capacitance $2C$ are connected to $Z_1/2$).

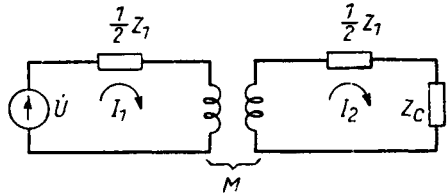



FIG. 29. Magnetic coupling.

The loop equations for such a circuit have the form

$$\left.\begin{aligned} \dot{U} &= \frac{Z_1}{2} \dot{I}_1 \pm j\omega M \dot{I}_2, \\ 0 &= \left(\frac{Z_1}{2} + Z_c\right) \dot{I}_2 \pm j\omega M \dot{I}_1. \end{aligned}\right\} \tag{8.61}$$

The joint solution of equations (61) gives

$$\dot{I}_1 = \frac{\dot{U}}{\frac{Z_1}{2} + \frac{(\omega M)^2}{\frac{Z_1}{2} + Z_c}},$$

from which, when the load is matched,

$$Z_c = \frac{\dot{U}}{\dot{I}_1} = \frac{Z_1}{2} + \frac{(\omega M)^2}{\frac{Z_1}{2} + Z_c}$$

or

$$(\omega M)^2 = Z_c^2 - \left(\frac{Z_1}{2}\right)^2.$$

From which

$$Z_c = \sqrt{\left[(\omega M)^2 + \frac{Z_1^2}{4}\right]}. \tag{8.62}$$

The pass band corresponds to the part of the spectrum in which $Z_c$ is real.

Expression (62) can be transcribed in this form:

$$Z_c = \tfrac{1}{2}\sqrt{[(Z_1 + j2\omega M)(Z_1 - j2\omega M)]}$$

$$= \frac{1}{2}\sqrt{\left[-\left(\omega L - \frac{1}{\omega C} + 2\omega M\right)\left(\omega L - \frac{1}{\omega C} - 2\omega M\right)\right]}$$

$$= \frac{1}{2\,\omega C}\sqrt{[-(\omega^2 LC - 1 + 2\omega^2 MC)(\omega^2 LC - 1 - 2\omega^2 MC)]}.$$

Taking $\omega_0^2 = 1/LC$ and $2M/L = k$ ($k$ is the coupling factor of the loops), we get

$$Z_c = \frac{1}{2\omega C}\sqrt{\left\{-\left[\left(\frac{\omega}{\omega_0}\right)^2(1+k) - 1\right]\left[\left(\frac{\omega}{\omega_0}\right)^2(1-k) - 1\right]\right\}}.$$

The characteristic impedance $Z_c$ is real on the condition that the expressions in the square brackets have different signs, namely

$$\left(\frac{\omega}{\omega_0}\right)^2(1+k) - 1 > 0, \quad \left(\frac{\omega}{\omega_0}\right)^2(1-k) - 1 < 0,$$

from which

$$\frac{\omega_0}{\sqrt{(1+k)}} < \omega < \frac{\omega_0}{\sqrt{(1-k)}}, \quad \omega_1 = \frac{\omega_0}{\sqrt{(1+k)}}, \quad \omega_2 = \frac{\omega_0}{\sqrt{(1-k)}};$$

$\omega_1$ and $\omega_2$ are the cut-off frequencies of the given band-pass filter.

In the region of frequencies below $\omega_1$, the characteristic impedance of the filter becomes capacitive, approaching the value $-j(1/2\omega C)$.

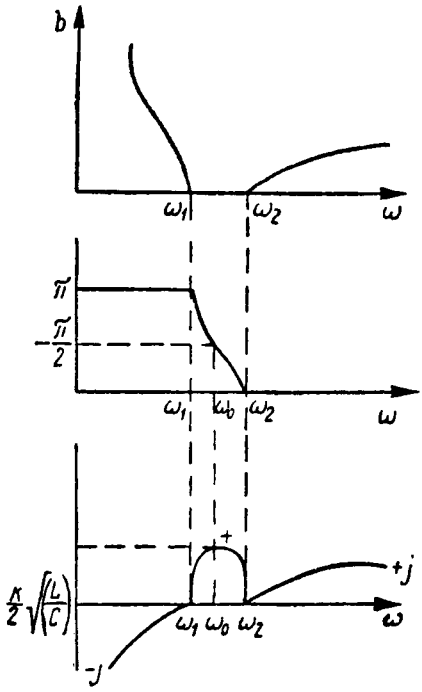



FIG. 30. Frequency characteristics of magnetically coupled circuits.

In the region of frequencies above $\omega_2$, the characteristic impedance of the filter becomes inductive and, as the frequency increases, tends to the value

$$j\omega \frac{L}{2}\sqrt{(1-k^2)}.$$

At the frequency $\omega = \omega_0 = [1/\sqrt{(LC)}]$ $Z_1 = 0$ and in accordance with (62)

$$Z_c = \omega_0 M = \frac{M}{\sqrt{(LC)}} = \frac{kL}{2\sqrt{(LC)}} = \frac{k}{2}\sqrt{\left(\frac{L}{C}\right)}.$$

Figure 30 shows the frequency characteristics of the given filter.

## 7. Lattice Filters. Piezoelectric Resonators

According to Chapter VII, Section 15, the characteristic impedance and propagation function of a lattice filter (Fig. 31) are found from the formulae

$$\left.\begin{aligned} Z_M &= \sqrt{(Z_1 Z_2)}, \\ \tanh\frac{g}{2} &= \sqrt{\left(\frac{Z_1}{Z_2}\right)}. \end{aligned}\right\} \tag{8.63}$$

Here $Z_1$ and $Z_2$ are the impedances of the arms of the lattice. In its turn

$$\tanh\frac{g}{2} = \frac{\sinh\frac{g}{2}}{\cosh\frac{g}{2}} = \frac{2\sinh^2\frac{g}{2}}{\sinh g} = \frac{\cosh g - 1}{\sinh g}. \tag{8.64}$$

If we substitute the value $g = b + ja$ in (64), after conversions we get

$$\tanh\frac{g}{2} = \frac{\sinh b + j\sin a}{\cosh b + \cos a} = \sqrt{\left(\frac{Z_1}{Z_2}\right)}. \tag{8.65}$$

The pass band of the filter ($b = 0$) corresponds to imaginary values of expression (65). Then $Z_1$ and $Z_2$ have different signs.

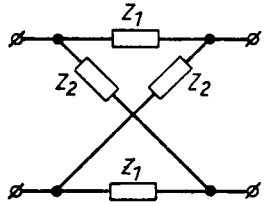



FIG. 31. Lattice filter.

The phase function is found in this case from the formula

$$\frac{\sin a}{1 + \cos a} = \tanh\frac{a}{2} = \pm\sqrt{\left|\frac{Z_1}{Z_2}\right|}. \tag{8.66}$$

The filter's attenuation band corresponds to the real values of equation (65). $Z_1$ and $Z_2$ then have the same sign and the frequency characteristic of the attenuation can be found from the formula

$$\frac{\sinh b}{\cosh b + \cos\alpha} = \sqrt{\left|\frac{Z_1}{Z_2}\right|}.$$

In its turn $\sin a/(\cosh b + \cos a) = 0$ from which $a = 0$ or $\pm\pi$.

In the first case

$$\frac{\sinh b}{\cosh b + 1} = \tanh\frac{b}{2} = \sqrt{\left|\frac{Z_1}{Z_2}\right|},$$

which corresponds to the condition $|Z_1| < |Z_2|$, since a hyperbolic tangent cannot exceed unity.

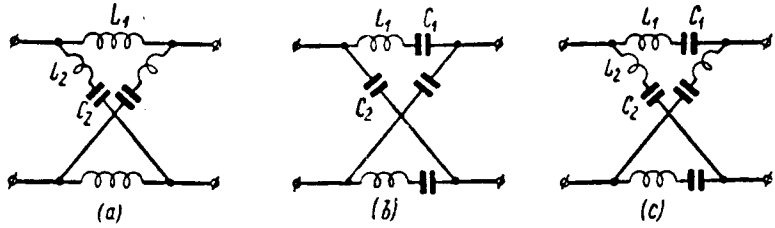



FIG. 32. Lattice filters.

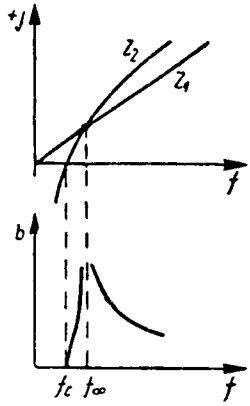



FIG. 33. Finding cut-off frequency and infinite-attenuation frequency of a lattice filter.

In the second case

$$\frac{\sinh b}{\cosh b - 1} = \coth\frac{b}{2} = \sqrt{\left|\frac{Z_1}{Z_2}\right|},$$

which corresponds to the condition $|Z_1| > |Z_2|$, since a hyperbolic cotangent cannot be less than unity.

When $Z_1 = Z_2$, attenuation $b = \infty$, since $\tanh \frac{1}{2}b = 1$. Physically, it means that the bridge is in equilibrium condition and hence the output voltage equals zero.

Figure 32 shows the simplest circuits of low-pass (Fig. 32*a*) and high-pass (Fig. 32*b*) lattice filters, and a band-pass filter (Fig. 32*c*).

Starting from the fact that $Z_1$ and $Z_2$ have different signs in the pass band, and that $b = \infty$ at the point where curves $Z_1$ and $Z_2$ intersect, i.e. when $Z_1 = Z_2$, it is easy to find cut-off frequency $f_c$ and the infinite attenuation frequency $f_\infty$.

As an example this is shown with application to a low-pass filter in Fig. 33. When $Z_1$ and $Z_2$ do not intersect, there is no infinite attenuation

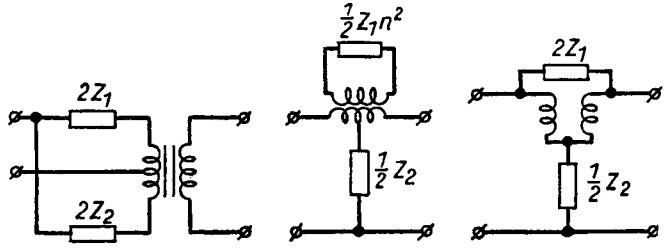



FIG. 34. Differential lattice circuits.

frequency, or it retreats to infinity (for an asymptotic approximation of $Z_1$ and $Z_2$).

In practice we also use lattice networks with a differential transformer (differential lattice networks [Refs. 1 and 9]) which are equivalent to lattice networks, with respect to their electrical values, but have fewer elements (Fig. 34).

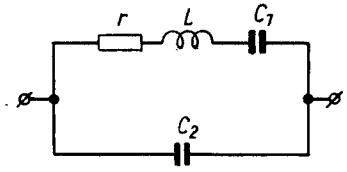



FIG. 35. Equivalent circuit of a piezoelectric resonator.

It should be noted that, in accordance with (63), $Z_M$ depends on the product and $g$ on the ratio of the impedances of the arms of the lattice filter. The characteristic parameter $Z_M$ and $g$ of a lattice network are therefore not related to each other as in, for example, T- and pi-filters.

Lattice filters are frequently constructed with the aid of *piezoelectric resonators*. The latter form an electromechanical system consisting of a piezoelectric plate*, electrodes and a support. Under the effect of an alternating electrical field the piezoelectric plate produces mechanical oscillations at the frequency of the applied voltage. Electrical charges then develop on its surfaces. When the frequency of the field coincides with the frequency of the plate's natural oscillations resonance occurs. The amplitude of the oscillations and, correspondingly, the magnitude of the charges on the plate reach a maximum.

* Quartz plates are mainly used in communication engineering.

The two-terminal network in Fig. 35 is an equivalent electrical circuit of a piezoelectric resonator.

The frequency characteristic of the reactance of the resonator is shown in Fig. 36.

As opposed to induction coils, the $Q$-factor of which is measured in hundreds at best, the $Q$-factor of piezoelectric resonators is calculated in tens of thousands. By using these resonators it is possible to obtain band-pass filters with a very narrow pass band and an attenuation curve which is

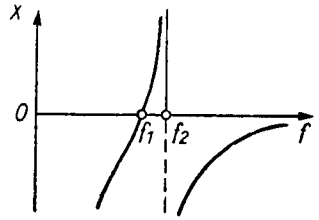



FIG. 36. Frequency characteristic of reactance of resonator.

extremely steep in the vicinity of the cut-off frequency (quartz lattice filters permit a pass band in the order of 20–50 periods at a frequency of 500 kc/s).

Capacitance $C_2$ contained in the equivalent electrical circuit of the piezoelectric resonator (Fig. 35) is considerably bigger than capacitance $C_1$; therefore the resonant frequencies $f_1$ and $f_2$ of the circuit lie very close together (Fig. 36).

Connecting an auxiliary capacitance in parallel with the resonator approximates the frequencies even more closely. By connecting an induction coil in series with the resonator it is possible, if necessary, to widen the filter's pass band.

## 8. Filters Based on the Chebyshev Approximation*

### Chebyshev and Maximally Flat Approximation Curves

Methods of designing filters are closely connected with the problem of approximating functions. The most convenient approximation methods are those developed by the eminent Russian mathematician P. L. Chebyshev. Besides the Chebyshev approximation curve, a maximally flat (or Butterworth) approximation curve is also used. At present both approximation methods are widely used; but the Chebyshev approximation possesses a number of advantages.

The maximally flat curve, located to one side of the approximated curve, does not make full use of the permissible deviations, whereas the Chebyshev curve, with an oscillatory character within the limits of the approximation section, attains the upper and lower permissible limits alternately,

* Written by K. A. Melik-Vartanian.

assuming the exact value of the function between the two adjacent extreme points (Fig. 37*a*). At the end of the approximation section the flat curve gives a very rough approximation; the Chebyshev curve gives a much smaller deviation in the same conditions. When the flat curve lies on both sides of the approximated curve the error is smaller (Fig. 37*b*).

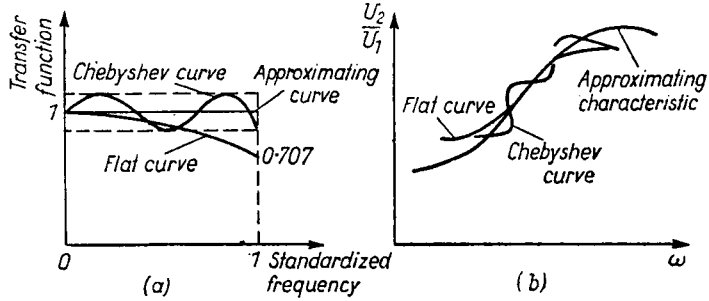



FIG. 37. Approximation curves.

Lattice filters based on a Chebyshev approximation always allow an equivalent conversion into other forms of filter, e.g. ladder filters. This is of considerable significance since various methods of designing filters based on lattice networks have been developed. Lattice networks are not suitable for practical purposes because they contain more elements than ladder networks.

The disadvantage of the Chebyshev approximation is the extreme nonlinearity of the phase characteristic in the pass band.

**Chebyshev Polynomials**

In algebraic form Chebyshev *polynomials* used as approximating functions have the appearance

$$T_n(x) = \tfrac{1}{2}\{[x + \sqrt{(x^2 - 1)}]^n + [x - \sqrt{(x^2 - 1)}]^n\} \qquad (8.67)$$

and are polynomials of the $n$-th power with a factor at the highest term $2^{n-1}$. Ascribing to $n$ the values of the series of integers 1, 2, ..., we can obtain Chebyshev polynomials in normalized form (Fig. 38):

$$T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1,$$
$$T_5(c) = 16x^5 - 20x^3 + 5x,$$
$$T_6(x) = 32x^6 - 48x^4 + 18x - 1.$$

Supposing $x = \cos\varphi$, in so far as $x$ is regarded on the section $[-1, 1]$, it is easy to obtain the trigonometrical form of the Chebyshev polynomials

$$T_n(x) = \tfrac{1}{2}\{[\cos\varphi + \sqrt{(\cos^2\varphi - 1)}]^n\} + \{[\cos\varphi - \sqrt{(\cos^2\varphi - 1)}]^n\}$$

$$= \tfrac{1}{2}(\cos n\varphi + j\sin n\varphi + \cos n\varphi - j\sin n\varphi)$$

$$= \cos n\varphi = \cos n \operatorname{arc}\cos x.$$

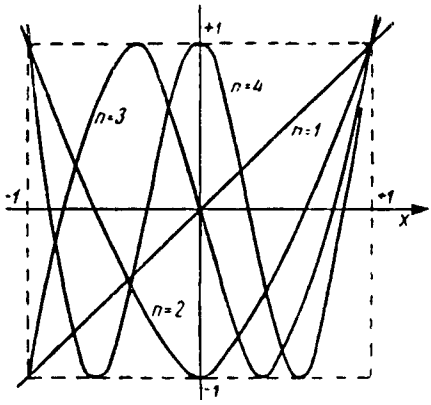



FIG. 38. Plots of Chebyshev polynomials.

The zeros of the polynomial $T_n(x)$ can be found by equating the right-hand side of the last equation to zero

$$\cos n \operatorname{arc}\cos x = 0,$$

from which

$$n \operatorname{arc}\cos x = \frac{1}{2}\pi; \quad \frac{3}{2}\pi; \quad \frac{5}{2}\pi; \quad \frac{2k-1}{2}\pi;$$

and the zeros of the polynomial will be the points

$$x = \cos\frac{2k-1}{2n}\pi \quad (k = 1, 2, \ldots n).$$

The points of the extreme can be found analogously

$$|\cos n \operatorname{arc}\cos x| = 1,$$

from which

$$n \operatorname{arc}\cos x = 0; \quad 1\pi; \quad 2\pi; \quad 3\pi; \quad \ldots \; l\pi$$

and

$$x = \cos\frac{l\pi}{n} \; (l = 0, 1, 2 \ldots n).$$

When $l$ is even, $T_n(x)$ passes through the maximum:

$$T_n(x) = \cos l\pi = +1,$$

and when $l$ is odd, it passes through the minimum:

$$T_n(x) = -1.$$

Consequently, the zeros of the polynomial are so arranged that the extreme points are strictly balanced: all the maxima equal $+1$ and all the minima equal $-1$.

At the end of the main interval (at points $\pm 1$) the Chebyshev polynomials also assume the value $\pm 1$:

$$T_n(+1) = +1$$

$$T_n(-1) = \begin{cases} -1 & \text{when } n \text{ is odd;} \\ +1 & \text{when } n \text{ is even.} \end{cases}$$

Outside the section $[-1, 1]$ the Chebyshev polynomials no longer have an oscillatory character but increase or decrease monotonally, the steepness of the rise or fall of the curve being directly proportional to the square of the power of the polynomial $n$ (Fig. 39).

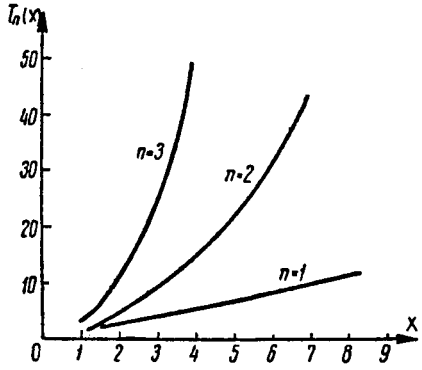



FIG. 39. Chebyshev polynomials outside approximation section ($x \gg 1$).

The value of the polynomials $T_n(x)$ when $|x| > 1$ can be found from the formula

$$T_n(x) = \cosh (n \operatorname{arc\,cosh} x).$$

In order to find the zeros and extreme points of Chebyshev polynomials the following graphic construction may be used: a semicircle with a diameter of the section $[-1, 1]$ should be divided into $2n$ parts and these parts

plotted on the abscissa axis and numbered from right to left (Fig. 40). The points with odd numbers give the position of the zeros and those with even numbers the position of the maxima and minima of the polynomial. It is

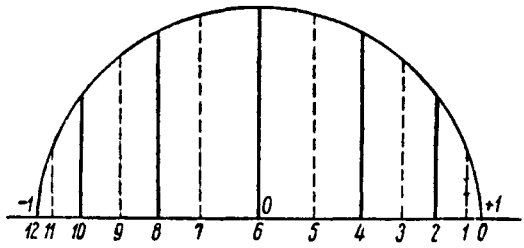



FIG. 40. Finding zeros and extreme points of Chebyshev polynomial $T_6(x)$.

evident from the construction that the zeros and the extreme points are located unevenly on the section $[-1, 1]$, being densest at its ends.

The recurrent formula for the series calculation of Chebyshev polynomials is as follows:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{8.68}$$

**Approximating Functions by Chebyshev Polynomials**

Let us pose the problem of reproducing approximately a given characteristic with a prescribed degree of accuracy, using Chebyshev polynomials for this purpose.

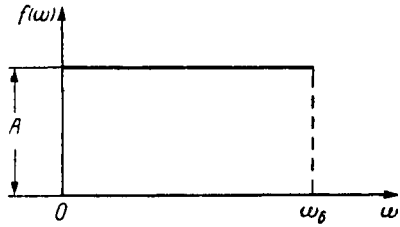



FIG. 41. Ideal frequency characteristic.

Let us approximate the ideal characteristic given by the graph in Fig. 41. In the band of frequencies $0$–$\omega_c$ the characteristic has a constant value A, in the band $\omega_c$–$\infty$ it equals zero. We shall take $T_5(x)$ and $T_6(x)$ as the approximating polynomials. Since Chebyshev polynomials have an oscillatory character only within the limits of the section $[-1, 1]$, the ideal characteristic should be represented in that range by the appropriate selec-

tion of a new variable. It is easy to see that the new variable is

$$x = \frac{\omega}{\omega_c}.$$

Thus, when the frequency is correspondingly normalized, the problem is reduced to approximating a horizontal straight line on the section [0, 1].

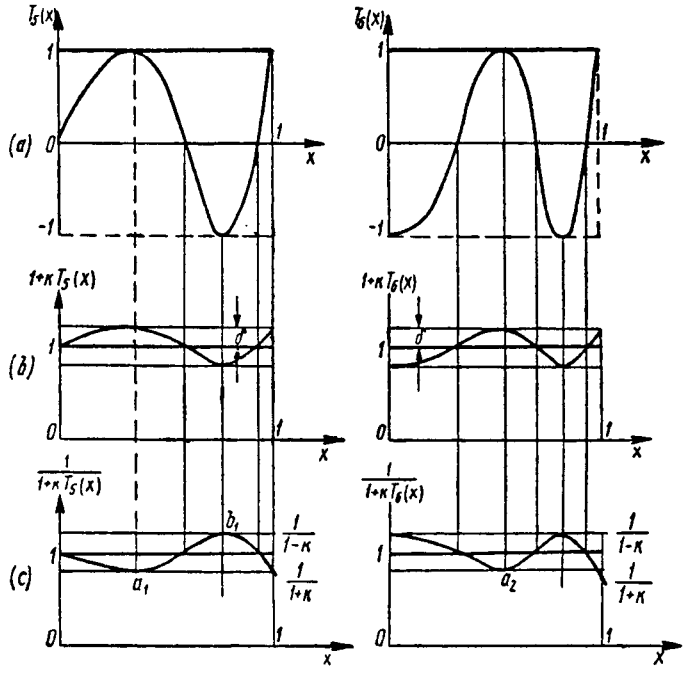



FIG. 42. Approximation of a horizontal straight line of height equal to unity using polynomials $T_5(x)$ and $T_6(x)$.

Figure 42*a* shows the horizontal straight line of height equal to unity which is the ideal characteristic, and the polynomials $T_5$ and $T_6$ in the range [0, 1]. Functions

$$1 + T_5(x) \quad \text{and} \quad 1 + T_6(x)$$

are the same curves displaced by one unit along the ordinates axis.

Reproducing the straight line of height equal to unity with the given accuracy means approximating it so that the maximum values of the approximation curve's deviations from unity do not exceed the permissible value $\delta$. For this reason the polynomials $T_5$ and $T_6$ should first be multiplied by a factor $k$ smaller than unity.

Consequently, the functions

$$1 + kT_5(x) \quad \text{and} \quad 1 + kT_6(x) \tag{8.69}$$

when $k$ is selected accordingly, approximate unity in the Chebyshev manner on section [0, 1] with the given accuracy; the ripple of these curves does not then exceed the value $\delta$ in amplitude and occurs between the upper and lower permissible limits (Fig. 42*b*). However, the progress of the curve is interesting not only in the range [0, 1] but also outside it. It must be ensured that, as far as possible, the curve beyond section [0, 1] falls steeply to zero. The functions which are reciprocals of (69) possess this property.

The approximating functions are thus represented in the form (Fig. 42*c*)

$$f_5(x) = \frac{1}{1 + kT_5(x)} \quad \text{and} \quad f_6(x) = \frac{1}{1 + kT_6(x)}. \tag{8.70}$$

It should be noted that for an approximation of this kind full use is made of the properties of Chebyshev polynomials. As is generally known, Chebyshev polynomials not only deviate least from zero in the section $[-1,1]$ but also exceed in modulus all polynomials of the same kind (of the same power of $n$ and with the same highest factor) beyond the principal section. Therefore, for the synthesis of filters they ensure the best approximation of the frequency characteristic in the pass band and the steepest drop in the characteristic at the cut-off frequency, thereby effecting the optimum separation of frequencies. The steepness of the drop in the characteristic also depends on the power of the polynomial $T_n(x)$; the higher it is the steeper the droop of the curve.

It will be seen from a comparison of Fig. 42*a* and *b* that the approximating functions $f_5$ and $f_6$ reach a minimum at points $a_1$ and $a_2$, where the Chebyshev polynomials reach their maximum value – unity. Consequently, the minimum value of functions $f_5(x)$ and $f_6(x)$ is $1/(1 - k)$.

Similarly, functions $f_5$ and $f_6$ reach a maximum at points $b_1$ and $b_2$ since the polynomials $T_5$ and $T_6$ have their minimum value – minus one. The maximum value of functions $f_5(x)$ and $f_6(x)$ is therefore $1/(1 - k)$.

At the given upper limit of the permissible deviation $\delta$ the value of factor $k$ can be determined from the equality

$$\delta = \frac{1}{1 - k} - 1 = \frac{k}{1 - k},$$

from which

$$k = \frac{\delta}{1 + \delta}.$$

The lower limit of the deviation of functions $f_5$ and $f_6$ can then easily be found from the formula

$$1 - \frac{1}{1+k} = \frac{k}{1+k} = \frac{\delta}{1+2\delta} < \delta,$$

from which it is evident that the upper and lower limits of the deviations of the approximating functions can be found at various distances from the straight line.

For calculating electrical filters as well as approximating by Chebyshev polynomials they also use *Chebyshev fractions* which deviate least from zero. Since we cannot examine these functions in greater detail, we shall simply mention that Chebyshev functions are analogous to the polynomials with respect to their properties.

## Normalized Parameters of a Lattice Network. Conditions of Passing and Attenuating

The possibility of selecting the characteristic parameters $Z_M$ and $g$ of a lattice filter independently, which is one of its important advantages over L-, T- and pi-type constant-$k$ and $m$-derived filters, is a result of the fact that parameters $Z_M$ and $g$ of lattice filters are functions of different values: the characteristic impedance depends on the cut-off frequencies and resonant frequencies of the arms of the lattice in the attenuation band and the propagation function depends on the cut-off frequencies and resonant frequencies of the arms in the pass band. These parameters are determined from formulae (65).

For the convenience of the calculations we convert these to normalized parameters by dividing all the impedances by the value $R$, which is the nominal value of the characteristic impedance in the filter's pass band and equal to the load impedance (with matched connexion).

Thus the *normalized* characteristic impedance

$$\frac{Z_M}{R} = \sqrt{\left(\frac{Z_1 Z_2}{R^2}\right)} = \sqrt{(Z_1' Z_2')}; \tag{8.71}$$

and

$$\tanh \frac{g}{2} = \sqrt{\left(\frac{Z_1}{Z_2}\right)} = \sqrt{\left(\frac{Z_1 : R}{Z_2 : R}\right)} = \sqrt{\left(\frac{Z_1'}{Z_2'}\right)}. \tag{8.72}$$

In these formulae

$$Z_1' = \frac{Z_1}{R} \quad \text{and} \quad Z_2' = \frac{Z_2}{R} \tag{8.73}$$

are the normalized impedances of the arms of the lattice network.

Let us denote

$$\sqrt{(Z_1' \cdot Z_2')} = \alpha, \tag{8.74}$$

$$\sqrt{\left(\frac{Z_1'}{Z_2'}\right)} = \beta. \tag{8.75}$$

Functions $\alpha$ and $\beta$ are used as the criterion on which many writers base the classification of filters according to their impedances or attenuation. The joint solution of equations (74) and (75) with respect to $Z_1'$ and $Z_2'$ gives the normalized impedances of the arms of the lattice:

$$\alpha\beta = Z_1',$$

$$\frac{\alpha}{\beta} = Z_2'.$$

Function $\alpha$, the normalized characteristic impedance is real in the pass band and imaginary in the attenuation band; function $\beta$ (the normalized propagation function) is real in the attenuation band and imaginary in the pass band. $\alpha$ and $\beta$ are functions of the resonant frequencies and cut-off frequencies of the arms of the lattice.

The purpose of the calculation given below is to match these frequencies in such a way that within the range of their real values functions $\alpha$ and $\beta$ differ least from unity. In other words it is necessary to ensure that functions $\alpha$ and $\beta$ approximate unity in the Chebyshev manner.

It is essential to obtain these functions close to unity for the following reasons.

When $\alpha = 1$, $Z_M = R$, i.e. the characteristic impedance equals its nominal value throughout the filter's pass band. This is the ideal case which ensures that the filter operates under matched conditions throughout the pass band.

When $\beta = 1$

$$\tanh \frac{g}{2} = 1,$$

from which it is easy to find that

$$g = b = \infty,$$

i.e. in the stop band the attenuation is infinitely great. This is also an ideal case since all frequencies are completely attenuated by the filter.

Actual filters can be approximated with varying degrees of accuracy to ideal filters.

### Calculating Filters by Cauer's Method

The Cauer method of calculating filters is based on an approximation using a modified Chebyshev fraction. The essence of the method is as follows.

By substituting a new variable $x$ for the variable $\omega$ in functions $\alpha$ and $\beta$ and new parameters $x_i$ for the resonant frequencies $\omega_i$ it is possible to reduce them completely to one form for all classes of filter.

The new variable and new parameters for a low-pass filter are determined from the formulae:

$$\text{for functions } \alpha \quad x = \frac{\omega}{\omega_c}; \quad x_i = \frac{\omega_i}{\omega_c}, \tag{8.76}$$

$$\text{for functions } \beta \quad x = \frac{\omega_c}{\omega}; \quad x_i = \frac{\omega_c}{\omega_i}. \tag{8.77}$$

With this selection of variables the range of approximation [0, 1] corresponds to the pass band for function $\alpha$, and the attenuation band for function $\beta$. In the case of other classes of filter $x$ and $x_i$ are determined from corresponding formulae: e.g. if a high-pass filter is being examined, formula (76) is used for function $\beta$ and (77) for function $\alpha$.

Normalizing the frequency thus makes it possible to reduce functions $\alpha$ and $\beta$ to the form

$$f_{2n}(x) = H^{\mp 1} \frac{\sqrt{\left(1 - \frac{x^2}{x_1^2}\right)\left(1 - \frac{x^2}{x_3^2}\right) \cdots \left(1 - \frac{x^2}{x_{n-1}^2}\right)}}{\left(1 - \frac{x^2}{x_2^2}\right)\left(1 - \frac{x^2}{x_4^2}\right) \cdots \left(1 - \frac{x^2}{x_n^2}\right)} \tag{8.78}$$

for even functions, and to the form

$$f_{2n-1}(x) = \frac{\sqrt{\left(1 - \frac{x}{x_1}\right)\left(1 + \frac{x}{x_2}\right)\left(1 - \frac{x}{x_3}\right) \cdots \left(1 \pm \frac{x}{x_n}\right)}}{\sqrt{\left(1 + \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right)\left(1 + \frac{x}{x_3}\right) \cdots \left(1 \mp \frac{x}{x_n}\right)}} \tag{8.79}$$

for odd functions; and

$$1 = x_1 < x_2 < x_3 < \cdots < x_n < \infty.$$

The upper signs in the factors of (78) and (79) relate to an even $n$, the lower to an odd $n$.

In order that functions $f_{2n}$ and $f_{2n-1}$ approximate a single straight line by the Chebyshev method, the normalized resonant frequencies $x_i$ should be determined by the formulae

$$x_i = \frac{1}{\operatorname{sn}\left(\frac{n - i + 1}{n}\right) K} \quad (i = 1, 2, \ldots, n) \tag{8.80}$$

for even functions, and

$$x_i = \frac{1}{\operatorname{sn}\left[\dfrac{2(n-1)+1}{2n-1}\right]K} \quad (i = 1, 2, \ldots, n) \tag{8.81}$$

for odd functions.

In these formulae, sn is a Jacobi elliptical function (elliptical sine) and $K$ a complete elliptical integral of the first kind having its value when the upper limit is equal to $\pi/2$, i.e.

$$K = \int_0^{\pi/2} \frac{d\psi}{\sqrt{(1 - k^2 \sin^2 \psi)}};$$

$$\text{when} \quad k = 0 \quad K = \frac{\pi}{2},$$

$$\text{when} \quad k = 1 \quad K = \int_0^{\pi/2} \frac{d\psi}{\cos \psi} = \infty.$$

Value $k$, a modulus of the elliptical sine, is a normalized limit frequency. A Chebyshev approximation of the function to unity takes place at the section $\pm k$.

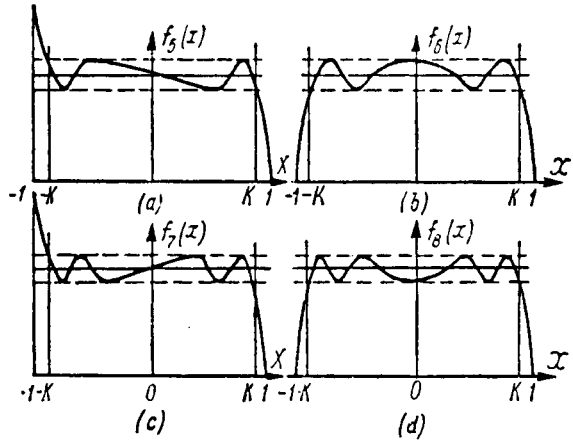



FIG. 43. Plots of functions $f(x)$.

The nature of the approximation to a horizontal straight line by functions

$$f_5(x), \quad f_6(x), \quad f_7(x) \quad \text{and} \quad f_8(x)$$

is given in Fig. 43.

**The Attenuation Frequency Characteristic of a Filter Using a Chebyshev Approximation**

Let, for example, the function

$$f_8(x) = \frac{\sqrt{\left(1 - \frac{x^2}{x_1^2}\right)\left(1 - \frac{x^2}{x_3^2}\right)}}{H\left(1 - \frac{x^2}{x_2^2}\right)\left(1 - \frac{x^2}{x_4^2}\right)}$$

be selected as functions $\alpha$ and $\beta$.

In accordance with Fig. 43*d* the Chebyshev approximation of the functions $f_8(x)$ to a horizontal straight line is shown in Fig. 44.

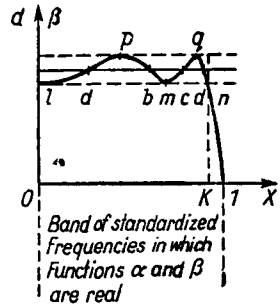



FIG. 44. Approximation of unity in the Chebyshev manner by the function $f_8(x)$.

Approximating function $\alpha$ to a horizontal straight line ensures that the characteristic impedance is constant in the pass band, matched conditions thus being achieved.

The points of intersection of functions $\beta$ of the single straight line denoted by letters *a*, *b*, *c* and *d* correspond to the frequencies of infinite attenuation. The points at which function $\beta$ deviates most from unity, denoted by *l*, *m*, *n*, *p* and *q* (regardless of sign) correspond to frequencies of minimum attenuation. It should be noted that the permissible deviations of functions $f(x)$ are such that apart from the individual equality of all the maxima and minima, their geometrical mean equals unity

$$\beta_{\max}\,\beta_{\min} = 1.$$

This fact ensures that the minima of attenuation throughout the attenuation band are equal—a typical feature of Chebyshev attenuation

behaviour. In fact, by solving equation (72) with respect to $g$ we find:

$$g = \ln \frac{\sqrt{\left(\frac{Z_1}{Z_2}\right)} + 1}{\sqrt{\left(\frac{Z_1}{Z_2}\right)} - 1} = \ln \frac{\beta + 1}{\beta - 1},$$

from which attenuation $b$ is determined by the formula

$$b = \ln \left| \frac{\beta + 1}{\beta - 1} \right|.$$

It will be seen from the last equation that $\beta_{max} = 1/\beta_{min}$ gives the same attenuation as $\beta_{max}$.

Figure 45 shows the frequency characteristic of attenuation in the stop band of the low-pass filter examined above. A value which is the reciprocal of the normalized frequency, i.e. $1/x$, is marked on the abscissa axis, or (on

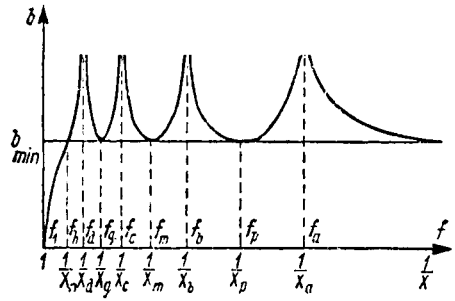



FIG. 45. Attenuation when approximating function $f_8(x)$.

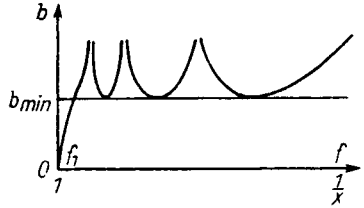



FIG. 46. Attenuation when approximating function $f_7(x)$.

another scale) the current frequency $f$. When frequency $f_n(x_n^{-1})$ corresponds to point $n$ (Fig. 44), attenuation reaches the permissible maximum for the first time. The range of frequencies from $f_1$ to $f_n$ should be regarded as a transition region in which the cut-off frequency also lies.

The number of frequencies of infinite attenuation, which equals the number of frequencies of minimum attenuation (including the minimum

when $f = \infty$), is determined by the degree of the approximating function $\beta$. In the present case (an even degree) it equals $n = 8/2 = 4$. As it was remarked above, for the purpose of separating frequencies better, the degree of the approximating function should be made higher. This ensures a steeper droop in the curve.

If the approximation curve is $f_7(x)$, then it will be seen from Fig. 43*c* that the last section of the attenuation characteristic goes to infinity (Fig. 46). Here, too, the number of frequencies of infinite attenuation is equal to the number of frequencies of minimum attenuation, if we only consider finite frequencies and do not take the pole of the attenuation function at infinity into account.

The frequency characteristics given in Figs. 45 and 46 show the typical shape of the attenuation curve in the attenuation band of filters during a Chebyshev approximation. Analogous attenuation curves are also obtained for Chebyshev approximations for ladder networks.

From this point of view we can introduce the concept of Chebyshev conditions of change in a variable quantity. *When any quantity fluctuates between two fixed limits it is said to have Chebyshev behaviour.* For example, Chebyshev polynomials and fractions in the range $[-1, 1]$ have Chebyshev behaviour. In a particular case the upper limit may equal $\infty$. This case occurs in Chebyshev attenuation conditions in the filters examined above. The lower limit may also equal zero exactly. This case may occur in Chebyshev attenuation behaviour in the pass-band of the filter (in this case attenuation takes place as a result of energy reflexion caused by the mismatching of the filter's load impedance and characteristic impedance.

It should be noted that function $\beta$, which determines the curve of the image attenuation constant in the attenuation band of the filter, at the same time describes the phase function in the pass band. Consequently, the phase response in a Chebyshev approximation is non-linear.

### Conversion of Normalized Functions $\alpha$ and $\beta$ into a Filter Network

When the normalized functions of the impedance and the propagation function are known, a filter is realized physically by the following method.

As it has been remarked the product and ratio of these functions give the normalized impedances of the arms of a lattice network:

$$\alpha\beta = Z_1'.$$

$$\frac{\alpha}{\beta} = Z_2'.$$

$Z_1'$ and $Z_2'$ will have the form of fractional functions of $x$ of the type of (78) or (79) or their product and ratio. In order to obtain canonic equations of

the impedances of the arms of the lattice ($Z_1$ and $Z_2$), all the mathematical conversions which were applied to the normalized functions should be carried out in the reverse order: by substituting (76) and (77) we convert from normalized frequencies $x$ and $x_1$ to ordinary frequencies $\omega$ and $\omega_t$ and by multiplying functions $Z_1'$ and $Z_2'$ by $R$ we convert to the impedances of the arms of the lattice, $Z_1$ and $Z_2$. The problem is then reduced to realizing two-terminal networks with given functions of input impedance in the form of canonical circuits.

It should be noted that in practical problems of filter design the selection of the normalized functions $\alpha$ and $\beta$ is preceded by a preliminary calculation starting with the conditions imposed on the filter.

The permissible oscillation of the impedance $\delta\%$ in the pass band from $f_1$ to $f_2$ and the minimum attenuation $b_{\min}$ for frequencies below $f_3$ and above $f_4$ are usually given.

By using these data and nomograms, filters can be classified according to impedance and attenuation, and the expressions for functions $\alpha$ and $\beta$ can be found from the appropriate tables. Accurate cut-off frequencies can be found from the corresponding formulae.

CHAPTER IX

# CIRCUITS WITH DISTRIBUTED PARAMETERS

## 1. Primary Parameters of a Uniform Line

It was said in Chapter I, Section 2 that electrical circuits can be conventionally divided in circuits with lumped parameters and circuits with distributed parameters.

For an electrical circuit with low frequencies and small geometrical dimensions it is possible to disregard the propagation time of the electromagnetic state without detriment to the accuracy of the calculation, since it is incommensurably small compared with the alteration time of the electromagnetic state itself. Such a circuit can be regarded as an aggregate of independent $r$, $L$ and $C$ elements concentrated at various points of the circuit. The electrical processes which take place in these elements are described by the equations

$$u_r = ri, \quad u_L = L\frac{\mathrm{d}i}{\mathrm{d}t}, \quad u_C = \frac{1}{C}\int i\,\mathrm{d}t,$$

which contain no spatial coordinate: the current entering each circuit element equals the current leaving it.

At high frequencies the same circuit takes the form of a complex electromagnetic system, each elementary section of which is characterized by a series resistance, shunt leakage conductance, series inductance and capacitance between conductors, which cannot be separated from each other.

The propagation time of the electromagnetic state in this type of system is commensurable with the alteration time of the electromagnetic state itself; the system's electrical quantities therefore not only vary as time functions but also according to the distance between the observation point and any fixed point, for example, where the source is applied.

If we imagine this circuit split up into a finite number of sections, then the currents at the beginning and end of each section will be different owing to displacement currents caused by the capacitance between the conductors and also to leakage currents through the insulation. Only when the sections of the circuit are reduced infinitely will the currents at their ends approach each other in value.

As an example of a circuit with distributed parameters we shall now examine a uniform two-wire line, i.e. *a line, the resistance and inductance of whose wires, and the conductance (caused by imperfect insulation) and capacitance between whose wires, are uniformly distributed throughout.*

Electrical energy is transmitted from the source to the load along the wo wires of the circuit. In communication engineering these lines are made in the form of *parallel wires* (Fig. 1 *a*) or a *coaxial cable* (Fig. 1 *b*).

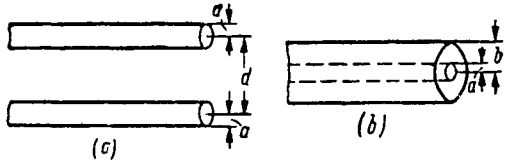



FIG. 1. Lines: (*a*) open-wire; (*b*) cable.

In accordance with the above, a uniform two-wire line is described by four *primary* electrical parameters per unit length of line, namely: the resistance of both wires ($r$); the shunt conductance (leakage conductance) between the wires ($g$); the inductance of the wires ($L$); and the capacitance between the wires ($C$).

These parameters depend on the design of the circuit and on frequency. In the region of radio frequencies the primary parameters of a uniform parallel-wire line with copper wires are calculated from the following formulae.

Open-wire line:

$$r \cong \frac{8{\cdot}33\sqrt{f}}{a}\,10^{-8}\,\frac{\Omega}{\mathrm{m}};$$

$$L \cong 0{\cdot}921\,\lg\frac{\mathrm{d}}{a}\,10^{-6}\,\frac{H}{\mathrm{m}};$$

$$C \cong \frac{12{\cdot}07}{\lg\dfrac{\mathrm{d}}{a}}\,10^{-12}\,\frac{F}{\mathrm{m}}.$$

Coaxial line:

$$r \cong 4{\cdot}16\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{f}\;\;10^{-8}\,\frac{\Omega}{\mathrm{m}};$$

$$g = \omega C \tan\delta\,\frac{1}{\Omega\mathrm{m}}\;(\delta - \text{loss angle});$$

$$L \cong 0{\cdot}46\,\lg\frac{b}{a}\,10^{-6}\,\frac{H}{\mathrm{m}};$$

$$C \cong \frac{0{\cdot}241\,\varepsilon_r}{\lg\dfrac{b}{a}}\,10^{-10}\,\frac{F}{\mathrm{m}}$$

($\varepsilon_r$ is the permittivity of the insulation).

As the frequency increases, the loss angle $\delta$ decreases. The insulation used for coaxial cables usually has its tan $\delta$ in the order of $10^{-3}$–$10^{-4}$. The conductance $g$ between the parallel wires, which depends on meteorological conditions, the state of the insulators on which the wires are suspended, and on other factors, is determined experimentally. In many cases it is in practice possible to take $g = 0$. In addition, the resistance of the wires $r$ may be disregarded in a number of cases on account of the prevalence of the inductive reactance at high frequencies.

## 2. Differential Equations of a Uniform Line

The voltage and current in a line are functions of two independent variables: the spatial coordinate $x$ which determines the observation point, and the time coordinate $t$ which determines the instant of observation. Thus

$$i = i(x, t), \qquad u = u(x, t).$$

If the origin of the line is taken as the zero reference $x$, the positive direction of axis $x$ will coincide with the reference direction for the current (from source to load).

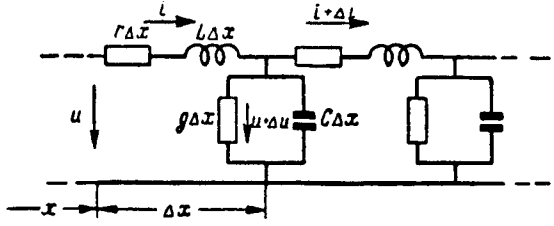



FIG. 2. Elementary section of a circuit with uniformly distributed parameters.

Let us isolate a simple section of a line $\Delta x$ in length and at a distance $x$ from the origin. Using parameters $r$, $g$, $L$ and $C$ related to a unit length of the line we shall represent the simple section of the line under examination in the form of a resistance $r\Delta x$ and an inductance $L\Delta x$ connected in series and a conductance $g\Delta x$ and capacitance $C\Delta x$ connected in parallel (Fig. 2).

Let us denote:

$u$ the voltage between the upper and lower wires at point $x$;
$\Delta u$ the voltage increment on section $\Delta x$;
$i$ the current at point $x$;
$\Delta i$ the current increment in section $\Delta x$.

The equations for the voltage and current increments on an element $\Delta x$ in length are written thus:

$$\left.\begin{aligned} -\Delta u &= \left(ri + L\frac{\partial i}{\partial t}\right)\Delta x, \\ -\Delta i &= \left(gu + C\frac{\partial u}{\partial t}\right)\Delta x. \end{aligned}\right\} \tag{9.1}$$

The minus sign in the left-hand side of equations (1) indicates that the voltage and current in the line decrease (in the algebraic sense) in the reference direction assumed here.

These equations are approximate (for finite values of $\Delta x$), as a value of the second order of smallness

$$\left[g\Delta u + C\frac{\partial(\Delta u)}{\partial t}\right]\Delta x$$

is omitted from the right-hand side of the lower equation.

As $\Delta x$ tends to zero, the degree of accuracy of these equations increases.

In view of the presence of the two independent variables ($x$ and $t$) the equations are written in partial derivatives.

By dividing both parts of equations (1) by $\Delta x$ and proceeding to the limit with $\Delta x$ tending to zero, we get differential equations of the line

$$\left.\begin{aligned} -\frac{\partial u}{\partial x} &= ri + L\frac{\partial i}{\partial t}, \\ -\frac{\partial i}{\partial x} &= gu + C\frac{\partial u}{\partial t}. \end{aligned}\right\} \tag{9.2}$$

These equations are known in technical literature as *transmission line equations.*

If the end of the line is taken as the zero reference $x$, i.e. the axis $x$ is taken in the direction from receiver to source, then the sign will change in the left-hand of (2) and the equations will take the form

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= ri + L\frac{\partial i}{\partial t}, \\ \frac{\partial i}{\partial x} &= gu + C\frac{\partial u}{\partial t}. \end{aligned}\right\} \tag{9.3}$$

Equations (2) or (3) can be solved, depending on initial and boundary conditions. The *initial conditions* are the values of the voltage and current in the line at an instant coinciding with the commencement of the process

being examined. The *boundary conditions* are determined by the values of the voltage and current at the beginning or at the end of the line.

The solution of the said equations gives the functional relationships of the voltage and current in the line to the variables $x$ and $t$.

### 3. Steady-state Solution

During periodic conditions, when a sinusoidal voltage is applied to a line the voltage and current change sinusoidally at any point on the line, with the frequency of the source. Let us denote the effective values of the voltage and current in complex form at a distance $x$ from the beginning of the line by

$$\dot{U} = \dot{U}(x) \quad \text{and} \quad \dot{I} = \dot{I}(x).$$

Using the complex notation, from (2) we get

$$\left.\begin{aligned} -\frac{\mathrm{d}\dot{U}}{\mathrm{d}x} &= (r + j\omega L)\,\dot{I}, \\ -\frac{\mathrm{d}\dot{I}}{\mathrm{d}x} &= (g + j\omega C)\,\dot{U}. \end{aligned}\right\} \tag{9.4}$$

In view of the fact that the complex values (sinors) $\dot{U}$ and $\dot{I}$ do not depend on $t$ and are only functions of $x$, the partial derivatives of $x$ are replaced by the ordinary derivatives when converting from equations (2) to (4).

Excluding current $\dot{I}$ from set (4) we get an equation with respect to $\dot{U}$

$$\frac{\mathrm{d}^2\dot{U}}{\mathrm{d}x^2} = (r + j\omega L)\,(g + j\omega C)\,\dot{U}. \tag{9.5}$$

Analogously, excluding voltage $\dot{U}$ from (4) we get an equation with respect to $\dot{I}$

$$\frac{\mathrm{d}^2\dot{I}}{\mathrm{d}x^2} = (r + j\omega L)\,(g + j\omega C)\,\dot{I}. \tag{9.6}$$

Let us denote*

$$\gamma = \sqrt{[(r + j\omega L)\,(g + j\omega C)]} = \beta + j\alpha \tag{9.7}$$

and call this quantity the *propagation factor.*

* In English technical literature it is customary to take $\gamma = \alpha + j\beta$.

Thus

$$\left.\begin{aligned}\frac{d^2\dot{U}}{dx^2} &= \gamma^2\dot{U},\\ \frac{d^2\dot{I}}{dx^2} &= \gamma^2\dot{I}.\end{aligned}\right\} \tag{9.8}$$

The general solution of the homogeneous linear differential equation of the second order (8) has the form

$$\dot{U} = A_1e^{-\gamma x} + A_2e^{\gamma x}. \tag{9.9}$$

The simplest method of finding current $\dot{I}$ is to substitute (9) in the first equation of (4):

$$\dot{I} = \frac{1}{r + j\omega L}(A_1\gamma e^{-\gamma x} - A_2\gamma e^{\gamma x}) = \sqrt{\left(\frac{g + j\omega C}{r + j\omega L}\right)}(A_1e^{-\gamma x} - A_2e^{\gamma x})$$

or

$$\dot{I} = \frac{1}{Z_c}(A_1e^{-\gamma x} - A_2e^{\gamma x}), \tag{9.10}$$

where

$$Z_c = \sqrt{\left(\frac{r + j\omega L}{g + j\omega C}\right)} \tag{9.11}$$

is called the *characteristic impedance* of the line.

Substituting (7) in (9) we get

$$\dot{U} = A_1e^{-\beta x}e^{-j\alpha x} + A_2e^{\beta x}e^{j\alpha x}. \tag{9.12}$$

The instantaneous value of the voltage at point $x$ is equal to the imaginary part of the expression $\sqrt{2}\,\dot{U}e^{j\omega t}$:

$$\begin{aligned}u(x, t) &= \mathrm{Im}[\sqrt{2}A_1e^{-\beta x}e^{-j\alpha x}e^{j\omega t} + \sqrt{2}A_2e^{\beta x}e^{j\alpha x}e^{j\omega t}]\\ &= \sqrt{2}\,|A_1|\,e^{-\beta x}\sin(\omega t + \psi_1 - \alpha x)\\ &\quad + \sqrt{2}\,|A_2|\,e^{\beta x}\sin(\omega t + \psi_2 + \alpha x);\end{aligned} \tag{9.13}$$

where $\psi_1$ and $\psi_2$ are arguments of the complex values $A_1$ and $A_2$.

Thus, the voltage at point $x$ is composed of two sinusoidal functions.

The first component is a wave whose amplitude decreases and phase lag increases as $x$ increases. This wave travels in time from the source to the load (Fig. 3*a*) and is called an *incident wave*.

The second component is a wave whose amplitude and phase lead increase as $x$ increases. This wave travels in time from the load to the source (Fig. 3*b*) and is called a *reflected wave.*

The incident wave and the reflected wave travel along the line with the same phase velocity. *By the phase velocity of a wave we mean the velocity of the movement of a point whose oscillation phase remains constant.* This

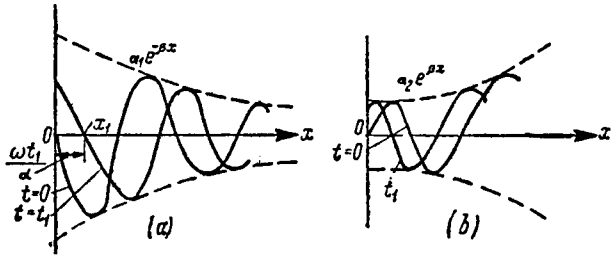



FIG. 3. Waves: (*a*) incident; (*b*) reflected.

condition is written for an incident wave in the form

$$\omega t + \psi_1 - \alpha x = \text{const},$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega t + \psi_1 - \alpha x) = 0$$

and, consequently,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v = \frac{\omega}{\alpha}. \tag{9.14}$$

For a reflected wave the same equation is obtained with a minus sign indicating that it moves in the opposite direction to the incident wave.

In accordance with above, the incident voltage wave and reflected voltage wave are written in complex form thus:

$$\dot{U}_1 = A_1 e^{-\gamma x}, \quad \dot{U}_r = A_2 e^{\gamma x}.$$

On the basis of (9) and (10)

$$\dot{U} = U_1 + \dot{U}_r,$$

$$\dot{I} = \frac{\dot{U}_1}{Z_c} - \frac{\dot{U}_r}{Z_c} = \dot{I}_1 - \dot{I}_r. \tag{9.15}$$

The voltage and current of the incident wave and, correspondingly, the reflected wave are related by Ohm's law

$$\frac{\dot{U}_1}{\dot{I}_1} = \frac{\dot{U}_r}{\dot{I}_r} = Z_c. \tag{9.16}$$

In addition to phase velocity $v$, *a wave is characterized by its length* $\lambda$, *which is the shortest distance between two points on the line at which the oscillation phase of the wave is the same at any instant.* This means that the phase angles at the points $x$ and $x + \lambda$ differ by $2\pi$:

$$[\omega t + \psi_1 - \alpha x] - [\omega t + \psi_1 - \alpha (x + \lambda)] = 2\pi,$$

from which

$$\lambda = \frac{2\pi}{\alpha}. \tag{9.17}$$

Since from (14)

$$\alpha = \frac{\omega}{v} = \frac{2\pi f}{v},$$

then

$$\lambda = \frac{v}{f} = vT, \tag{9.18}$$

i.e. in the duration of one period the wave travels a distance equal to the wavelength.

*Lines with a physical length commensurable with the wavelength are regarded as long lines.* In practice, at sufficiently high frequencies any electrical circuit becomes long with respect to the wavelength.

If we take the propagation velocity of an electromagnetic wave as being equal to the speed of light ($3 \times 10^8$ m/sec) then, for example, a wavelength of 6000 km will correspond to a frequency of 50 kc/s and a wavelength of 10 cm will correspond to a frequency of $3 \times 10^9$ kc/s. Consequently, in the first case the line will be long, measured in hundreds or thousands of kilometres, and in the second will be only a few centimetres.

Expression (13) contains the factor $\sqrt{2}|A_1|\, e^{-\beta x}$, which represents the amplitude of the sinusoidal voltage of an incident wave at any point at a distance $x$ from the beginning of the line. The factor $e^{-\beta x}$ denotes that the amplitude of the wave decreases as it travels from the beginning of the line. This phenomenon is known as the *attenuation* of the wave.

The value $\beta$ which characterizes the change in the amplitude of a wave per unit length of line is called the *attenuation factor*; the value $\alpha$ which characterizes the phase difference per unit length of line is called the *phase constant.*

The integration constants $A_1$ and $A_2$ contained in (9) and (10) are found in accordance with the boundary conditions, i.e. the voltage and current at the beginning of the line.

When $x = 0$

$$\dot{U}(0) = \dot{U}_1 = A_1 + A_2,$$

$$Z_c\dot{I}(0) = Z_c\dot{I}_1 = A_1 - A_2,$$

from which

$$A_1 = \frac{\dot{U}_1 + Z_c\dot{I}_1}{2}, \quad A_2 = \frac{\dot{U}_1 - Z_c\dot{I}_1}{2}. \tag{9.19}$$

Substituting (19) in (9) and (10) gives

$$\left.\begin{aligned} \dot{U} &= \frac{\dot{U}_1 + Z_c\dot{I}_1}{2}(e^{-\gamma x} + n_1 e^{\gamma x}), \\ \dot{I} &= \frac{\dot{U}_1 + Z_c\dot{I}_1}{2Z_c}(e^{-\gamma x} - n_1 e^{\gamma x}), \end{aligned}\right\} \tag{9.20}$$

where $n_1$ is the reflexion factor at the beginning of the line

$$n_1 = \frac{\dot{U}_1 - Z_c\dot{I}_1}{\dot{U}_1 + Z_c\dot{I}_1} = \frac{Z_1 - Z_c}{Z_1 + Z_c},$$

and $Z_1 = \dot{U}_1/\dot{I}_1$ is the input impedance.

If we start from equations (3), i.e. read coordinate $x$ not from the beginning but from the end of the line and use the boundary conditions $\dot{U}(0) = \dot{U}_2$ and $\dot{I}(0) = \dot{I}_2$, then instead of (20) we get a set of equations

$$\left.\begin{aligned} \dot{U} &= \frac{\dot{U}_2 + Z_c\dot{I}_2}{2}(e^{\gamma x} + n_2 e^{-\gamma x}), \\ \dot{I} &= \frac{\dot{U}_2 + Z_c\dot{I}_2}{2Z_c}(e^{\gamma x} - n_2 e^{-\gamma x}), \end{aligned}\right\} \tag{9.21}$$

where $n_2$ is the *reflexion factor* at the end of the line:

$$n_2 = \frac{\dot{U}_2 - Z_c\dot{I}_2}{\dot{U}_2 + Z_c\dot{I}_2} = \frac{Z_2 - Z_c}{Z_2 + Z_c}. \tag{9.22}$$

In that case when the impedance of the load $Z_2 = \dot{U}_2/\dot{I}_2$ equals the characteristic impedance of the line $Z_c$, the reflexion factor is zero: $n_2 = 0$. Only the incident wave then occurs in the line; there is no reflexion wave.

This important property is realized in high-frequency feeders transmitting energy from generator to load. In these lines reflexions are not desirable since, if the attenuation in the line is not great, the reflected wave creates an echo effect at the beginning of the line. The reflexions also entail energy losses and reduce efficiency. A part of the energy which reaches the receiving end does not enter the receiver but returns along the line in the form of the energy of a reflected wave, causing additional energy losses in the line's impedance and admittance.

If the impedance of the source supplying the line does not equal the characteristic impedance, the reflected wave, having reached the beginning of the line, undergoes reflexion yet again, and so on. As a result of repeated reflexions undesirable energy losses occur in the line. In addition an undesirable increase in voltage or current may occur in the line in this case (Section 6). In view of this, we try in practice to match the impedance of the load with the characteristic impedance of the line.

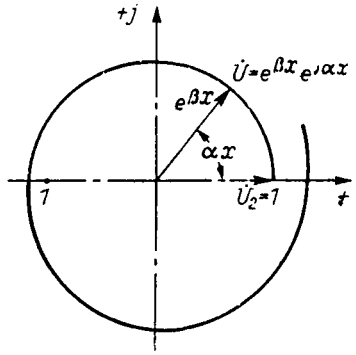



FIG. 4. Locus of a voltage vector for a matched load.

When the load is thus matched with the line expressions (21) are simplified; taking into account that $Z_c\dot{I}_2 = \dot{U}_2$, we find

$$\left.\begin{aligned} \dot{U} &= \dot{U}_2 e^{\gamma x}, \\ \dot{I} &= \dot{I}_2 e^{\gamma x}. \end{aligned}\right\} \tag{9.23}$$

The resulting formulae show that when the load is matched ($Z_2 = Z_c$), a logarithmic spiral is the locus of the tip of the vector $\dot{U}$. In Fig. 4, which illustrates this, we have taken $\dot{U}_2 = 1$.

It follows from (23) that when the observation point is moved along a line with matched loading in the direction from load to source, the voltage changes $e^{\gamma x}$ times; the modulus of the voltage increases $e^{\beta x}$ times and the phase increases by $\alpha x$ radians

$$\beta x = \ln \frac{U}{U_2}, \quad \alpha x = \arg \frac{\dot{U}}{\dot{U}_2}.$$

If a unit length of line is regarded as a four-terminal network, the propagation factor $\gamma$ will be the same as the characteristic propagation function of the given network.

When the line is open at the end (open-circuit conditions $z_2 = \infty$) the reflexion factor $n_2 = 1$, and when the line is closed at the end (short-circuit conditions $z_2 = 0$) $n_2 = -1$.

The set of equations (21) can be transcribed thus:

$$\left.\begin{aligned} \dot{U} &= \dot{U}_2 \frac{e^{\gamma x} + e^{-\gamma x}}{2} + Z_c \dot{I}_2 \frac{e^{\gamma x} - e^{-\gamma x}}{2}, \\ \dot{I} &= \frac{\dot{U}_2}{Z_c} \frac{e^{\gamma x} - e^{-\gamma x}}{2} + I_2 \frac{e^{\gamma x} + e^{-\gamma x}}{2}. \end{aligned}\right\} \tag{9.24}$$

Formulae (21) and (24) are equations for the line in exponential (or wave) form, reading distance $x$ from the end of the line. Equations (24) can be converted by using hyperbolic functions

$$\left.\begin{aligned} \dot{U} &= \dot{U}_2 \cosh \gamma x + Z_c \dot{I}_2 \sinh \gamma x, \\ \dot{I} &= \dot{I}_2 \cosh \gamma x + \frac{1}{Z_c} \dot{U}_2 \sinh \gamma x. \end{aligned}\right\} \tag{9.25}$$

Substituting $l = x$ in (25) we get equations for the line in hyperbolic form, expressing the voltage and current at the beginning by the voltage and current at the end of the line

$$\left.\begin{aligned} \dot{U}_1 &= \dot{U}_2 \cosh \gamma l + \dot{Z}_c I_2 \sinh \gamma l, \\ \dot{I}_1 &= \dot{I}_2 \cosh \gamma l + \frac{\dot{U}_2}{Z_c} \sinh \gamma l. \end{aligned}\right\} \tag{9.26}$$

The property of the resulting equations (26), together with the equations for the symmetrical four-terminal network (equation (63)) given in Chapter VII, Section 4, deserve attention. These equations show that a uniform line is a symmetrical four-terminal network with image parameters $\gamma l$ and $Z_c$. Using the $A$ parameters we get:

$$A_{11} = A_{22} = \cosh \gamma l, \quad A_{12} = Z_c \sinh \gamma l, \quad A_{21} = \frac{1}{Z_c} \sinh \gamma l.$$

Substituting $\dot{I}_2 = \dot{U}_2 / Z_2$ in the first equation of (26) we get

$$\dot{U}_1 = \dot{U}_2 \left( \cosh \gamma l + \frac{Z_c}{Z_2} \sinh \gamma l \right),$$

from which

$$\dot{U}_2 = \frac{\dot{U}_1}{\cosh \gamma l + \dfrac{Z_c}{Z_2} \sinh \gamma l}, \qquad \dot{I}_2 = \frac{\dot{U}_1}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}.$$

In connexion with this, from (25)

$$\left.\begin{aligned} \dot{U} &= \dot{U}_1 \frac{Z_2 \cosh \gamma x + Z_c \sinh \gamma x}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}, \\ \dot{I} &= \dot{U}_1 \frac{\cosh \gamma x + \dfrac{Z_2}{Z_c} \sinh \gamma x}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}. \end{aligned}\right\} \tag{9.27}$$

These formulae express the voltage and current at an arbitrary point $x$, with a variable voltage $\dot{U}_1$ at the beginning of the line; in addition they make it easily possible to obtain analogous correlations for the case where $x$ is read from the beginning of the line. For this purpose it is sufficient to substitute $l - x$ for $x$ in the right-hand side of (27):

$$\left.\begin{aligned} \dot{U} &= \dot{U}_1 \frac{Z_2 \cosh \gamma (l - x) + Z_c \sinh \gamma (l - x)}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}, \\ \dot{I} &= \dot{U}_1 \frac{\cosh \gamma (l - x) + \dfrac{Z_2}{Z_c} \sinh \gamma (l - x)}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}. \end{aligned}\right\} \tag{9.28}$$

The exponential and hyperbolic forms of writing equations for the lines are mutually complementary.

The advantage of the exponential form of writing the equations consists in the fact that the physical processes in the line are made graphically clear with the aid of incident and reflected waves and that it is convenient to construct loci in a complex plane. Equations (21) are therefore widely used in the following sections of this chapter.

The hyperbolic form of writing the equations is also recognized as being convenient in several cases from the point of view of examining and calculating electrical quantities in the line and their phase relationships. When the line is regarded as a four-terminal network it is usual to base the examination on the hyperbolic form of writing the equations.

### 4. Secondary (Image) Parameters of a Uniform Line

The attenuation factor $\beta$, the phase factor $\alpha$ and the characteristic impedance $Z_c$ which, in their turn, are expressed by the primary parameters of the line and frequency are the *secondary* (or image) parameters.

It follows from the expression

$$\gamma = \sqrt{[(r + j\omega L)(g + j\omega C)]} = \sqrt{[rg - \omega^2 LC + j\omega (Lg + Cr)]}$$

that

$$\beta^2 + j2\alpha\beta - \alpha^2 = rg - \omega^2 LC + j\omega(Lg + Cr),$$

from which

$$\left.\begin{aligned} \beta^2 - \alpha^2 &= rg - \omega^2 LC, \\ 2\alpha\beta &= \omega Lg + \omega Cr. \end{aligned}\right\}$$

The joint solution of these equations gives

$$\beta = \sqrt{\left[\tfrac{1}{2}\{rg - \omega^2 LC + \sqrt{[(r^2 + \omega^2 L^2)(g^2 + \omega^2 C^2)]}\}\right]}, \quad (9.29)$$

$$\alpha = \sqrt{\left[\tfrac{1}{2}\{\omega^2 LC - rg + \sqrt{[(r^2 + \omega^2 L^2)(g^2 + \omega^2 C^2)]}\}\right]}. \quad (9.30)$$

The resulting formulae show that $\beta$ and $\alpha$ generally depend on frequency. But, as opposed to the attenuation factor which varies within relatively fixed limits, the phase factor can increase infinitely with the frequency.

Formula (30) makes it possible to express the phase velocity of an electromagnetic wave by the primary parameters of the line. Substituting (30) in (14) gives

$$v = \frac{\omega}{\sqrt{\left[\tfrac{1}{2}\{\omega^2 LC - rg + \sqrt{[(r^2 + \omega^2 L^2)(g^2 + \omega^2 C^2)]}\}\right]}}. \quad (9.31)$$

This formula shows that, generally speaking, the phase velocity does not remain constant at different frequencies but changes in accordance with the frequency.

Expressions (29)–(31) are not suitable for practical application because they are unwieldy. There are several approximate formulae for calculating secondary line parameters, which take into account the fact that at high frequencies (1 Mc/s and above) resistance $r$ is extremely small compared with $\omega L$ and conductance $g$ is negligibly small compared with $\omega C$. The first assumption ($\omega L \gg r$) is based on the fact that the inductive reactance is directly proportional to frequency whereas the resistance of the wires $r$ is proportional to the square root of the frequency (as a result of the skin effect). The second assumption is true for high-frequency feeders which, being "long" compared with the wavelength, have an extremely small physical length and can therefore have a reliable insulation between the wires. The conductance $g$ of cable lines is particularly small.

Using a Newton binomial for the expression

$$\gamma = j\omega\sqrt{(LC)}\left(1 - j\frac{r}{\omega L}\right)^{\frac{1}{2}}\left(1 - j\frac{g}{\omega C}\right)^{\frac{1}{2}}$$

and concentrating on the first two terms of the series, we find

$$\gamma \cong j\omega\sqrt{(LC)}\left(1 - j\frac{r}{2\omega L}\right)\left(1 - j\frac{g}{2\omega C}\right)$$

or, disregarding the term $-jrg/[4\omega\sqrt{(LC)}]$ since it is so small,

$$\gamma \cong \frac{r}{2}\sqrt{\left(\frac{C}{L}\right)} + \frac{g}{2}\sqrt{\left(\frac{L}{C}\right)} + j\omega\sqrt{(LC)}. \tag{9.32}$$

The real part of (32) is the limit to which the attenuation factor tends when the frequency tends to infinity

$$\beta = \frac{r}{2}\sqrt{\left(\frac{C}{L}\right)} + \frac{g}{2}\sqrt{\left(\frac{L}{C}\right)}. \tag{9.33}$$

Expression (33) should not be taken to mean that $\beta$ does not depend on frequency; the parameters contained in (33) are frequency functions.

The first term determines the part of the attenuation which is caused by the series resistance of the line. The second term determines the part of the attenuation which is contained in transmission as a result of the shunt conductance of the line.

In order to reduce irreversible losses when transmitting electromagnetic energy through a line it is desirable to keep the resistance of the line and the shunt conductance as small as possible.

The imaginary part of (32) is the limit to which the phase function tends as the frequency increases:

$$\alpha \cong \omega\sqrt{(LC)}. \tag{9.34}$$

The phase velocity is, in accordance with (14) and (34),

$$v \cong \frac{1}{\sqrt{(LC)}}.$$

This is the critical propagation velocity of a wave along the line, at an infinitely high frequency; it is easy to confirm this by taking the limit of equation (31) when $\omega \to \infty$.

On the condition that $g \cong 0$, the binomial expansion of the propagation factor results in the expression:

$$\begin{aligned}
\gamma &\cong \sqrt{[(r + j\omega L)\, j\omega C]} = (-\omega^2 LC + j\omega Cr)^{\frac{1}{2}} \\
&= (-\omega^2 LC)^{\frac{1}{2}} + j\tfrac{1}{2}(-\omega^2 LC)^{-\frac{1}{2}}\omega Cr + \tfrac{1}{8}(-\omega^2 LC)^{-\frac{3}{2}}(\omega Cr)^2 \\
&\quad - j\tfrac{1}{16}(-\omega^2 LC)^{-\frac{5}{2}}(\omega Cr)^3 + \cdots \\
&= j\omega\sqrt{(LC)} + \frac{r}{2}\sqrt{\left(\frac{C}{L}\right)} + j\frac{r^2}{8\omega L}\sqrt{\left(\frac{C}{L}\right)} - \frac{1}{16}\frac{r^3}{\omega^2 L^2}\sqrt{\left(\frac{C}{L}\right)} + \cdots \\
&= \frac{r}{2}\sqrt{\left(\frac{C}{L}\right)}\left(1 - \frac{r^2}{8\omega^2 L^2}\right) + j\omega\sqrt{(LC)}\left(1 + \frac{r^2}{8\omega^2 L^2}\right) + \cdots
\end{aligned}$$

In this case

$$\beta \cong \frac{r}{2}\sqrt{\left(\frac{C}{L}\right)}\left(1 - \frac{r^2}{8\omega^2 L^2}\right),$$

$$\alpha \cong \omega\sqrt{(LC)}\left(1 + \frac{r^2}{8\omega^2 L^2}\right),$$

$$v \cong \frac{1}{\sqrt{(LC)}\left(1 + \frac{r^2}{8\omega^2 L^2}\right)}.$$

In many cases the quantity $r^2/8\omega^2L^2$ is insignificant compared with unity. And the same is true of the second term in (33) compared with the

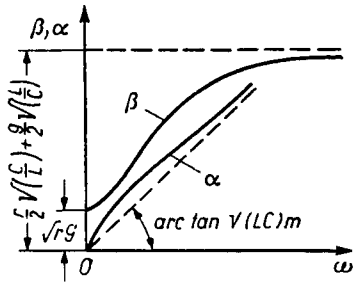



FIG. 5. Frequency characteristics $\beta$ and $\alpha$.

first, in view of the negligibly small value of $g$. Therefore for radio frequencies a simplified formula is used:

$$\beta \cong \frac{r}{2}\sqrt{\left(\frac{C}{L}\right)} = \frac{r}{2\varrho}. \tag{9.35}$$

For direct current ($\omega = 0$) the concepts "phase function" and "phase velocity" lose their physical meaning; from the general formulae (29) and (30) educed above it follows that when $\omega = 0$

$$\beta = \sqrt{(rg)}, \quad \alpha = 0. \tag{9.36}$$

Figure 5 illustrates the nature of the change in $\beta$ and $\alpha$ in the range of frequencies from zero to infinity. As the frequency increases, $\alpha$ asymptotically approaches the straight line which forms an angle $\arctan\sqrt{(LCm)}$ with the abscissa axis; $m$ is the scaling factor.

It is characteristic of cable lines to have a sharply expressed capacitive susceptance $\omega C$ in comparison with which the conductance of the insulation $g$ is negligibly small. In addition, if the frequency is not very high,

the inductive reactance $\omega L$ is small compared with the resistance $r$. Therefore, in the case of cable lines, by disregarding parameters $g$ and $L$ compared with $r$ and $C$, i.e. taking $g \cong 0$ and $L \cong 0$, we get the simplified calculation formulae

$$\gamma \cong \sqrt{(jr\omega C)} = \sqrt{(r\omega C)}\, e^{j45^\circ}$$

or

$$\gamma = \sqrt{(r\omega C)}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right).$$

Consequently,

$$\alpha \cong \beta \cong \sqrt{\frac{(r\omega C)}{2}}. \tag{9.37}$$

Correspondingly, the phase velocity of an electromagnetic wave in a cable line is

$$v \cong \sqrt{\left(\frac{2\omega}{rC}\right)}, \tag{9.38}$$

i.e. it is directly proportional to the square root of the frequency.

A wave is propagated more slowly along a cable line than along an open-wire line. It will be shown later (in Sections 5 and 6) that for distortionless lines and dissipationless lines

$$v \cong \frac{c}{\sqrt{(\varepsilon_r \mu_r)}}, \tag{9.39}$$

where $c$ is the velocity of light in a vacuum ($\approx 3 \times 10^8$ m/sec); $\varepsilon_r$ and $\mu_r$ are, respectively, the permittivity and the magnetic permeability of the material covering the conductor.

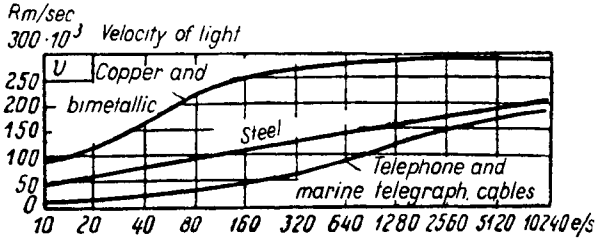



FIG. 6. Phase velocity of waves in open-wire and cable lines.

In the case of open-wire lines $\varepsilon_r \cong 1$ and $\mu_r \cong 1$ and, in accordance with (39), the wave is propagated at a velocity almost equal to that of light in a vacuum.

In the case of cable lines the value of $\varepsilon_r$ lies between 4 and 5 and therefore the phase velocity is approximately 2 to 2·5 times lower than the velocity of light in a vacuum.

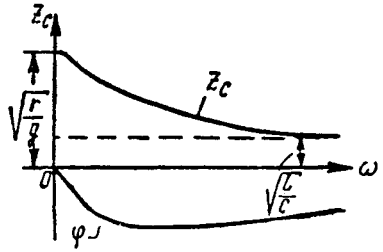



FIG. 7. Frequency characteristics of the modulus and angle of the characteristic impedance of a line.

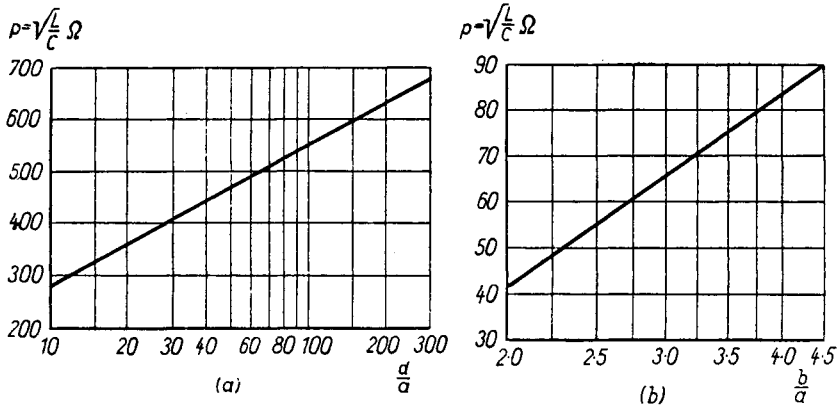



FIG. 8. Characteristic impedance of lines: (*a*) open-wire; (*b*) cable.

Figure 6 illustrates the change in the phase velocity, depending on the frequency and the type of line.

The characteristic impedance of the line

$$Z_c = \sqrt{\left(\frac{r + j\omega L}{g + j\omega C}\right)} = \sqrt{\left(\frac{r}{g}\,\frac{1 + j\omega\dfrac{L}{r}}{1 + j\omega\dfrac{C}{g}}\right)} \qquad (9.40)$$

with direct current ($\omega = 0$) and at an infinite frequency ($\omega = \infty$) has the

real values

$$\left.\begin{aligned} Z_c = \varrho &= \sqrt{\left(\frac{r}{g}\right)} \quad (\omega = 0). \\ Z_c = \varrho &= \sqrt{\left(\frac{L}{C}\right)} \quad (\omega = \infty). \end{aligned}\right\} \tag{9.41}$$

In the remaining part of the frequency range the characteristic impedance of the line has a capacitive character, since usually $C/g > L/r$ (the argument of the complex denominator in the right-hand side of (40) is greater than the argument of the numerator).

Figure 7 shows the curves for the change in the modulus ($z_c$) and the angle ($\varphi$) of a line's characteristic impedance, depending on frequency.

Substituting the expressions for $L$ and $C$ in the formula $\varrho = \sqrt{(L/C)}$, we get approximate calculation formulae for high frequencies:

$$\left.\begin{aligned} \varrho &= 276 \lg \frac{d}{a} \Omega \quad \text{(open-wire line)}, \\ \varrho &= \frac{138}{\sqrt{\varepsilon_r}} \lg \frac{b}{a} \Omega \quad \text{(coaxial cable)}. \end{aligned}\right\} \tag{9.42}$$

The average value of $\varrho$ for open-wire lines is 400–500 Ω, and for cables 50–70 Ω.

Figure 8 illustrates the graphical dependences of $\varrho$ on $d/a$ and $b/a$ for aerial and cable lines, plotted in accordance with (42).

## 5. Distortionless Line

Signals transmitted along a communication line are a combination of many different frequencies which form a line spectrum in the case of periodic nonsinusoidal signals (Chapter X) and a continuous spectrum in the case of nonperiodic signals (Chapter XII).

Signal transmission is called *undistorted transmission* when the shape of the signal at the beginning of the line is the same as that at the end, i.e. all the ordinates of the voltage or current curve at the end of the line are directly proportional to the corresponding ordinates of the curve at the beginning.

This occurs when the line's attenuation factor and, correspondingly, phase velocity are the same at all frequencies.

When attenuation varies at different frequencies, so-called amplitude distortions are produced, and variable phase velocities at different frequencies cause phase distortions.

According to (37) and (38) in the case of cable lines the attenuation factor and the phase velocity are proportional to the square root of the frequency. In the case of aerial lines $\beta$ and $v$ also depend on frequency. As a result of this amplitude and phase distortions occur.

Thus, *in order to obtain undistorted transmission, the attenuation factor $\beta$ must not be dependent on frequency and the phase factor $\alpha$ must be directly proportional to frequency.* In the latter case the phase velocity $v = \omega/\alpha$ is obtained independent of frequency.

This takes place on the condition that

$$\frac{L}{r} = \frac{C}{g}. \tag{9.43}$$

In fact, the propagation function is

$$\gamma = \sqrt{[(r + j\omega L)(g + j\omega C)]} = \sqrt{\left[rg\left(1 + j\omega\frac{L}{r}\right)\left(1 + j\omega\frac{C}{g}\right)\right]}$$

and, taking (43) into account,

$$\gamma = \sqrt{(rg)} + j\omega\sqrt{(rg)}\frac{L}{r}$$

or

$$\gamma = \sqrt{(rg)} + j\omega\sqrt{(LC)}.$$

If we regard the primary parameters as being independent of frequency, then the attenuation factor will, in this case, be constant:

$$\beta = \sqrt{(rg)}, \tag{9.44}$$

and the phase factor will be directly proportional to frequency:

$$\alpha = \omega\sqrt{(LC)}. \tag{9.45}$$

A line whose parameters satisfy condition (43) is called *distortionless* in so far as any signal propagated through it keeps its shape. A distortionless line is, at the same time, a minimum attenuation line which is only possible for certain parameters of $r$ and $g$.

The relationships given in (43) can be obtained from (29) and (30). In order to observe equations (44) and (45) it is essential that the inner root in (29) and (30) should equal

$$rg + \omega^2 LC.$$

In this case

$$r^2g^2 + \omega^4L^2C^2 + \omega^2L^2g^2 + \omega^2C^2r^2 = (rg + \omega^2LC)^2$$

or

$$\omega^2L^2g^2 - 2\omega^2LCrg + \omega^2C^2r^2 = 0,$$

from which

$$Lg = Cr,$$

which agrees with (43).

The characteristic impedance of a distortionless line is a pure resistance. In accordance with (40)

$$Z_c = \varrho = \sqrt{\left(\frac{r}{g}\right)} = \sqrt{\left(\frac{L}{C}\right)}. \tag{9.46}$$

The phase velocity in a distortionless line

$$v = \frac{1}{\sqrt{(LC)}}. \tag{9.47}$$

In electromagnetic field theory it can be proved that

$$LC = \frac{\mu_r \varepsilon_r}{c^2}. \tag{9.48}$$

Substituting (48) in (47) gives the expression

$$v = \frac{c}{\sqrt{\varepsilon_r \mu_r}} \tag{9.49}$$

which was used in the previous section.

In order to eliminate distortions caused by the load impedance being mismatched with the line, i.e. to avoid reflexions at the receiving end, the load impedance should be $\varrho$. In this case the efficiency of the line has its highest possible value, $e^{-2\beta l}$.

It should be noted that condition (43) does not obtain as a rule, since $L/r$ is usually smaller than $C/g$. As a result, the attenuation of the line is always greater than the minimum. Cable lines are the furthest from satisfying condition (43).

In order to approximate a line to (43), any primary parameter should be changed, e.g. $r$ or $C$ should be increased, or $g$ or $L$ reduced.

It is possible to increase the resistance $r$ by using wires of large diameter; but this raises the cost of the line considerably.

There is no advantage in increasing the conductance $g$ of the insulation as this also increases the attenuation of the line.

The best method of approximating the primary parameters to their optimum ratios (43) is to increase the inductance artificially by connecting induction coils in the line over a certain distance or by using a cable, the conductors of which are bound round by a thin tape made of a material with a high magnetic permeability.

## 6. Dissipationless Line

In Section 4 we considered some calculation formulae for the secondary parameters of aerial and cable lines, assuming that the losses in these lines were extremely small. In practice it is sometimes possible to disregard completely the line's resistance $r$ and conductance $g$ compared with the inductive reactance $\omega L$ and capacitive susceptance $\omega C$, i.e. regard *the line as being dissipationless*, without much detriment to the accuracy of the calculations. This kind of idealization is permitted in radio engineering for making approximate qualitative and quantitive estimate of phenomena under examination. The calculation formulae are then simplified and the hyperbolic equations for the line become trigonometric.

When $r = 0$ and $g = 0$, formulae (17) and (11) take the form:

$$\left.\begin{aligned} &\gamma = j\omega\sqrt{(LC)}, \quad \beta = 0, \quad \alpha = \omega\sqrt{(LC)}, \\ &Z_c = \varrho = \sqrt{\left(\frac{L}{C}\right)}. \end{aligned}\right\} \tag{9.50}$$

Consequently, *there is no attenuation in a dissipationless line nor, as a result of the phase velocity being constant* $v = 1/\sqrt{(LC)}$, *are there any phase distortions.*

In the case of a dissipationless line, the formulae for the phase factor and phase velocity coincide with formulae (45) and (47) which are obtained for a distortionless line. Formula (49), relating the phase velocity to the velocity of light, therefore, also remains valid.

In view of the fact that hyperbolic functions with an imaginary argument are converted into trigonometrical functions, the hyperbolic line equations (25) assume the trigonometrical form

$$\left.\begin{aligned} \dot{U} &= \dot{U}_2 \cos \alpha x + j\varrho \dot{I}_2 \sin \alpha x, \\ \dot{I} &= \dot{I}_2 \cos \alpha x + j \frac{\dot{U}_2}{\varrho} \sin \alpha x. \end{aligned}\right\} \tag{9.51}$$

These equations are used below for examining standing waves in a dissipationless line (Section 7).

The energy transmitted along the line consists of the energy of electrical and magnetic fields.

If a dissipationless line terminates in the characteristic impedance, the voltage-to-current ratio at any point on the line is equal to the characteristic impedance

$$\frac{\dot{U}}{\dot{I}} = \frac{U}{I} = \sqrt{\left(\frac{L}{C}\right)}.$$

Squaring this equality we get $U^2/I^2 = L/C$, from which $LI_m^2/2 = CU_m^2/2$.

Consequently, when the load is matched, the maximum energy of the magnetic and electrical fields is the same for every unit section of the line. All the energy supplied by the incident wave is then absorbed in the load impedance.

If the load impedance differs from the characteristic impedance then the energy is redistributed between the fields at the junction of the line to the load, as a result of which reflexions occur.

In an extreme case, when the line is open at the end, the incident wave meets an infinitely high impedance; the current at the end of the line reverts to zero and, correspondingly, the magnetic field is forced to zero. The energy that was conveyed by the magnetic field cannot be dissipated, so it will be transfered to the energy of the electrical field; the voltage at the open end of the line will be doubled, causing a reflected wave with the same sign as the incident wave ($n_2 = 1$) to appear.

In the other extreme case, when the line is closed at the end, the incident wave meets an impedance equal to zero; the voltage at the end of the line reverts to zero and, correspondingly, the energy of the electrical field is transferred to the energy of the magnetic field; the current at the short-circuited end of the line is doubled and a reflected wave with a sign opposite to that of the incident wave ($n_2 = -1$) appears.

If the line terminates in a resistance greater than the line's characteristic impedance $\varrho$, the voltage increases; and in the opposite case, when the line terminates in a resistance less than $\varrho$, the current in the line increases (compared with the matched conditions).

## 7. Standing Waves

Let us now examine the law of the distribution of the effective values of voltage and current along a dissipationless line. For this purpose we shall use current equations (21) and (51) in a complex and hyperbolic form.

Taking the imaginary propagation factor in (21) to be $\gamma = j\alpha$, we get, for any point of the line a distance $x$ from the end,

$$\left.\begin{aligned} \dot{U} &= \frac{1}{2}\dot{U}_2\left(1 + \frac{\varrho}{Z_2}\right)\left(e^{j\frac{2\pi}{\lambda}x} + n_2 e^{-j\frac{2\pi}{\lambda}x}\right), \\ \dot{I} &= \frac{1}{2\varrho}\dot{U}_2\left(1 + \frac{\varrho}{Z_2}\right)\left(e^{j\frac{2\pi}{\lambda}x} - n_2 e^{-j\frac{2\pi}{\lambda}x}\right). \end{aligned}\right\} \tag{9.52}$$

The reflexion factor contained in (52)

$$n_2 = \frac{Z_2 - \varrho}{Z_2 + \varrho} \tag{9.53}$$

is usually a complex quantity.

Equations (52) give visual proof of the fact that the complex voltage at any point $x$ is composed of an incident and a reflected voltage wave, whose absolute amplitude values are found in the ratio $1:|n_2|$. In its turn the complex current is equal to the difference between the incident and the reflected current wave, in the same amplitude ratio.

The maximum effective value of $U$ corresponds to the points $x + k\lambda/2$ ($k$ is an integer) which satisfy the equation

$$\left|e^{j\frac{2\pi}{\lambda}x} + n_2 e^{-j\frac{2\pi}{\lambda}x}\right| = 1 + |n_2| \tag{9.54}$$

since the phases of the incident wave and reflected wave then coincide. At a distance $\lambda/4$ from the points the incident and reflected waves are in antiphase and the effective value of the voltage attains a minimum. The equation

$$\left|e^{j\frac{2\pi}{\lambda}x} - n_2 e^{-j\frac{2\pi}{\lambda}x}\right| = 1 - |n_2| \tag{9.55}$$

is then satisfied.

The coordinates of the maxima and minima of $U$ are multivalent functions $n_2$ and $\lambda$, independent of time, i.e. they always remain in the same place. The minimum $U$ is located directly between two adjacent maximum

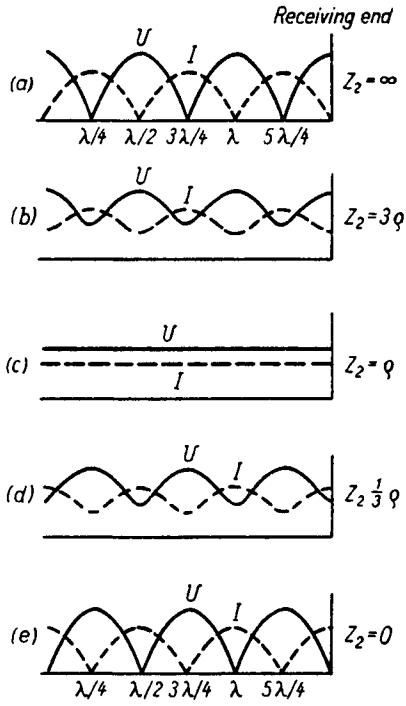



FIG. 9. Distribution of voltage $U$ along a line.

values of $U$, the distance between the nearest maximum (or minimum) values being $\lambda/2$.

The curve of the effective values of voltage along a dissipationless line is thus an undulating curve whose maximum and minimum values differ more in value as $|n_2|$ approaches unity.

On the strength of the fact that the said undulating curve does not travel along the line in time, but remains in the same place, it is called a *standing wave*.

When $|n_2| = 1$, the minimum of the effective value of $U$ is zero: the standing wave is in this case a "rectified" sine curve. The points on the line at which $U$ equals zero are called the *nodes*, and the points at which $U$ is at its maximum are called the *antinodes* of the voltage.

When there is no reflected wave ($n_2 = 0$), no standing wave is formed and the effective value of the voltage $U$ is the same at all points of the dissipationless line.

The same is true with respect to the standing current wave; the voltage nodes coinciding with the current antinodes and, vice versa, the current nodes coinciding with the voltage antinodes.

Correspondingly, the antinodes (or nodes) of the voltage and current are shifted by a quarter of a wave with respect to each other.

This is illustrated by Fig. 9, in which the following typical cases are selected:

(a) $n_2 = 1$ ($Z_2 = \infty$ is open-circuit conditions);

(b) $n_2 = 0{\cdot}5$ ($Z_2 = 3\varrho$);

(c) $n_2 = 0$ ($Z_2 = \varrho$ is matched loading);

(d) $n_2 = -0{\cdot}5$ ($Z_2 = \frac{1}{3}\varrho$);

(e) $n_2 = -1$ ($Z_2 = 0$ is a short-circuit).

Standing waves can easily be studied with the aid of equations (51) for a dissipationless line.

For open-circuit conditions ($I_2 = 0$)

$$\left.\begin{aligned} \dot{U} &= \dot{U}_2 \cos \frac{2\pi}{\lambda} x, \\ \dot{I} &= j \frac{\dot{U}_2}{\varrho} \sin \frac{2\pi}{\lambda} x. \end{aligned}\right\} \tag{9.56}$$

The nodes of the voltage are located at the points at which

$$\cos \frac{2\pi}{\lambda} x = 0, \quad \text{i.e.} \quad x = \frac{2k+1}{4} \lambda.$$

The antinodes of the voltage are located at the points for which

$$\cos \frac{2\pi}{\lambda} x = \pm 1, \quad \text{i.e.} \quad x = \frac{k}{2} \lambda.$$

The open end of the line coincides with a current node and a voltage antinode (Fig. 9*a*).

As (56) shows, the current leads the voltage by 90°, when $(2\pi/\lambda)\, x$ and $\cos (2\pi/\lambda)\, x$ have the same sign

$$0 \leqq x \leqq \frac{\lambda}{4}, \quad \frac{\lambda}{2} \leqq x \leqq \frac{3}{4} \lambda, \quad \text{etc.}$$

and lags by 90° when the signs of $\sin (2\pi/\lambda)\, x$ and $\cos (2\pi/\lambda)\, x$ are different

$$\left( \frac{\lambda}{4} \leqq x \leqq \frac{\lambda}{2}, \quad \frac{3}{4} \lambda \leqq x \leqq \lambda, \text{ etc.} \right).$$

In the first case the impedance of the line, i.e. the ratio of $\dot{U}$ to $\dot{I}$, is purely capacitive; in the second case it is inductive. This also follows from (56):

$$Z_0 = \frac{\dot{U}}{\dot{I}} = -j\varrho \cot \frac{2\pi}{\lambda} x. \tag{9.57}$$

The cotangent assumes any value from $-\infty$ to $+\infty$, depending on $x$. Any reactance at a certain given frequency can therefore be replaced by a section of line of corresponding length, open at the end, and dissipationless.

For a short-circuit, assuming $U_2 = 0$ in (51), we get

$$\left.\begin{aligned} \dot{U} &= j\varrho \dot{I}_2 \sin \frac{2\pi}{\lambda} x, \\ \dot{I} &= \dot{I}_2 \cos \frac{2\pi}{\lambda} x. \end{aligned}\right\} \tag{9.58}$$

The voltage nodes and current antinodes are located at the closed end of the line ($x = 0$) and at points an integer of half-waves ($k = \lambda/2$) away from it, and the voltage antinodes and current nodes are located at the points an odd number of quarter-waves $(2k + 1)\, \lambda/4$ away (Fig. 9*e*).

As (58) shows, the current lags behind the voltage by 90° when $\sin (2\pi/\lambda)\, x$ and $\cos (2\pi/\lambda)\, x$ have the same sign

$$\left( 0 \leqq x \leqq \frac{\lambda}{4}, \quad \frac{\lambda}{2} \leqq x \leqq \frac{3}{4} \lambda, \text{ etc.} \right),$$

and leads by 90° when the signs of $\sin(2\pi/\lambda)\,x$ and $\cos(2\pi/\lambda)\,x$ are different

$$\left(\frac{\lambda}{4} \leqq x \leqq \frac{\lambda}{2}, \quad \frac{3}{4}\lambda \leqq x \leqq \lambda, \text{ etc.}\right).$$

In the first case the line has an inductive reactance, and in the second a capacitive reactance. This also follows from (58) since

$$Z_s = \frac{\dot{U}}{\dot{I}} = j\varrho \tan \frac{2\pi}{\lambda} x. \tag{9.59}$$

Thus the impedance of a dissipationless line is reactive for a short circuit, the magnitude of the reactance and its sign depending on the ratio of the length of the line to the wavelength.

It should be noted that the presence of even the smallest losses in actual lines means that the effective values of $U$ and $I$ do not reach zero but attain certain minimum values at points corresponding to the nodes.

With a *matched load* ($Z_2 = \varrho$)

$$\left.\begin{aligned} \dot{U} &= \dot{U}_2 e^{j\frac{2\pi}{\lambda}x}, \\ \dot{I} &= \dot{I}_2 e^{j\frac{2\pi}{\lambda}x}, \end{aligned}\right\} \tag{9.60}$$

and the curves of the effective values of voltage and current are represented by the straight horizontal lines plotted in Fig. 9*c*. The impedance of the line is $\varrho$ for any value of $x$.

For resistive loads greater than $\varrho$, the maximum and minimum values of $U$ and $I$ coincide in their position on the line with the antinodes and nodes of the standing waves of $U$ and $I$ in open-circuit conditions (Fig. 9*b*), and for resistive loads with an impedance less than $\varrho$, they coincide with the antinodes and nodes of $U$ and $I$ with short-circuiting (Fig. 9*d*).

In a dissipationless line terminating in the reactance, the standing waves have nodes, i.e. minimum values of $U$ and $I$, equal to zero both for open-circuit and short-circuit conditions. It is then easy to ascertain whether to replace mentally the reactance at the end of the line by a section of open- or short-circuit line. In this case, in accordance with the above, standing waves are produced, although there can be no node or antinode at the junction of the line and the reactive load.

In order to estimate in radio engineering the degree to which line and load are matched, the *travelling wave factor* is widely used, by which we mean the ratio of the minimum value of the standing wave of $U$ and $I$ to the maximum value of the same wave

$$K = \frac{U_{\min}}{U_{\max}} = \frac{I_{\min}}{I_{\max}}. \tag{9.61}$$

The reciprocal $S = 1/k$ is called the *standing wave ratio*.

Taking (54) and (55) into account we get:

$$K = \frac{1 - |n_2|}{1 + |n_2|}, \tag{9.62}$$

from which

$$|n_2| = \frac{1 - K}{1 + K} = \frac{S - 1}{S + 1} = \frac{U_{max} - U_{min}}{U_{min} + U_{min}}. \tag{9.63}$$

Formula (62) is simplified in the case of a resistive load. When $r_2 > \varrho$, $|n_2| = (r_2 - \varrho)/(r_2 + \varrho)$ and, according to (62),

$$K = \frac{\varrho}{r_2}; \tag{9.64}$$

when $r_2 < \varrho$, $|n_2| = (\varrho - r_2)/(\varrho + r_2)$ and, consequently,

$$K = \frac{r_2}{\varrho}. \tag{9.65}$$

In actual conditions the travelling wave factor is not usually less than from 0·5 to 0·6.

Standing waves are used in practice for measuring wavelength or frequency.

The wavelength is determined by doubling the distance between adjacent maximum or minimum values of the standing wave, and the frequency is calculated from the wave-length with the aid of (18).

## 8. Input Impedance of a Line

The input impedance of a line, measured at an arbitrary point at a distance $x$ from the end is determined by the ratio $Z = \dot{U}/\dot{I}$ and can be given in complex or hyperbolic form. Owing to the generality of the problem we can assume that the line is loaded at the end by an arbitrary impedance $Z_2$.

The complex form of the formula for the input impedance of the line is obtained from (21)

$$Z = \frac{\dot{U}}{\dot{I}} = Z_c \frac{e^{\gamma x} + n_2 e^{-\gamma x}}{e^{\gamma x} - n_2 e^{-\gamma x}}$$

or

$$Z = Z_c \frac{1 + n_2 e^{-2\gamma x}}{1 - n_2 e^{-2\gamma x}}. \tag{9.66}$$

The given formula shows that the modulus of the input impedance of the line reaches its maximum values at certain points

$$x + k\frac{\pi}{\alpha} = x + k\frac{\lambda}{2}$$

and its minimum values at a distance $\lambda/4$ from them.

The modulus of the input impedance of a line thus undulates in accordance with coordinate $x$, the maximum, and minimum, values of curve $x$ being shifted with respect to each other by $\lambda/2$.

If we vary the value by the phase factor $\alpha = 2\pi/\lambda$ instead of coordinate $x$, by changing the frequency of the source, then we get an analogous undulating curve, the maximum and, correspondingly, the minimum values being staggered by $\pi/x$ (here, $x = \text{const}$).

When measuring the input impedance of the line by increasing the frequency applied to the line evenly, it is possible to fix two consecutive maximum (or minimum) values corresponding to the frequencies $f_1$ and $f_2$.

In this case

$$\alpha_1 = \frac{\omega_1}{v_1} = \frac{2\pi f_1}{v_1}, \quad \alpha_2 = \frac{\omega_2}{v_2} = \frac{2\pi f_2}{v_2}$$

and, consequently,

$$\alpha_2 - \alpha_1 = 2\pi\left(\frac{f_2}{v_2} - \frac{f_1}{v_1}\right) = \frac{\pi}{x},$$

from which

$$x = \frac{v_1 v_2}{2(v_1 f_2 - v_2 f_1)}.$$

When the discrepancy between frequencies $f_1$ and $f_2$ is small, the phase velocities $v_1$ and $v_2$ are almost identical $v_1 \cong v_2 \cong v$.

Then

$$x \cong \frac{v}{2(f_2 - f_1)}. \tag{9.67}$$

The given formula makes it possible to determine the distance from the observation point to the nearest point on the line at which reflexion occurs (e.g. when there is a short circuit), by measuring at only one point.

The undulating character of curve $z$ is in general subject to the law of the change in the modulus of a hyperbolic tangent with a complex argument, in accordance with (69).

The hyperbolic form of the formula for $Z$ follows directly from (25):

$$Z = \frac{\dot{U}_2 \cosh \gamma x + Z_c \dot{I}_2 \sinh \gamma x}{\dot{I}_2 \cosh \gamma x + \dfrac{\dot{U}_2}{Z_c} \sinh \gamma x} = \frac{Z_2 + Z_c \tanh \gamma x}{1 + \dfrac{Z_2}{Z_c} \tanh \gamma x}. \tag{9.68}$$

Denoting $Z_2/Z_c = \tanh M$, we get:

$$Z = Z_c \frac{\tanh M + \tanh \gamma x}{1 + \tanh M \,.\, \tanh \gamma x} = Z_c \tanh (\gamma x + M). \qquad (9.69)$$

It is convenient to use this formula when $Z_2$ and $Z_c$ are given. The complex number $M = m_1 + jm_2$ can be found from the formula

$$M = \frac{1}{2} \ln \frac{Z_c + Z_2}{Z_c - Z_2},$$

or with the aid of the formulae

$$\tanh 2m_1 = \frac{2q_1}{1 + q_1^2 + q_2^2} = \frac{2Q \cos \varphi}{1 + Q^2},$$

$$\tan 2m_2 = \frac{2q_2}{1 - q_1^2 - q_2^2} = \frac{2Q \sin \varphi}{1 - Q^2},$$

on the assumption that

$$\frac{Z_2}{Z_c} = q_1 + jq_2 = Qe^{j\varphi}.$$

For open-circuit conditions ($z_2 = \infty$) the input impedance of the line, in accordance with (68), is

$$Z_o = Z_c \coth \gamma x, \qquad (9.70)$$

and for short-circuit conditions ($z_2 = 0$)

$$Z_s = Z_c \tanh \gamma x. \qquad (9.71)$$

These formulae agree with formula (54) in Chapter VII.

Taking (70) and (71) into account, equation (68) can be converted thus, by analogy with expression (77) in Chapter VII:

$$Z = Z_o \frac{Z_2 + Z_s}{Z_2 + Z_o}. \qquad (9.72)$$

This formula is used when $Z_0$ and $Z_s$ have been found experimentally in open-circuit and short-circuit conditions.

The given experiments with open-circuit and short-circuit conditions are also used for calculating the image parameters of the line. On the basis of (70) and (71)

$$\left.\begin{aligned} Z_c &= \sqrt{(Z_o Z_s)}, \\ \tanh \gamma x &= \sqrt{\left(\frac{Z_s}{Z_o}\right)}. \end{aligned}\right\} \qquad (9.73)$$

These formulae agree with formula (52) in Chapter VII. In view of the fact that the phase factor $\alpha$ is determined ambivalently from (73), a verification based on (14) is carried out when $\alpha$ is calculated, the phase velocity first $v$ being selected approximately. The calculation of the image parameters from formulae (73) is illustrated below in Example 1.

With direct current ($\gamma = \beta, \varrho = \sqrt{(r/g)}$) the relationships of the line's input impedances to $x$, in open- and short-circuit conditions, have the form of monotone curves which obey the laws of a hyperbolic cotangent ($z_0$) and tangent ($z_s$) with a real argument $\beta x$:

$$z_o = \coth \beta x, \quad z_s = \tanh \beta x.$$

With alternating current $z_0$ and $z_s$ undulate, as $x$ varies (Fig. 10).

The input impedances of a dissipationless line in open- and short-circuit conditions were examined in the previous section. These reactive

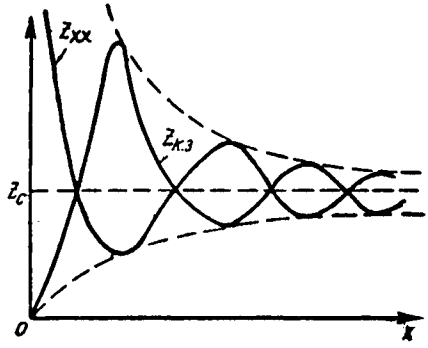



FIG. 10. Input impedances of a line under open- and short-circuit conditions.

input impedances, taking sign into account, are represented by cotangent and tangent curves (Fig. 11), in accordance with (57) and (59). The coordinate $x$ or the phase factor $\alpha$ (proportional to frequency) can be the argument.

By comparing these plots with the frequency characteristics of the impedances of reactive two-terminal networks (Chapter VI) it is easy to confirm their properties: series resonance and parallel resonance alternate but, as opposed to two-terminal networks with their limited number of resonances, a dissipationless line has an infinite number of resonance points.

The input impedance of a dissipationless line is inductive when $x \leqq \lambda/4$ in short-circuit conditions and capacitive in open-circuit conditions. In the first case when $x = \lambda/4$ parallel resonance occurs ($z = \infty$), and in the second series resonance occurs ($z = 0$). Consequently a quarter-wave line with a short-circuited end is similar to a two-terminal network with an inductance and capacitance connected in parallel, and the same line with an open end

is similar to a two-terminal network with an inductance and a capacitance connected in series.

The input impedance of a dissipationless line loaded by an arbitrary impedance $Z_2$ is, according to (66),

$$Z = \varrho \frac{1 + |n_2| e^{j(\Phi - 2\alpha x)}}{1 - |n_2| e^{j(\Phi - 2\alpha x)}}, \tag{9.74}$$

where $\Phi$ is the argument of the complex reflexion factor:

$$n_2 = |n_2| e^{j\Phi}.$$

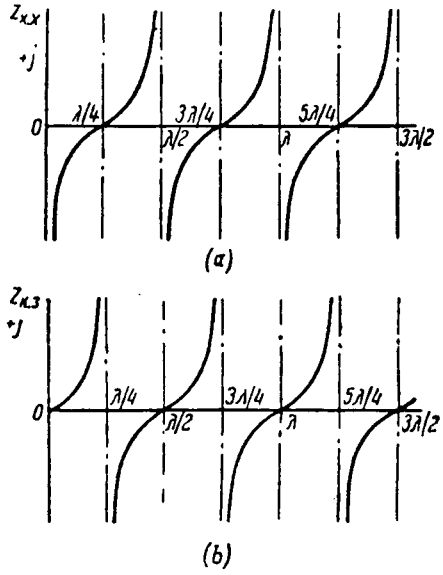



FIG. 11. Input impedances of a dissipationless line: (*a*) open circuit; (*b*) short circuit.

The input impedance of the line reaches a maximum when

$$\Phi - 2\alpha x + 2k\pi = 0 \quad \text{or} \quad x = \frac{\Phi}{2\alpha} + k\frac{\lambda}{2};$$

and, taking (62) into account,

$$Z = r_{\max} = \varrho \frac{1 + |n_2|}{1 - |n_2|} = \frac{\varrho}{K}. \tag{9.75}$$

The minimum input impedance is reached when

$$x = \frac{\Phi}{2\alpha} + \frac{2k - 1}{4}\lambda;$$

and then

$$Z = r_{\min} = \varrho \frac{1 - |n_2|}{1 + |n_2|} = K\varrho. \tag{9.76}$$

The characteristic impedance of a dissipationless line can be defined as the average geometrical value of the maximum and minimum values of the input impedance of the line

$$\varrho = \sqrt{(r_{\max} r_{\min})}. \tag{9.77}$$

It should be noted that in actual conditions, owing to losses the input impedance of the line never falls to zero and never reaches an infinite value.

Approximate calculation formulae were given in Section 4 for lines with small losses:

$$\varrho \cong \sqrt{\left(\frac{L}{C}\right)}, \quad \beta \cong \frac{r}{2\varrho}, \quad \alpha \cong \omega\sqrt{(LC)}.$$

If $\beta x$ is small enough, it is possible to suppose that

$$\cosh \beta x \cong 1, \quad \sinh \beta x \cong \beta x.$$

In this case

$$\cosh \gamma x \cong \cos \alpha x + j\beta x \sin \alpha x,$$

$$\sinh \gamma x \cong \beta x \cos \alpha x + j \sin \alpha_c.$$

Substituting these equations in (68) gives

$$Z = \varrho \frac{(Z_2 + \varrho\beta x) \cos \alpha x + j(Z_2 \beta x + \varrho) \sin \alpha x}{(Z_2 \beta x + \varrho) \cos \alpha x + j(Z_2 + \varrho\beta x) \sin \alpha x}.$$

For *short-circuit conditions* ($Z_2 = 0$)

$$Z_s = \varrho \frac{\beta x \cos \alpha x + j \sin \alpha x}{\cos \alpha x + j\beta x \sin \alpha x} = \varrho \frac{\beta x + j(1 - \beta^2 x^2) \sin \alpha x \cos \beta x}{1 - (1 - \beta^2 x^2) \sin^2 \alpha x}.$$

When $0 < x < \lambda/4$ the input impedance has an inductive character and when

$$\frac{\lambda}{4} < x < \frac{\lambda}{2}$$

it has a capacitive character.

In the case when $x = \lambda/4$ the input impedance is a pure resistance and maximal*:

$$Z_s \cong \frac{\varrho}{\beta x} \cong \frac{8\varrho^2}{r\lambda}.$$

* In connexion with this, a quarter-wave line can be used as a base insulator (known as a "quarter-wave insulator").

If the length of the line is a multiple of the odd number $\lambda/4$, i.e.

$$x = (2k - 1)\frac{\lambda}{4}, \quad \text{then} \quad Z_s \cong \frac{4\varrho}{\beta(2k-1)\lambda} \cong \frac{8\varrho^2}{r(2k-1)\lambda}.$$

In the case where $x = \lambda/2$, the input impedance is a pure resistance and minimal:

$$Z_s \cong \varrho\beta x = \frac{r\lambda}{4}.$$

If $x = k\lambda/2$, then $Z_s = kr\lambda/4$.

For *open-circuit conditions* ($z_2 = \infty$)

$$Z_o = \varrho\frac{\cos\alpha x + j\beta x \sin\alpha x}{\beta x\cos\alpha x + j\sin\alpha x} = \varrho\frac{\beta x - j(1-\beta^2x^2)\sin\alpha x\cos\alpha x}{1-(1-\beta^2x^2)\cos^2\alpha x}.$$

When $0 < x < \lambda/4$ the input impedance has a capacitive character, and when $\lambda/4 < x < \lambda/2$ it has an inductive character.

When $x = \lambda/4$, $Z_0 \cong \varrho\beta x$, i.e. the input impedance is active and minimal.

If $x = (2k - 1)\lambda/4$, then

$$Z_o \cong \varrho\beta(2k-1)\frac{\lambda}{4} \cong \frac{r(2k-1)\lambda}{8}.$$

When $x = \lambda/2$,

$$Z_o \cong \frac{\varrho}{\beta x} \cong \frac{4\varrho^2}{r\lambda},$$

i.e. the impedance is a pure resistance and maximal.

If $x = k\lambda/2$, then $Z_0 \cong 4\varrho^2/kr\lambda$.

It should be noted that a short-circuit line with $x = \lambda/4$ has a greater input impedance than an open-circuit line with $x = \lambda/2$, but an open-circuit line with $x = \lambda/4$ has a lower input impedance than a short-circuit line with $x = \lambda/2$. The higher the factor $k$, i.e. the more times $\lambda/4$ or $\lambda/2$ are contained in the line, the lower the maximum values and the higher the minimum values of the input impedance become, tending to $\varrho$ as the length of the line increases infinitely (regardless of the conditions at the end).

EXAMPLE 1. The results of measuring the input impedance of a line 160 km in length at a frequency of 1000 c/s in open- and short-circuit conditions are given:

$$Z_o = 887e^{-j45°}, \quad Z_s = 540e^{j21°}.$$

We are required to calculate the primary and secondary parameters of the line. We begin by calculating the characteristic impedance and propagation factor

$$Z_c = \sqrt{(Z_oZ_s)} = \sqrt{(887e^{-j45°}540e^{j21}\,)} = 692e^{-j12°};$$

$$\tanh\gamma l = \sqrt{\left(\frac{Z_s}{Z_o}\right)} = \sqrt{\left(\frac{540e^{j21°}}{887e^{-j45°}}\right)} = 0{\cdot}78e^{j33°};$$

$$e^{2\gamma l} = \frac{1+\tanh\gamma l}{1-\tanh\gamma l} = \frac{1+0{\cdot}78e^{j33°}}{1-0{\cdot}78e^{j33°}} = 4{\cdot}53e^{j(48°+k360°)} = 4{\cdot}53e^{j(0{\cdot}82+k2\pi)}.$$

The whole number $k$ can be found by calculating the value $2\alpha l$ approximately, if we start with the approximate value of the phase velocity

$$v = 300000 \text{ km/sec}, \quad 2\alpha l = \frac{2l\omega}{v} \cong \frac{6280 \cdot 320}{300000} \cong 6{\cdot}69 \text{ rad};$$

consequently,

$$k = 1 \quad \text{and} \quad 2\alpha l = 0{\cdot}82 + 6{\cdot}28 = 7{\cdot}1 \text{ rad}.$$

Thus,

$$e^{2\gamma l} = e^{1{\cdot}51} e^{j7{\cdot}1},$$

from which

$$2\beta l = 1{\cdot}51 \text{ nepers}.$$

The attenuation factor is

$$\beta = \frac{2\beta l}{2l} = \frac{1{\cdot}51}{2 \cdot 160} = 0{\cdot}00472 \text{ nep/km}.$$

The phase factor is

$$\alpha = \frac{2\alpha l}{2l} = \frac{7{\cdot}1}{2 \cdot 160} = 0{\cdot}0222 \text{ rad/km}.$$

The propagation factor is

$$\gamma = \beta + j\alpha = 0{\cdot}0227 e^{j78^\circ}.$$

The phase velocity is

$$v = \frac{\omega}{\alpha} = \frac{6280}{0{\cdot}0222} = 283200 \text{ km/sec};$$

and the wavelength is

$$\lambda = \frac{2\pi}{\alpha} = \frac{6{\cdot}28}{0{\cdot}0222} = 283{\cdot}2 \text{ km}.$$

The primary parameters of the line can be found from the formulae

$$\gamma Z_c = r + j\omega L \quad \text{and} \quad \frac{\gamma}{Z_c} = g + j\omega C;$$

$$r + j\omega L = 0{\cdot}0227 e^{j78^\circ} \cdot 692 e^{-j12^\circ} = 15{\cdot}8 e^{j66^\circ} = 6{\cdot}5 + j14{\cdot}4;$$

$$g + j\omega C = \frac{0{\cdot}0227 e^{j78^\circ}}{692 e^{-j12^\circ}} = 32{\cdot}9 \cdot 10^{-6} e^{j90^\circ} = j32{\cdot}9 \cdot 10^{-6}.$$

Thus,

$$r = 6{\cdot}5 \Omega/\text{km}, \quad L = \frac{14{\cdot}4}{2\pi \cdot 1000} = 0{\cdot}0023 \text{ H/km},$$

$$g = 0, \quad C = \frac{32{\cdot}9 \cdot 10^{-6}}{2\pi \cdot 1000} = 0{\cdot}00523 \ \mu\text{F/km}.$$

## 9. Power in a Dissipationless Line

In the case of standing waves with nodes, when the amplitude of the incident wave is the same as that of the reflected wave, and particularly in open-circuit or short-circuit conditions or with a purely reactive load, the power in the voltage and current nodes is zero. At the other points on the

line the power is reactive since voltage and current are displaced with respect to each other by 90°.

In this case no energy is transmitted along the line, but only an exchange of energy takes place between the electrical and magnetic fields in the sections of line limited by voltage and current nodes.

If the line is dissipative or the load is consuming active power the nodes disappear; the amplitude of the incident wave exceeds the amplitude of the reflected wave; and on account of the difference between these amplitudes, energy is transmitted along the line.

It was shown in Section 7 that the maximum effective value of the voltage is obtained when the incident and reflected waves are in phase, and the minimum when there is a phase shift of 180° between them. In accordance with (52)

$$\left.\begin{aligned} U_{\max} &= \frac{1}{2} U \left| 1 + \frac{\varrho}{Z_2} \right| (1 + |n_2|), \\ U_{\min} &= \frac{1}{2} U \left| 1 + \frac{\varrho}{Z_2} \right| (1 - |n_2|). \end{aligned}\right\} \tag{9.78}$$

Maximum current is obtained when the incident and reflected waves are in antiphase, and the minimum when they are in phase:

$$\left.\begin{aligned} I_{\max} &= \frac{1}{2\varrho} U \left| 1 + \frac{\varrho}{Z_2} \right| (1 + |n_2|), \\ I_{\min} &= \frac{1}{2\varrho} U \left| 1 + \frac{\varrho}{Z_2} \right| (1 - |n_2|). \end{aligned}\right\} \tag{9.79}$$

It can be concluded from (78) and (79) that

$$\left.\begin{aligned} \frac{U_{\max}}{I_{\max}} &= \frac{U_{\min}}{I_{\min}} = \varrho, \\ \frac{U_{\max}}{I_{\min}} &= \varrho \frac{1 + |n_2|}{1 - |n_2|} = r_{\max}, \\ \frac{U_{\min}}{I_{\max}} &= \varrho \frac{1 - |n_2|}{1 + |n_2|} = r_{\min}. \end{aligned}\right\} \tag{9.80}$$

If the line is dissipationless, the total transmitted along the line is delivered to the load, and consequently,

$$P = r_{\max} I^2_{\min} = r_{\min} I^2_{\max}$$

or

$$P^2 = r_{\max} r_{\min} (I_{\max} I_{\min})^2,$$

from which, taking (77) into account,

$$P = \varrho I_{max} I_{min}. \tag{9.81}$$

Analogously, it can be proved that

$$P = \frac{1}{\varrho} U_{max} U_{min}. \tag{9.82}$$

It is easy to calculate the power in a dissipationless line by using formulae (81) and (82).

In addition, these formulae show that at the maximum voltage permissible (from the point of view of the dielectric strength of the insulation) $U_{max}$ the higher the travelling wave factor, i.e. the fewer reflexions, the greater the active power transmitted. The maximum transmitted power occurs when the load is matched ($Z_2 = \varrho$), with $U_{max} = U_{min}$ or $K = 1$.

If the load is not matched to the line, some of the energy is wasted in reflexions at the junction. Since the effective values of the voltage of the incident and reflected waves $U_i$ and $U_r$ are invariable at all points on a dissipationless line, then obviously

$$U_{max} = U_i + U_r,$$

$$U_{min} = U_i - U_r,$$

from which

$$K = \frac{U_{min}}{U_{max}} = \frac{U_i - U_r}{U_i + U_r}. \tag{9.83}$$

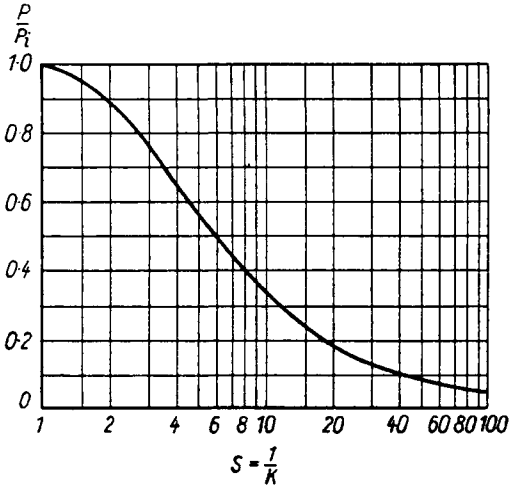



FIG. 12. Relationship of $P/P_i$ to $S = 1/K$.

On the basis of (82)

$$P = \frac{1}{\varrho}(U_i + U_r)(U_i - U_r) = \frac{1}{\varrho}(U_i^2 - U_r^2).$$

It is evident from this that the power transmitted in the incident wave $P_i = U_i^2/\varrho$ is the sum of the useful power absorbed by the load ($P$) and the power travelling back in the reflected wave $P_r = U_r^2/\varrho$.

The ratio of the power delivered to the load to the power transmitted by the incident wave is

$$\frac{P}{P_i} = \frac{P_i - P_r}{P_i} = \frac{U_i^2 - U_r^2}{U_i^2} = 1 - \frac{U_r^2}{U_i^2} = 1 - |n_2|^2$$

or, taking (63) into account,

$$\frac{P}{P_i} = 1 - \left(\frac{1-K}{1+K}\right)^2 = \frac{4K}{(1+K)^2}. \tag{9.84}$$

The ratio of the power absorbed by the load to the power transmitted is plotted as a function of $S = 1/K$ in Fig. 12 (Ref. 99).

## 10. Line as a Matching Transformer

It was shown in Chapter IV, Section 13 that a transformer with a transformation ratio $n = w_1/w_2$ loaded with an impedance $Z_2$ has an input impedance $Z = n^2 Z_2$. It can therefore be used as a *matching device* between load and receiver. A dissipationless line, $\lambda/8$ or $\lambda/4$ in length, can serve the same function.

The input impedance of dissipationless line $x = \lambda/8$ in length and with a $Z_2$ termination is

$$Z = \varrho \frac{Z_2 + j\varrho \tan\frac{\pi}{4}}{\varrho + jZ_2 \tan\frac{\pi}{4}} = \varrho \frac{Z_2 + j\varrho}{\varrho + jZ_2}. \tag{9.85}$$

In the case of a purely resistive load ($Z_2 = r_2$) the moduli of the numerator and the denominator of (85) are equal and therefore the modulus of the input impedance of the line is equal to the characteristic impedance $|Z| = \varrho$.

Thus, by using an eight-wave line, an arbitrary resistance can be matched in modulus with a source which has an internal resistance $\varrho$.

When $x = \lambda/4$, the input impedance of a dissipationless line loaded by an impedance $Z_2$, is

$$Z = \varrho \frac{\dfrac{Z_2}{\tan \dfrac{\pi}{2}} + j\varrho}{\dfrac{\varrho}{\tan \dfrac{\pi}{2}} + jZ_2} = \frac{\varrho^2}{Z_2}. \tag{9.86}$$

If the impedance $Z_2$ is a pure resistance, the input impedance of the line is also resistive. Consequently, a quarter-wave line functions as a matching transformer, the characteristic impedance of the line being selected equal to the geometric mean of the source and load impedances

$$\varrho = \sqrt{(ZZ_2)}. \tag{9.87}$$

As (86) shows, a quarter-wave line is capable of transforming a large impedance into a small impedance, and vice versa.

The extreme cases of $Z_2$ are short-circuit and open-circuit conditions in which the input impedance of a dissipationless quarter-wave line reverts to infinity (when $z_2 = 0$) and zero (when $z_2 = \infty$).

The quarter-wave line finds its practical application as, for example, a matching device between an antenna and a high-frequency transmission line. In this case the characteristic impedance $\varrho$ of the quarter-wave line is selected in accordance with formula (87) in which $Z_2$ denotes the resistance of the antenna, and $Z$ is the characteristic impedance of the transmission line.

In practice the maximum correlation between the impedances being matched which permits quarter-wave lines to be used as matching devices, is approximately ten.

The input impedance of a dissipationless half-wave line ($x = \lambda/2$) equals the load impedance

$$Z = \varrho \frac{Z_2 + j\varrho \tan \pi}{\varrho + jZ_2 \tan \pi} = Z_2.$$

Such a line is therefore equivalent to an ideal transformer with a transformation ratio $n = 1$. Cascading a half-wave and a quarter-wave line ($x = \lambda/2 + \lambda/4 = 3/4\lambda$) does not alter the matching properties of the latter. What has been said above with respect to quarter-wave lines is still valid, therefore, for lines whose length is a multiple of the odd number $\lambda/4$.

In practice, "exponential" lines, whose inductance and capacitance vary along the line in accordance with the exponential function law (Ref. 99) are also for matching the impedances of the source and load.

## 11. Single-stub Impedance Matching on a Line

If a high-frequency transmission line which is several times longer the wavelength is terminated by an impedance $Z_2$, e.g. an untuned aerial, the load and the line can be matched by connecting one or two sections of the same line in parallel with the line. Let us familiarize ourselves with methods of connecting one such section closed at the end. A short-circuited section has an advantage over an open section in that radiation losses are reduced and it is simpler from the point of view of design. The term "stub" is used to denote such a section of transmission line.

The stub is connected across the transmission line at a designated distance from the load where the input admittance of the circuit looking toward the load is $Y = \varrho^{-1} - jb$. By selecting the length of the stub so that its conductance equals $jb$ we get a total admittance at the point of connexion equal to $\varrho^{-1}$, i.e. the input impedance at this point is equal to the characteristic impedance of the transmission line.

When this condition obtains there will be no reflexions or standing waves in the basic section of line – between the source and the connexion point – and, consequently, losses are reduced.

Before the stub is connected up, the input admittance of the line at any arbitrary point is, according to (74),

$$Y = \frac{1}{\varrho} \frac{1 - |n_2| e^{j(\Phi - 2\alpha x)}}{1 + |n_2| e^{j(\Phi - 2\alpha x)}}.$$

Omitting the subscript 2 in the reflexion factor for the sake of simplification, we get

$$Y = \frac{1}{\varrho} \frac{1 - |n| \cos(\Phi - 2\alpha x) - j|n| \sin(\Phi - 2\alpha x)}{1 + |n| \cos(\Phi - 2\alpha x) + j|n| \sin(\Phi - 2\alpha x)}$$

or

$$Y = \frac{1}{\varrho} \frac{1 - |n|^2 - j2|n| \sin(\Phi - 2\alpha x)}{1 + |n|^2 + 2|n| \cos(\Phi - 2\alpha x)}.$$

Assuming $Y = g = jb$, we find

$$g\varrho = \frac{1 - |n|^2}{1 + |n|^2 + 2|n| \cos(\Phi - 2\alpha x)}, \tag{9.88}$$

$$b\varrho = \frac{2|n| \sin(\Phi - 2\alpha x)}{1 + |n|^2 + 2|n| \cos(\Phi - 2\alpha x)}. \tag{9.89}$$

Equations (88) and (89) are plotted in Fig. 13 for a value of $|n|$ arbitrarily chosen as 0·5.

The quantity $g\varrho$ attains its maximum when $\Phi - 2\alpha x_2 = -\pi$ from which

$$x_2 = \frac{\Phi + \pi}{2\alpha}. \tag{9.90}$$

At the point $x_2$, $b = 0$ and

$$g = \frac{1}{\varrho} \cdot \frac{1 + |n|}{1 - |n|} = \frac{1}{K\varrho}.$$

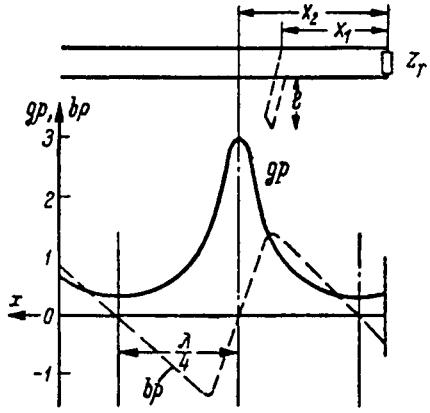



FIG. 13. Single-stub impedance matching on a line: (*a*) location of the stub; (*b*) curves $g\varrho$ and $b\varrho$ in a function of $x$.

Consequently, at this point $Z = r_{\min} = K\varrho$, in accordance with (80), i.e. voltage minimum exists.

The point $x_1$ (Fig. 14) at which $g = 1/\varrho$ can be found from (88):

$$1 = \frac{1 - |n|^2}{1 + |n|^2 + 2|n| \cos(\Phi - 2\alpha x_1)},$$

from which $\cos(\Phi - 2\alpha x_1) = -|n|$. Since $\arccos(-|n|) = -\pi + \arccos|n|$ we get

$$x_1 = \frac{\Phi + \pi - \arccos|n|}{2\alpha}. \tag{9.91}$$

Substituting (91) in (89) we find the susceptance at point $x$ from the formula

$$b\varrho = \frac{2|n| \sin(\pi + \arccos|n|)}{1 + |n|^2 + 2|n| \cos(\pi + \arccos|n|)} = -\frac{2|n|}{\sqrt{(1 - |n|^2)}}. \tag{9.92}$$

At point $x_1$ the stub is connected, its input impedance, in accordance with (59),

$$Y_{\text{sec}} = -j \frac{1}{\varrho} \cot \frac{2\pi}{\lambda} l \tag{9.93}$$

satisfying the condition

$$Y_{sec} = jb. \tag{9.94}$$

As a result of substituting (92) and (93) in (94) we get the equation

$$\cot \frac{2\pi}{\lambda} l = \frac{2|n|}{\sqrt{(1 - |n|^2)}},$$

from which

$$l = \frac{\lambda}{2\pi} \operatorname{arc\,tan} \frac{\sqrt{(1 - |n|^2)}}{2|n|} = \frac{\lambda}{2\pi} \operatorname{arc\,tan} \frac{\sqrt{K}}{1 - K}. \tag{9.95}$$

The resulting equation makes it possible to find the length of the short-circuited stub connected in parallel to the basic line at point $x_1$, i.e. at a distance

$$x_2 - x_1 = \frac{\operatorname{arc\,cos} |n|}{2\alpha} = \frac{\lambda}{4\pi} \operatorname{arc\,cos} \frac{1 - K}{1 + K}$$

from the point of the voltage minimum.

As a result of connecting up the stub, the resulting impedance of the circuit at point $x_1$ equals the characteristic impedance $\varrho$ and therefore energy is transmitted in the line without reflexions between the source and point $x_1$.

## 12. Circle Diagrams for a Dissipationless Line

According to (74)

$$\frac{Z}{\varrho} = \frac{1 + |n| e^{j(\Phi - 2\alpha x)}}{1 - |n| e^{j(\Phi - 2\alpha x)}} = m_1 + jm_2$$

or, taking (63) into account,

$$(1 + m_1 + jm_2) \frac{1 - K}{1 + K} e^{j(\Phi - 2\alpha x)} = m_1 - 1 + jm_2. \tag{9.96}$$

As a result of conversions, we get

$$m_1^2 - m_1 \frac{1 + K^2}{K} + m_2^2 = -1,$$

which can be written in the form of a circle equation

$$\left(m_1 - \frac{1 + K^2}{2K}\right)^2 + m_2^2 = \left(\frac{1 - K^2}{2K}\right)^2,$$

which represents the locus of the vector $Z/\varrho$, with $K$ constant. The centre of this circle lies on the abscissa at the point

$$\frac{1 + K^2}{2K} = \frac{1}{2}\left(\frac{1}{K} + K\right),$$

the radius being

$$\frac{1 - K^2}{2K} = \frac{1}{2}\left(\frac{1}{K} - K\right).$$

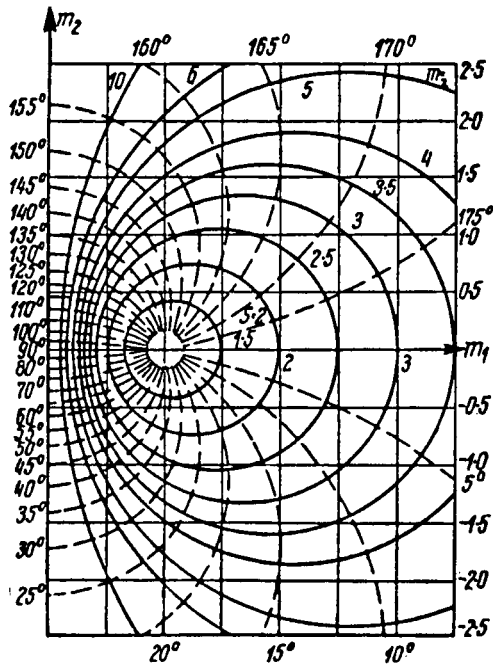



FIG. 14. Circle diagrams of a dissipationless line on axes $m_1$ and $m_2$.

As one might expect, the coordinate of the centre of the circle is determined by half the sum of, and the magnitude of the radius by half the difference between, the maximum and minimum values $Z/\varrho$. According to (75) and (76)

$$\frac{r_{\max}}{\varrho} = \frac{1}{K} = S, \quad \frac{r_{\min}}{\varrho} = K = \frac{1}{S}.$$

The family of circles shown in Fig. 14 (continuous lines) can be constructed for different values of $S$.

Writing the initial equation (96) in the form

$$\frac{S - 1}{S + 1} e^{j(\Phi - 2\alpha x)} = \frac{m_1 - 1 + jm_2}{m_1 + 1 + jm_2}$$

and discarding the imaginary part from the denominator we get

$$\frac{S-1}{S+1}\, e^{j(\Phi-2\alpha x)} = \frac{m_1^2 - 1 + m_2^2 + j2m_2}{(1+m_1)^2 + m_2^2}. \tag{9.97}$$

Assuming that $\Phi = 0$ we can read the angle $\alpha x$ from the abscissa axis. According to (97)

$$\tan(-2\alpha x) = \frac{2m_2}{m_1^2 - 1 + m_2^2} \quad \text{and} \quad m_1^2 - 1 + m_2^2 + \frac{2m_2}{\tan 2\alpha x} = 0$$

or

$$m_1^2 + \left(m_2 + \frac{1}{\tan 2\alpha x}\right)^2 = 1 + \frac{1}{\tan^2 2\alpha x} = \frac{1}{\sin^2 2\alpha x}.$$

Consequently the locus of points $Z/\varrho$ for a constant value of $\alpha x$ is a circle of radius $1/\sin 2\alpha x$. The centre of this circle lies on the ordinate axis at the point $1/\tan 2\alpha x$.

The family of curves shown in Fig. 14 (broken lines) can be constructed for different values of $\alpha x$. All these circles pass through the point [1; 0], and for the convenience of calculation the numerical values of the angles are given for $\alpha x$ in Fig. 14.

The circle diagrams shown in Fig. 14 can also be used for the reciprocal of $Z/\varrho$, namely, $\varrho Y$. In this case $m_1$ and $m_2$ will correspond to the real and imaginary parts of $\varrho Y$. The positive (capacitive) susceptance will be plotted above the abscissa axis and the negative (inductive) susceptance below it.

Various problems can be solved with the aid of the circle diagrams in Fig. 14. Thus, if any two of the three values – load impedance $Z_2$, argument $\alpha x$ and the input impedance of the line $Z$ – are known, then the third value can easily be found from the Fig. 14 diagram.

The use of the circle diagram is illustrated by the example below.

EXAMPLE 2. An impedance $Z_2 = 1000 + j500$ is connected to the end of a dissipationless line in which $\varrho = 500\Omega$ and $\alpha x = 120°$.

The normalized terminating impedance is $Z_2 p/\varrho = 2 + j1$. Consequently, in the given case, $m_1 = 2$ and $m_2 = 1$. The point $2 + j1$ in the upper halfplane corresponds to the end of the line ($x = 0$). It lies on the circle $S = 2{\cdot}63$. The angle $\alpha x = 120°$ should be marked off from this point on the circle $S = 2{\cdot}63$ in the direction of the negative angles, i.e. clockwise.

Thus a point $Z/\varrho = 0{\cdot}41 + j0{\cdot}25$ will be obtained. Consequently the unknown input impedance of the line is $Z = 205 + j125 = 240ej31°24'$. When calculating analytically from formula (68) we get $Z = 208 + j123\Omega$.

It should be noted that when there is no standing wave ($K = 1$), a circle of zero radius corresponds to the value $S = 1/k = 1$, i.e. point 1 is on the abscissa axis. Circles for all other values of $S$ circumscribe this point and the circle of infinite radius coinciding with the ordinate axis in Fig. 14

corresponds to the extreme value $S = \infty$. The circles $S$ and $\alpha x$ are not concentric and, correspondingly, the reading scales of $S$ and $\alpha x$ are not identical. This introduces inaccuracies into integration. In addition, the restricted dimensions of the diagram make calculations with high impedances difficult.

A modification of the circle diagram of a dissipationless line suggested by F. Smith in 1939 has been widely used in practice. In this chart all values of impedances are circumscribed by a circle of unit radius (Refs. 90 and 99).

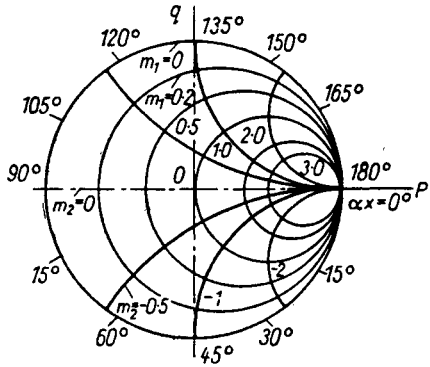



FIG. 15. Circle diagram of a dissipationless line on axes $p$ and $q$.

The diagram is constructed from (97) on the coordinates $p$ and $q$.

$$|n|\, e^{j(\Phi - 2\alpha x)} = p + jq = \frac{m_1^2 - 1 + m_2^2}{(m_1 + 1)^2 + m_2^2} + j\frac{2m_2}{(m_1 + 1)^2 + m_2^2},$$

from which

$$p = \frac{m_1^2 - 1 + m_2^2}{(m_1 + 1)^2 + m_2^2}, \qquad q = \frac{2m_2}{(m_1 + 1)^2 + m_2^2}.$$

Eliminating $m_1$ and $m_2$ in turn from these two equations, we find

$$\left.\begin{aligned} \left(p - \frac{m_1}{m_1 + 1}\right)^2 + g^2 &= \frac{1}{(m_1 + 1)^2}, \\ (p - 1)^2 + \left(q - \frac{1}{m_2}\right)^2 &= \frac{1}{m_2^2}. \end{aligned}\right\} \tag{9.98}$$

The first equation in (98) expresses a family of circles for constant values of $m_1$. Their centres lie at points $m_1/(m_1 + 1)$ on the abscissa axis, and their radii are $1/(m_1 + 1)$.

The second equation in (98) expresses a family of circles for constant values of $m_2$. Their centres lie at points $1 + j/m_2$, and their radii are $1/m_2$. These two families of circles are shown in Fig. 15.

## 13. Line as a Resonant Circuit Element

If a line with small losses is used as an element in a resonant circuit then, by analogy with the method described in Chapter VIII, Section 1, it is convenient to use the concepts of the band width and the line's $Q$ factor.

Here, as in Chapter VIII, by *band width* we mean the relative width of the resonance curve between the points which correspond to half the maximum power.

Let us examine a quarter-wave line open at the end, as an analogue of a resonant circuit with $r$, $L$ and $C$ connected in series. At the edges of the bandwidth of the $rLC$ circuit the current equals (Chapter VIII, Section 1)

$$I = \frac{U}{\sqrt{(r^2 + x^2)}} = \frac{I_0}{\sqrt{2}} = \frac{U}{\sqrt{2}r},$$

from which

$$r = x.$$

By equating the resistance and inductive reactance of the input impedance of the line open at the end, in accordance with Section 8 we get, for a frequency $f_2$ or, the same thing, a phase factor $\alpha_2$,

$$\varrho\beta x = -\varrho(1 - \beta^2 x^2) \sin \alpha_2 x \cos \alpha_2 x$$

or

$$\sin 2\alpha_2 x = -\frac{2\beta x}{1 - \beta^2 x^2}.$$

Quantity $\beta^2 x^2$ is negligibly small in comparison with unity. Therefore

$$\sin 2\alpha_2 x \cong -2\beta x$$

or

$$2\alpha_2 x \cong \text{arc sin}\,(-2\beta x).$$

When $\sin 2\beta x \cong 2\beta x$, $\sin(\pi + 2\beta x) = -\sin 2\beta x \cong -2\beta x$. Therefore

$$2\alpha_2 x \cong \pi + 2\beta x,$$

from which

$$\alpha_2 x \cong \frac{\pi}{2} + \beta x$$

or

$$f_2 = \frac{v}{2\pi x}\left(\frac{\pi}{2} + \beta x\right).$$

When $x = \lambda/4$ the resonant frequency $f_0 = v/\lambda = v/4x$. Consequently,

$$f_2 = f_0 + \frac{v\beta}{2\pi}.$$

The second cut-off frequency is

$$f_1 = f_0 - \frac{v\beta}{2\pi}.$$

The bandwidth, with respect to the resonant frequency, is

$$\delta_2 - \delta_1 = \frac{f_2 - f_1}{f_0} = \frac{\frac{v\beta}{\pi}}{\frac{v}{4x}} = \frac{4\beta x}{\pi};$$

By substituting $x = \lambda/4 = \pi/2\alpha$ we find $\delta_2 - \delta_1 = 2\beta/\alpha$.

The *Q factor* is a quantity which is the reciprocal of the bandwidth:

$$Q = \frac{1}{\delta_2 - \delta_1} = \frac{\alpha}{2\beta}.$$

At small values of the attenuation factor $\beta$ the $Q$ factor is high; it reaches approximately 1000 to 4000 which is considerably higher than the $Q$ factor of *rLC* circuits. In conjunction with this the sharpness of the tuning also increases.

## 14. Artificial Line

A circuit which has concentrated parameters and approximates in its frequency characteristics (within a given band of frequencies) to a circuit with distributed parameters is called an *artificial line.*

Artificial lines are widely used for experimental purposes in laboratory conditions, and in particular in present-day pulse radio engineering, in order to obtain a required delay of signals, and as wave-shaping circuits.

It was noted in Section 3 that all uniform lines are symmetrical four-terminal networks with the image parameters

$$\gamma l = l\sqrt{[(r + j\omega L)(g + j\omega C)]}$$

and

$$Z_c = \sqrt{\left(\frac{r + j\omega L}{g + j\omega C}\right)}.$$

By replacing a line by an equivalent T-network in accordance with Fig. 7*a* and formulae (33) and (54) in Chapter VII, we get the calculation formulae

$$\frac{Z_1}{2} = Z_c \tanh\frac{\gamma l}{2}, \quad Z_2 = \frac{Z_c}{\sinh \gamma l}. \tag{9.99}$$

Such a T-network can be made for any fixed frequency. But when signals are transmitted in a certain fixed band of frequencies, the quantities $Z_1$ and $Z_2$ are complex functions of frequency, not realizable in the form of the simplest elements. In this case the artificial line is produced in the form of a ladder network, each section of which replaces, with a sufficient degree of accuracy, an extremely small section of uniform line.

Supposing that, for small values of $\beta x$ and $\alpha x$,

$$\sinh \gamma x \cong \gamma x, \quad \tanh \frac{\gamma x}{2} \cong \frac{\gamma x}{2},$$

we can find for each T-section

$$\frac{Z_1}{2} \cong Z_c \frac{\gamma x}{2} = \frac{(r + j\omega L)x}{2}, \quad Z_2 \cong \frac{Z_c}{\gamma x} = \frac{1}{(g + j\omega C)\, x}.$$

The resulting values are physically realizable for a given band of frequencies with the aid of the simplest elements—resistance, inductance and capacitance.

The characteristic impedance and propagation function of each section are determined from the formulae

$$Z_T = \sqrt{\left[Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)\right]} = Z_c \sqrt{\left[1 + \left(\frac{\gamma x}{2}\right)^2\right]},$$

$$g_T = 2 \operatorname{arc\,sinh} \sqrt{\left(\frac{Z_1}{4Z_2}\right)} = 2 \operatorname{arc\,sinh} \left(\frac{\gamma x}{2}\right).$$

A ladder network consisting of $n = 1/x$ such T-sections connected in cascade forms an artificial line. The image parameters of such a circuit are

$$\left.\begin{aligned} Z_T &= Z_c \sqrt{\left[1 + \left(\frac{\gamma l}{2n}\right)^2\right]}, \\ n g_T &= 2n \arcsin \frac{\gamma l}{2n}. \end{aligned}\right\} \tag{9.100}$$

Expanding (100) into a series with $|\gamma l/2n| \ll 1$ gives

$$Z_T = Z_c \left[1 + \frac{1}{2}\left(\frac{\gamma l}{2n}\right)^2 - \frac{1}{8}\left(\frac{\gamma l}{2n}\right)^4 + \cdots\right] = Z_c(1 + \delta_z);$$

$$n g_T = \gamma l \left[1 - \frac{1}{6}\left(\frac{\gamma l}{2n}\right)^2 + \frac{3}{40}\left(\frac{\gamma l}{2n}\right)^4 - \cdots\right] = \gamma l\,(1 + \delta_g).$$

Here $\delta_z$ and $\delta_\varphi$ are the relative errors $Z_T$ and $n\varphi_T$:

$$\delta_z \cong -3\delta_\varphi = \frac{1}{8}\left(\frac{\gamma l}{n}\right)^2 = \frac{1}{8}\left(\frac{l}{n}\right)^2 (r + j\omega L)(g + j\omega C).$$

The error $\delta_z$ is equal to the triple value $\delta_\varphi$ with the opposite sign and increases with frequency. Disregarding $r$ and $g$ in comparison with $\omega L$ and $\omega C$, in the field of high frequencies, we get

$$\delta_z \cong -\frac{LC\omega^2}{8}\left(\frac{l}{n}\right)^2,$$

from which, at a given maximum value of frequency $\omega$,

$$n = \omega l \sqrt{\left(\frac{LC}{8\,|\delta_z|}\right)}.$$

The given equation makes it possible to find the number of sections $n$ in the ladder network, depending on the required degree of approximation of the parameters of the artificial line to the parameters of a real uniform line.

In conclusion it should be noted that in certain cases a line can be represented by an equivalent T-network (or pi-network) with a sufficient degree of accuracy. Example 3 below gives a case where the branches of an equivalent circuit calculated from formulae (99) can be replaced with a certain degree of approximation by uniform $r$ and $C$ elements in the band of audio frequencies. This case obtains in certain conditions because the imaginary part of $Z_1/2$ and the real part of $Z_2$ are so small in absolute value that they can be disregarded. The circuit would otherwise be impracticable on account of the negative sign of the real part of $Z_2$.

EXAMPLE 3 (Ref. 67). To verify the possibility of constructing an equivalent T-network for a cable 6·4 km in length for frequencies of 1000 and 2000 c/s. The primary parameters of the cable are:

$$r = 55\Omega/\text{km}, \quad C = 0{\cdot}0375\ \mu\text{F/km}, \quad L \cong 0, \quad g \cong 0.$$

The results of the calculations are given in the table below:

| $f$ | 1000 c/s | 2000 c/s |
|---|---|---|
| $Z_c$ | $488 \angle -45°$ | $342 \angle -45°$ |
| $\gamma$ | $0{\cdot}114 \angle 45°$ | $0{\cdot}161 \angle 45°$ |
| $\gamma l$ | $0{\cdot}729 \angle 45°$ | $1{\cdot}030 \angle 45°$ |
| $\tanh \gamma l/2$ | $0{\cdot}37 \angle 42°$ | $0{\cdot}52 \angle 39°$ |
| $\sinh \gamma l$ | $0{\cdot}735 \angle 50°$ | $1{\cdot}04 \angle 55°$ |
| $Z_1/2$ | $179 - j9$ | $177 + j17$ |
| $Z_2$ | $-6 - j655$ | $-57 - j324$ |

It will be seen from the table that the real part of $Z_2$ can be disregarded in comparison with the imaginary part which, in this band of frequencies, changes in inverse proportion to frequency. We can take $Z_2 = 1/j\omega C$ approximately, where $C = 0{\cdot}244\ \mu F$.

In its turn $Z_1/2$ can be represented in the form of a resistance $r = 178\Omega$. The resulting T-structure is shown in Fig. 16.

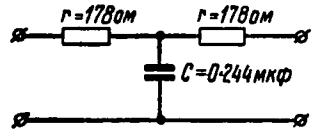



FIG. 16. Example 3.

CHAPTER X

# NONSINUSOIDAL PERIODIC WAVES (FOURIER SERIES)

## 1. Trigonometric Fourier Series

The study of the sinusoidal response of a linear electrical network is based on the assumption that the electrical energy sources supplying the network in question have *sinusoidal voltages* of the same frequency across their terminals, as a result of which all the voltages and currents in the given circuit also change sinusoidally with the same frequency. If the sources connected to the circuit have different frequencies, or if the shape of the e.m.f. curve of the voltage source and, correspondingly, the shape of the curve of the current in the current source, are *nonsinusoidal*, the voltages and currents in the circuit are also nonsinusoidal.

In many electrical and radio engineering devices nonsinusoidal operating conditions are the circuit's normal conditions. These conditions are particularly typical of non-linear circuits.

This chapter is devoted to an examination of nonsinusoidal periodic waves in linear electrical circuits under the effect of nonsinusoidal voltages. The electrical and magnetic quantities in such circuits are nonsinusoidal periodic functions.

A periodic function, generally nonsinusoidal, with a period $T$

$$f(t) = f(t + T) \tag{10.1}$$

can be studied for any interval of time $T$.

An arbitrary instant $t_0$ can be selected as the beginning of the interval. An instant $t_0 + T$ then corresponds to the end of the interval. For example, when $t_0 = 0$ or $t_0 = -T/2$, an interval $(0, T)$ or $(-T/2, T/2)$ is obtained.

Within the limits of an interval of time $T$ the function $f(t)$ can be either continuous or have discontinuities. Let us assume that all these discontinuities possess the following property: if function $f(t)$ is discontinuous at point $t = t_1$ (Fig. 1), then there are finite limits to function $f(t)$ as it approaches the discontinuity point both from the right (from values $t > t_1$) and from the left (from values $t < t_1$). The limits are denoted convention-

ally by $f(t_1 + 0)$ and $f(t_1 - 0)$ or $f(t_1 -)$ and $f(t_1 +)$. Discontinuities of this kind are called *finite discontinuities*. Let us also assume that, within an interval of time $T$, the number of such discontinuities, and also the number of maxima and minima of function $f(t)$, are finite.

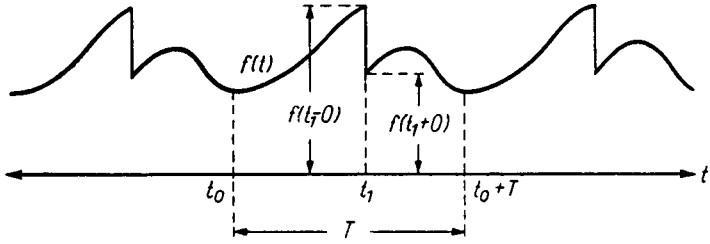



FIG. 1. Nonsinusoidal function.

We shall thus discuss functions which satisfy *Dirichlet conditions*. It should be noted that any function which corresponds to any real electromagnetic process satisfies Dirichlet conditions.

*A nonsinusoidal periodic wave which satisfies Dirichlet conditions can be represented in the form of an infinite harmonic series—a Fourier series.* This series converges to the value of the given function at all points of continuity, and to the mean of the left and right limit values of $f(t)$ at discontinuities, i.e.

$$\tfrac{1}{2}[f(t -) + f(t +)].$$

Let us denote $\omega_1 = 2\pi/T$ and call $\omega_1$ the *fundamental angular frequency*.

A harmonic series in trigonometric form has the appearance

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t), \tag{10.2}$$

where

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cos n\omega_1 t \, dt, \tag{10.3}$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \sin n\omega_1 t \, dt. \tag{10.4}$$

Here $a_0/2$ is a constant, and $a_n$ and $b_n$ are the amplitudes of the cosinusoidal and sinusoidal terms of the series (and do not depend on the $t_0$ selected).

The constant $a_0/2$, which is determined with the aid of (3), with $n = 0$, represents the average or d.c. component of a wave. It is zero when the areas of the positive and negative values of $f(t)$ are the same (Fig. 2).

The value $t_0$ can be selected arbitrarily. We usually take $t_0 = 0$ or $t_0 = -T/2$.

Assuming $\omega_1 t = \alpha$ and $t_0 = 0$ (and taking it into account that $\omega_1 T = 2\pi$ and $d\alpha = \omega_1 dt$), we find

$$f(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\alpha + b_n \sin n\alpha), \tag{10.5}$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \cos n\alpha \, d\alpha, \tag{10.6}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin n\alpha \, d\alpha. \tag{10.7}$$

The Fourier series can be written in another form. Taking

$$a_n \cos n\alpha + b_n \sin n\alpha = F_n \cos (n\alpha - \psi_n),$$

where

$$F_n = \sqrt{(a_n^2 + b_n^2)}, \quad \psi_n = \arctan \frac{b_n}{a_n}, \tag{10.8a}$$

we get

$$f(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} F_n \cos (n\alpha - \psi_n). \tag{10.8b}$$

This form is convenient in those cases where, for example, it is required to find the percentage content of each harmonic ($F_n$ is the amplitude of the $n$-th harmonic).

In practice, expansions into Fourier series are limited by the finite number of terms in the series, depending on the rapidity of convergence (Ref. 23).

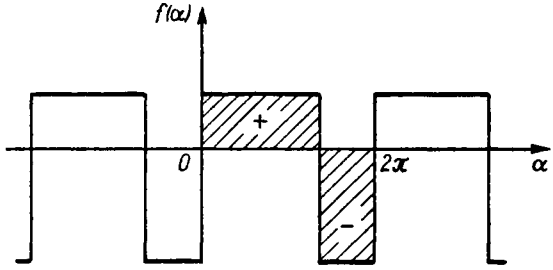



FIG. 2. Zero average value.

The nonsinusoidal periodic waves which occur in practice can be divided into two groups. The first group contains curves of a geometrically correct shape. The second contains curves of a more complex shape; they are expanded into a Fourier series by means of graphical evaluation of constants.

The graphical method of analysis is an approximate method of numerical integration in which, instead of calculating a certain integral one finds the sum of a finite number of terms. For this purpose a period of function $f(\alpha)$ equal to $2\pi$ is divided into $m$ equal intervals of $\Delta\alpha$, so that $m \times \Delta\alpha = 2\pi$.

The distance from the origin to the middle of the $k$-th interval is

$$\alpha = k\frac{2\pi}{m} - \frac{1}{2}\frac{2\pi}{m} = \left(k - \frac{1}{2}\right)\frac{2\pi}{m} \text{ rad.}$$

Let $f_k(\alpha)$ be the value of a given nonsinusoidal periodic function in the middle of the $k$-th interval.

The integrals in (6) and (7) are replaced by the sums

$$a_n \cong \frac{1}{\pi}\sum_0^{2\pi} f(\alpha)\cos n\alpha\Delta\alpha \cong \frac{2}{m}\sum_{k=1}^{m} f_k(\alpha)\cos n\left(k - \frac{1}{2}\right)\frac{2\pi}{m},$$

$$b_n \cong \frac{1}{\pi}\sum_0^{2\pi} f(\alpha)\sin n\alpha\Delta\alpha \cong \frac{2}{m}\sum_{k=1}^{m} f_k(\alpha)\sin n\left(q - \frac{1}{2}\right)\frac{2\pi}{m}.$$

## 2. Cases of Symmetry

Nonsinusoidal periodic waves describing electrical and magnetic quantities usually possess symmetries of various types, and this facilitates expanding them into a Fourier series because it leads to simplification of the formulae for the Fourier coefficients.

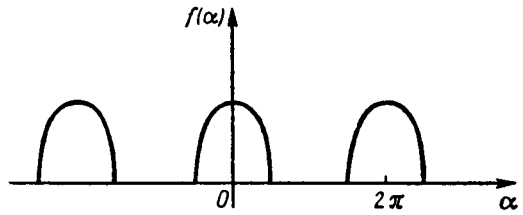



FIG. 3. Zero-axis symmetry.

We shall examine the following cases of symmetry.

*Zero-axis symmetry.* The wave possesses zero-axis symmetry if it is symmetrical about the axis $\alpha = 0$ (Fig. 3), i.e. $f(\alpha) = f(-\alpha)$. Such a function is called an *even function.*

In accordance with (5)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos n\alpha + b_n\sin n\alpha] = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(-n\alpha) + b_n\sin(-n\alpha)],$$

from which, taking into account that $\cos(-n\alpha) = \cos n\alpha$ and $\sin(-n\alpha) = -\sin n\alpha$, we get

$$\sum_{n=1}^{\infty} b_n \sin n\alpha = 0. \tag{10.9}$$

Equation (9) is satisfied for arbitrary values of $\alpha$ only if $b_n = 0$ for any $n$, therefore, for a given form of symmetry

$$f(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\alpha, \tag{10.10}$$

i.e. *an even wave can only contain cosine curves and a constant component*

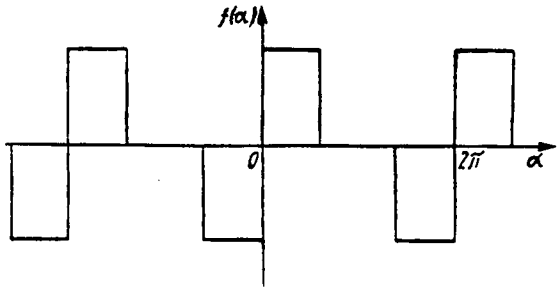



FIG. 4. Zero-point symmetry.

Another important property of even functions is that it is sufficient to use the curve $f(\alpha)$ for a half-period in order to determine the coefficients $a_n$, i.e.

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\alpha) \cos n\alpha \, d\alpha. \tag{10.11}$$

For proof we use the formula

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\alpha) \cos n\alpha \, d\alpha,$$

from which

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} f(\alpha) \cos n\alpha \, d\alpha + \frac{1}{\pi} \int_0^{\pi} f(\alpha) \cos n\alpha \, d\alpha.$$

Substituting $-\alpha$ for $\alpha$ in the first term gives

$$a_n = -\frac{1}{\pi} \int_{\pi}^{0} f(\alpha) \cos n\alpha \, d\alpha + \frac{1}{\pi} \int_0^{\pi} f(\alpha) \cos n\alpha \, d\alpha$$

$$= \frac{2}{\pi} \int_0^{\pi} f(\alpha) \cos n\alpha \, d\alpha.$$

*Zero-point symmetry.* A wave possesses zero-point symmetry if it is symmetrical about the origin (Fig. 4), i.e. $f(\alpha) = -f(-\alpha)$. This is called an *odd function.*

Taking the property of these functions into account, equation (5) can be written in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\alpha + b_n \sin n\alpha]$$

$$= -\frac{a_0}{2} - \sum_{n=1}^{\infty} [a_n \cos(-n\alpha) + b_n \sin(-n\alpha)],$$

from which

$$a_0 + 2\sum_{n=1}^{\infty} a_n \cos n\alpha = 0. \tag{10.12}$$

Equation (12) is only satisfied for arbitrary values of $\alpha$ when $a_n = 0$ for any $n$.

For the given form of symmetry, therefore,

$$f(\alpha) = \sum_{n=1}^{\infty} b_n \sin n\alpha, \tag{10.13}$$

i.e. *an odd wave can only contain sine curves.*

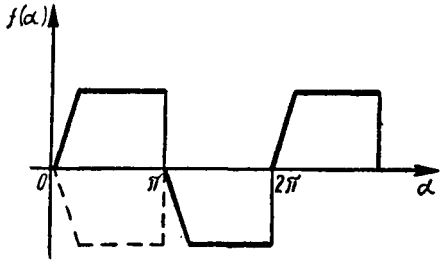



FIG. 5. Half-wave symmetry.

In this case, as in the previous case, it is sufficient to use the curve $f(\alpha)$ for half a period in order to determine the coefficients $b_n$, i.e.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\alpha) \sin n\alpha \, d\alpha. \tag{10.14}$$

For proof we use the formula

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\alpha) \sin n\alpha \, d\alpha = \frac{1}{\pi}\int_{-\pi}^{0} f(\alpha) \sin n\alpha \, d\alpha + \frac{1}{\pi}\int_0^{\pi} f(\alpha) \sin n\alpha \, d\alpha.$$

Substituting $-\alpha$ for $\alpha$ in the first term gives

$$b_n = -\frac{1}{\pi}\int_{\pi}^{0} f(\alpha) \sin n\alpha \, d\alpha + \frac{1}{\pi}\int_0^{\pi} f(\alpha) \sin n\alpha \, d\alpha = \frac{2}{\pi}\int_0^{\pi} f(\alpha) \sin n\alpha \, d\alpha.$$

*Half-wave (or mirror) symmetry.* A wave possesses half-wave symmetry when the negative portion of the wave is the mirror image of the positive portion, shifted by a half-period, i.e.

$$f(\alpha) = -f(\alpha + \pi).$$

Replacing $f(\alpha)$ in accordance with (5) we get

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\alpha + b_n \sin n\alpha]$$

$$= -\frac{a_0}{2} - \sum_{n=1}^{\infty} [a_n \cos n(\alpha + \pi) + b_n \sin n(\alpha + \pi)],$$

from which, for even values of $n$

$$a_0 + 2\sum_{n=2}^{\infty} [a_n \cos n\alpha + b_n \sin n\alpha] = 0. \tag{10.15}$$

Equation (15) is only satisfied for arbitrary values of $\alpha$ when $a_0 = 0$ and $a_n = b_n = 0$ for even values of $n$.

For the given form of symmetry, therefore,

$$f(\alpha) = \sum_{n=1}^{\infty} [a_n \cos n\alpha + b_n \sin n\alpha], \tag{10.16}$$

i.e. *a function which possesses half-wave symmetry contains odd harmonics only.*

In this case coefficients $a_n$ and $b_n$ can be calculated from formulae (11) and (14).

When expanding a nonsinusoidal function into a Fourier series it should first be ascertained whether the given function possesses any form of symmetry. The presence of symmetry makes it possible to predict the harmonics to be expected in the expansion.

EXAMPLE 1. It is required to expand the saw-tooth wave given in Fig. 6 into a Fourier series. This wave satisfies the condition

$$f(\alpha) = 1 - \frac{\alpha}{\pi} \quad \text{when} \quad 0 < \alpha < 2\pi.$$

The given function is odd and therefore the expansion contains no constant component or cosine curves:

$$a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{\alpha}{\pi}\right)\sin n\alpha \, d\alpha = \frac{2}{n\pi}.$$

Consequently,

$$f(\alpha) = \frac{2}{\pi}\left(\frac{\sin\alpha}{1} + \frac{\sin 2\alpha}{2} + \frac{\sin 3\alpha}{3} + \cdots\right).$$

At points $\alpha = 0, 2\pi, 4\pi \ldots$ the sum of the series is zero, i.e. the arithmetic mean of the left and right extreme values of the function.

EXAMPLE 2. It is required to expand the square wave shown in Fig. 7 into a Fourier series. This wave satisfies the conditions

$$f(\alpha) = \begin{cases} 1 & \text{when } 0 < \alpha < \pi, \\ -1 & \text{when } \pi < \alpha < 2\pi. \end{cases}$$

The given function is even and, in addition, possesses half-wave symmetry. There are, therefore, no constant component, cosine curves or even sine curves in the expansion, i.e.

$$a_n = 0, \quad b_n = \frac{2}{\pi} \int_0^{\pi} \sin n\alpha \, d\alpha = \frac{4}{\pi n},$$

where $n$ is odd. From this

$$f(\alpha) = \frac{4}{\pi} \left( \frac{\sin \alpha}{1} + \frac{\sin 3\alpha}{3} + \frac{\sin 5\alpha}{5} + \cdots \right).$$

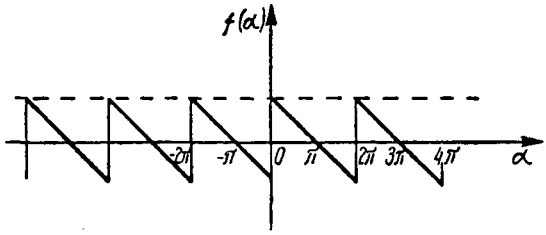



FIG. 6. Example 1.

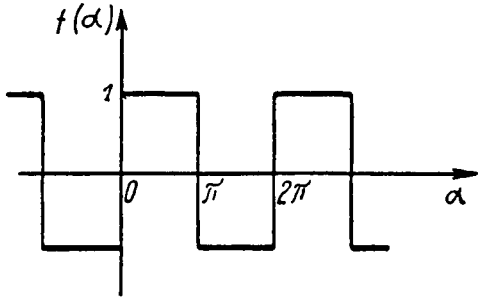



FIG. 7. Example 2.

## 3. Shift of Origin

Let us suppose that coefficients $a_n$ and $b_n$ are given for a certain non-sinusoidal periodic function $f(\alpha)$, i.e. the expansion of the given function into a Fourier series is known

$$f(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\alpha + b_n \sin n\alpha].$$

If we shift the zero reference by a certain amount $\beta$ to the right or left with respect to the initial origin, the expansion into a Fourier series with respect to the new set of coordinates can be obtained by substituting $\alpha_1 + \beta$ for $\alpha$, where $\alpha_1$ is the abscissa in the new set of coordinates, and a positive value $\beta$ corresponds to shifting the origin to the right and a negative to the left. Thus,

$$f(\alpha_1 + \beta) = f_1(\alpha_1) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n(a_1 + \beta) + b_n \sin n(\alpha_1 + \beta)]. \tag{10.17}$$

Formula (17) makes it possible to find the expansion of a function into a Fourier series quickly, if we know the coefficients of the Fourier series for another function, by shifting the first function along the abscissa axis

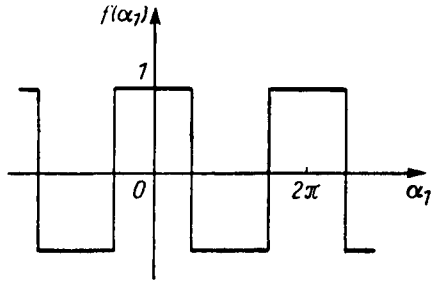



FIG. 8. Example 3.

by a constant amount. If a given nonsinusoidal periodic function is differentiated, integrated or subjected to some other process, the coefficients of the Fourier series change.

Calculation formulae for cases when periodic waves are subjected to different operations are given in the work of G. Ye. Pukhov (Ref. 48).

EXAMPLE 3. It is required to find the expansion into a Fourier series for the square wave shown in Fig. 8, using the results of Example 2.

The function in Fig. 8 is obtained by shifting the origin of Fig. 7 (point 0) to the right a distance $\beta = \pi/2$. Consequently, in accordance with (17),

$$f_1(\alpha_1) = \frac{4}{\pi}\left[\frac{\sin\left(\alpha_1 + \frac{\pi}{2}\right)}{1} + \frac{\sin 3\left(\alpha_1 + \frac{\pi}{2}\right)}{3} + \frac{\sin 5\left(\alpha_1 + \frac{2}{5}\right)}{5} + \cdots\right]$$

$$= \frac{4}{\pi}\left[\frac{\cos \alpha_1}{1} - \frac{\cos 3\alpha_1}{3} + \frac{\cos 5\alpha_1}{5} - \cdots\right].$$

## 4. Exponential Fourier Series

A trigonometric Fourier series can be converted into an exponential series thus. Starting with the fact that

$$\cos n\alpha = \frac{e^{jn\alpha} + e^{-jn\alpha}}{2} \quad \text{and} \quad \sin n\alpha = \frac{e^{jn\alpha} - e^{-jn\alpha}}{2j},$$

the expression contained in brackets in (5) can be reduced to the form

$$a_n \cos n\alpha + b_n \sin n\alpha = \frac{a_n - jb_n}{2} e^{jn\alpha} + \frac{a_n + jb_n}{2} e^{-jn\alpha},$$

and series (5) can be written thus:

$$f(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{a_n - jb_n}{2} e^{jn\alpha} + \frac{a_n + jb_n}{2} e^{-jn\alpha} \right].$$

On the basis of (6) and (7), $a_n$ is an even and $b_n$ an odd function with respect to $n$, i.e. $a_n$ retains its sign for negative values of $n$, whereas $b_n$ changes its signs:

$$a_n = a_{-n}, \quad b_n = -b_{-n}.$$

Therefore

$$f(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\alpha} + \sum_{n=-1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\alpha} = \sum_{-\infty}^{\infty} \frac{a_n - jb_n}{2} e^{jn\alpha}.$$

An exponential Fourier series thus has the form:

$$f(\alpha) = \tfrac{1}{2} \sum_{-\infty}^{+\infty} \dot{F}_n e^{jn\alpha}, \tag{10.18}$$

where the complex coefficient (complex amplitude)

$$F_n = a_n - jb_n = \frac{1}{\pi} \int_0^{2\pi} f(\alpha)\, e^{-jn\alpha}\, d\alpha. \tag{10.19}$$

As was shown above $\alpha = \omega_1 t$; consequently, series (18) contains two infinite series of vectors which are conjugate with respect to the real axis and rotate in opposite directions with an angular velocity of $n\omega_1$. The geometrical sum of each pair of conjugate vectors contains a real term only (Fig. 9). As a result of adding the two infinite series of conjugate vectors, a real function $f(\omega_1 t)$ is obtained.

In Fig. 9 the horizontal axis is the real axis. The geometrical sum of the conjugate vectors $\frac{1}{2}\dot{F}_n e^{jn\omega_1 t}$ and $\frac{1}{2}\dot{F}_{-n} e^{-jn\omega_1 t}$ coinciding with this axis at each given instant is equal to the value of the $n$-th harmonic of the Fourier series shown in Fig. 9 on the time axis $t$.

In other words, the $n$-th harmonic contains two identical components equal to plotting the conjugate vectors, rotating at angular velocities of $n\omega_1$ and $-n\omega_1$, on the real axis. On the frequency scale these components

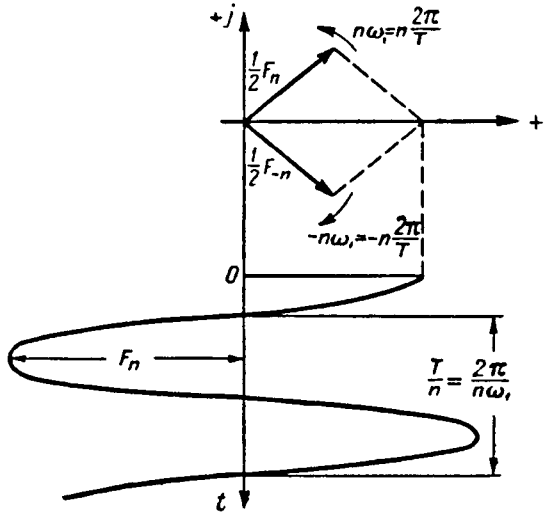



FIG. 9. Representation of the function of a real variable in the form of the sum of two functions of a complex variable.

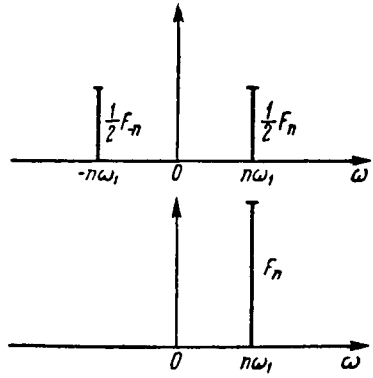



FIG. 10. Coefficients of an exponential Fourier series.

are located symmetrically with respect to the zero point (Fig. 10). Their amplitude values are equal. Obviously the $n$-th harmonic on the frequency scale can also be represented by one component $F_n$ with a frequency $n\omega_1$. We can thus eliminate negative frequencies.

The amplitude and phase of the $n$-th harmonic are expressed by $a_n$ and $b_n$ in accordance with (8a).

## 5. Spectra of Periodic Functions

Frequencies $n\omega_1$ and $-n\omega_1$ (when $n = 0, 1, 2, \ldots$) form a series of equidistant points on the frequency axis. The amplitudes of the corresponding harmonics ($\frac{1}{2}F_n = \frac{1}{2}F_{-n}$) can be plotted from each of these points, perpendicular to the frequency axis. The aggregate of the ordinates constructed in this way, i.e. the moduli of the exponential Fourier coefficients, plotted against the corresponding frequencies, is called the *amplitude spectrum* of the given periodic wave (Fig. 11). In view of the equality $\frac{1}{2}F_n = \frac{1}{2}F_{-n}$,

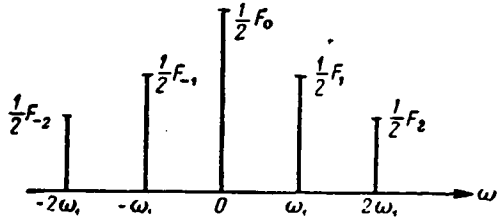



FIG. 11. Amplitude spectrum of a periodic wave.

the amplitude spectrum is symmetrical with respect to the line $\omega = 0$, i.e. the curve enveloping the points $n\omega_1$, $\frac{1}{2}F_n$ is an even function of $\omega$.

The aggregate of the ordinates equal to the arguments $\psi_n$ of the complex number $\dot{F}_n$, plotted against the corresponding frequencies, is called the *phase spectrum* of the given periodic function. It follows from (8) that the curve enveloping the points $n\omega_1$, $\psi_n$ is an odd function of $\omega$.

## 6. Parseval's Relation

*The average value for a period of the product of two periodic functions is equal to the sum of the series of the products of the exponential Fourier factors of one of these functions by the corresponding conjugate factors of the other*

$$\frac{1}{T}\int_0^T f(t)\, g(t)\, \mathrm{d}t = \frac{1}{4}\sum_{-\infty}^{+\infty} \dot{F}_n \overset{*}{G}_n. \tag{10.20}$$

This equality is called *Parseval's relation.* For proof let us examine two infinite Fourier series

$$f(t) = \tfrac{1}{2}\sum_{-\infty}^{+\infty} \dot{F}_n e^{jn\omega_1 t} \quad \text{and} \quad g(t) = \tfrac{1}{2}\sum_{-\infty}^{+\infty} \dot{G}_k e^{jk\omega_1 t}.$$

We shall form the average value of the product of these two series, for one period,

$$\frac{1}{T}\int_0^T f(t)\,g(t)\,\mathrm{d}t = \frac{1}{4T}\int_0^T \sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} \dot{F}_n\dot{G}_k e^{j(n+k)\omega_1 t}\,\mathrm{d}t$$

$$= \sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} \dot{F}_n\dot{G}_k \frac{1}{4T}\int_0^T e^{j(n+k)\omega_1 t}\,\mathrm{d}t.$$

But

$$\frac{1}{T}\int_0^T e^{j(n+k)\omega_1 t}\,\mathrm{d}t = \begin{cases} 0 \text{ when } n+k \neq 0, \\ 1 \text{ when } n+k = 0. \end{cases}$$

Therefore

$$\frac{1}{T}\int_0^T f(t)\,g(t)\,\mathrm{d}t = \frac{1}{4}\sum_{-\infty}^{+\infty} \dot{F}_n\dot{G}_{-n} = \frac{1}{4}\sum_{-\infty}^{+\infty} \dot{F}_n\overset{*}{\dot{G}}_n,$$

since, when $n$ is fixed, only the term of the second, inner, sum is other than zero, for which $k = -n$.

In the special case when $f(t) = g(t)$ and, correspondingly, $\dot{F}_n = \dot{G}_n$, the average value of the square of the periodic function for a period equals the sum of the series of the squares of the moduli of its Fourier coefficients:

$$\frac{1}{T}\int_0^T [f(t)]^2\,\mathrm{d}t = \sum_{-\infty}^{+\infty}\left(\frac{F_n}{2}\right)^2. \tag{10.21}$$

## 7. Effective Value and Average Value of a Nonsinusoidal Periodic Wave

The *effective* value (Chapter I, Section 9) of a nonsinusoidal periodic function $f(t)$ is determined with the aid of (32) in Chapter I and (21) above, from the formula

$$\sqrt{\left(\frac{1}{T}\int_0^T [f(t)]^2\,\mathrm{d}t\right)} = \sqrt{\left(\frac{1}{2\pi}\int_0^{2\pi} [f(t)]^2\,\mathrm{d}\alpha\right)}$$

$$= \sqrt{\left[\left(\frac{a_0}{2}\right)^2 + \sum_0^\infty \frac{F_n^2}{2}\right]}. \tag{10.22}$$

Formula (22) follows from (21), taking into account that

$$F_n = F_{-n}.$$

Since $F_n$ is the amplitude of the $n$-th harmonic, $F_n/\sqrt{2}$ is the effective value of the harmonic. Thus the resulting expression shows that *the effective value of a nonsinusoidal periodic wave equals the square root of the sum of*

*the squares of the effective values of the harmonics and the square of the constant component.*

It follows from this that the effective values of a wave which represents the sum of the harmonics of different frequencies does not depend on the initial phases of these harmonics but is determined wholly by their effective values.

The effective value of a nonsinusoidal periodic wave can be measured just as in sinusoidal conditions, by an electromagnetic, electrodynamic or thermal electrical measuring device.

EXAMPLE 4. To determine the effective value of a nonsinusoidal periodic current

$$i(t) = 3 + 4 \sin\left(\omega t + \frac{\pi}{4}\right) + 6 \sin\left(3\omega t - \frac{\pi}{6}\right).$$

On the basis of (22), the unknown effective current value is

$$I = \sqrt{\left(3^2 + \frac{4^2}{2} + \frac{6^2}{2}\right)} = \sqrt{35} \cong 5{\cdot}92a.$$

By the *average* value of a nonsinusoidal periodic function $f(t)$, considered absolutely, we understand the average value during a periodic of the function with respect to its absolute magnitude, i.e. an integral in the form

$$\frac{1}{T}\int_0^T |f(t)|\, \mathrm{d}t.$$

If function $f(t)$ possesses half-wave symmetry and is positive for half a period, then the above-mentioned value is equal to the average value of the function for a half-period (Chapter I, Section 9).

$$\frac{1}{T}\int_0^T |f(t)|\, \mathrm{d}t = \frac{2}{T}\int_0^{T/2} f(t)\, \mathrm{d}t. \tag{10.23}$$

The average value of the wave, considered absolutely, is measured by a magnetoelectrical instrument with a rectifier (a magnetoelectrical instrument without a rectifier measures the constant component).

## 8. Active (Average) Power with Nonsinusoidal Current and Voltage

The instantaneous power in a circuit with nonsinusoidal periodic voltages and currents is determined by the product of the instantaneous values of the voltages and currents

$$p(t) = u(t)\, i(t).$$

The active power is equal to the average value of the power for a period (Chapter II, Section 7)

$$P = \frac{1}{T}\int_0^T ui\,dt.$$

The given integral can be regarded as the left-hand side of Parseval's relation, supposing that $u = f(t)$ and $i = g(t)$. In accordance with (20)

$$P = \frac{1}{4}\sum_{-\infty}^{+\infty} \dot{F}_n \overset{*}{G}_n = \frac{F_0}{2}\frac{G_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\dot{F}_n}{\sqrt{2}}\frac{\overset{*}{G}_n}{\sqrt{2}} + \frac{1}{2}\sum_{n=-1}^{-\infty} \frac{\dot{F}_n}{\sqrt{2}}\frac{\overset{*}{G}_n}{\sqrt{2}}.$$

In the present case $F_0/2 = U_0$ and $G_0/2 = I_0$ are the constant components of voltage and current. When $n > 0$, $F_n/\sqrt{2} = U_n$ and $G_n/\sqrt{2} = I_n$ are the effective values of the $n$-th harmonics of voltage and current.

Consequently,

$$P = U_0 I_0 + \tfrac{1}{2}\sum_{n=1}^{\infty} \dot{U}_n \overset{*}{I}_n + \tfrac{1}{2}\sum_{n=-1}^{-\infty} \dot{U}_n \overset{*}{I}_n.$$

In view of the fact that $\dot{U}_{-n} = \overset{*}{U}_n$ and $\overset{*}{I}_{-n} = \dot{I}_n$,

$$P = U_0 I_0 + \tfrac{1}{2}\sum_{n=1}^{\infty} (\dot{U}_n \overset{*}{I}_n + \overset{*}{U}_n \dot{I}_n),$$

or, the same thing,

$$P = U_0 I_0 + \sum_{n=1}^{\infty} U_n I_n \cos \varphi_n. \tag{10.24}$$

Thus, *the active power of a nonsinusoidal periodic current is equal to the sum of the active powers of like harmonics plus the power of the constant components.*

In other words, the active power of the interaction of the like harmonics of voltages and currents, or of harmonics with constant components, is zero.

By analogy with the concept of reactive power for sinusoidal functions we can introduce the concept of reactive power in a circuit with nonsinusoidal periodic values. The latter is determined by the sum of the reactive powers of the separate harmonics

$$Q = \sum_{n=1}^{\infty} U_n I_n \sin \varphi_n.$$

If one of the curves (voltage or current) contains harmonics which are not present in the other, this does not affect the value of the active and reactive power, but it raises the effective value of the first function. Therefore, if the total power in this circuit is defined as the product of the effective values of voltage and current $S = UI$, then it can be concluded from the

above that, as opposed to sinusoidal conditions, the sum of the squares of the active and reactive power in a circuit with nonsinusoidal periodic values does not equal the square of the apparent power

$$P^2 + Q^2 = S^2 - T^2.$$

The quantity $T$ is called the *distortion power.* It characterizes the degree of difference between the shapes of the voltage curve $u$ and current curve $i$.

EXAMPLE 5. To determine the active power if

$$u = \sqrt{2} \cdot 10 \sin\left(\omega t - \frac{\pi}{4}\right) + \sqrt{2} \cdot 6 \sin 2\omega t + \sqrt{2} \cdot 4 \sin\left(3\omega t + \frac{\pi}{4}\right),$$

$$i = 10 + \sqrt{2} \cdot 4 \sin\left(\omega t + \frac{\pi}{4}\right) + \sqrt{2} \cdot 5 \sin\left(3\omega t + \frac{\pi}{4}\right).$$

According to (24)

$$P = 10 \cdot 4 \cos\frac{\pi}{2} + 4 \cdot 5 \cos 0 = 20 \text{ cm}.$$

## 9. Factors Characterizing Nonsinusoidal Periodic Waves

By analogy with sinusoidal functions the ratio of the active power of nonsinusoidal currents to the total (apparent) power is called the *power factor* and is denoted by $\cos\theta$

$$\cos\theta = \frac{P}{UI} = \frac{\int_0^T ui\,dt}{\sqrt{\left(\int_0^T u^2\,dt \int_0^T i^2\,dt\right)}}. \tag{10.25}$$

Here $\theta$ is a certain conventional angle which describes the nonsinusoidal functions.

The relationship in the right-hand side of (25) only becomes equal to unity when $u$ and $i$ are directly proportional.

Let us suppose that the voltage is sinusoidal and the current nonsinusoidal. In this case the active power is defined, in accordance with (24), as the power of the first harmonic

$$P = U_1 I_1 \cos\varphi_1 = U I_1 \cos\varphi_1.$$

Then, the effective value of the current is

$$I = \sqrt{(I_0^2 + I_1^2 + I_2^2 + \cdots)} > I_1.$$

Consequently, the power factor is

$$\cos\theta = \frac{UI_1\cos\varphi_1}{UI} = \frac{I_1}{I}\cos\varphi_1 = k_d\cos\varphi_1. \tag{10.26}$$

The factor $k_d = I_1/I < 1$ is known as the *distortion factor.*

In electrical and radio engineering *form factors* ($k_f$) and *amplitude factors* ($k_a$) are also used.

The *form factor* is defined as the ratio of the effective value of the wave to the average value,

$$k_f = \frac{\sqrt{\left(\frac{1}{T}\int_0^T [f(t)]^2\,dt\right)}}{\frac{1}{T}\int_0^T |f(t)|\,dt}. \tag{10.27}$$

For a sine curve

$$k_f = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11.$$

The *amplitude factor* is defined as the ratio of the maximum value of the wave to the effective value

$$k_a = \frac{f_{max}}{\sqrt{\left(\frac{1}{T}\int_0^T [f(t)]^2\,dt\right)}}. \tag{10.28}$$

For a sine curve

$$k_a = \sqrt{2} = 1{\cdot}41.$$

Nonsinusoidal voltage and current curves can be replaced by equivalent sine curves in many cases. The substitution is carried out in such a way that the effective values of the sinusoidal functions are taken as being equal to the effective values of the nonsinusoidal functions, and the angle of the phase difference between the equivalent sine curves is taken as being equal to the angle $\theta$, determined from (25).

## 10. Use of Fourier Series in Circuit Analysis

Let us suppose that a nonsinusoidal periodic voltage in the form (8*b*) is impressed across the terminals of a circuit

$$u = U_0 + \sum_{n=1}^{\infty} U_{nm}\cos(n\omega_1 t - \psi_n). \tag{10.29}$$

If the circuit is linear, i.e. parameters $r$, $L$ and $C$ are invariable, then the current in the circuit can be obtained in steady-state conditions by the

superposition method, by adding the currents produced by each voltage component individually

$$i = I_0 + \sum_{n=1}^{\infty} I_{nm} \cos(n\omega_1 t - \psi_n - \varphi_n), \tag{10.30}$$

where

$$I_0 = \frac{U_0}{z(0)}, \quad I_{nm} = \frac{U_{nm}}{z(n\omega_1)}.$$

By $z(0)$ we understand the impedance of the circuit at a frequency equal to zero, i.e. d.c. resistance; $z(n\omega_1)$ is the scalar impedance at the frequency $n\omega_1$.

The angle $\varphi_n$ is defined as the arc tangent of the ratio of the reactance of the circuit to its resistance at the frequency $n\omega_1$.

When the circuit consists of $r$, $L$ and $C$ elements connected in series,

$$z(n\omega_1) = \sqrt{\left[r^2 + \left(n\omega_1 L - \frac{1}{n\omega_1 C}\right)^2\right]}$$

and

$$\varphi_n = \arctan \frac{n\omega_1 L - \dfrac{1}{n\omega_1 C}}{r}.$$

Then $z(0) = \infty$, since the circuit is open for direct current.

As the number of the harmonic is raised, i.e. as frequency $n\omega_1$ increases, the inductive reactance $n\omega_1 L$ grows according to the law of straight lines and the capacitive reactance $1/(n\omega_1 C)$ decreases according to the law of hyperbolae. In a circuit consisting of $r$ and $L$ connected in series, therefore, the current harmonics are more weakly expressed than those in the voltage applied to the circuit.

On the other hand, in capacitive circuit the higher current harmonics may attain large values.

In a mixed inductive-capacitive circuit, when $n\omega_1 L - 1/(n\omega_1 C) = 0$ resonance occurs for the corresponding harmonic.

In general, both the reactance and resistance of a circuit consisting of $r$, $L$ and $C$ elements are functions of frequency.

The periods of the different harmonics are inversely proportional to the number of the harmonic. Therefore, if we plot not time $t$, but angle $\alpha_n = n\omega_1 t$ on the abscissa axis, then, when $\alpha_1 = \alpha_n$, a section $n$ times smaller than the section corresponding to angle $\alpha_1$ of the fundamental harmonic corresponds to angle $\alpha_n$ of the $n$-th harmonic (Fig. 12).

The calculation of nonsinusoidal currents in complicated circuits is carried out, as a rule, by the exponential method.

The trigonometric expansion into a Fourier series (8), like the exponential form (18), makes it possible to represent a nonsinusoidal periodic wave in the form of the real part of the series

$$f(t) = \mathrm{Re}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \dot{F}_n e^{jn\omega_1 t}\right],$$

where $F_n = F_n e^{-j\psi_n}$ is the complex amplitude of the $n$-th component.

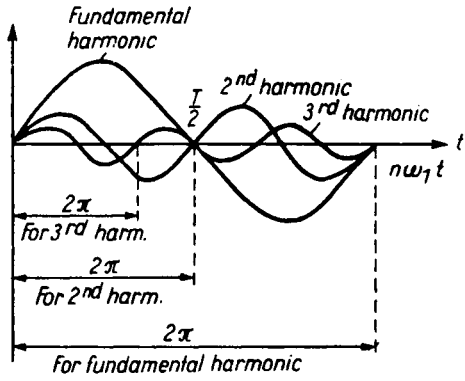



FIG. 12. Relation of harmonic scales, in terms of sine functions.

In connexion with this, equations (29) and (30) can be rewritten in the form

$$u(t) = U_0 + \mathrm{Re} \sum_{n=1}^{\infty} \dot{U}_{nm} e^{jn\omega_1 t}, \tag{10.31}$$

$$i(t) = I_0 + \mathrm{Re} \sum_{n=1}^{\infty} \dot{I}_{nm} e^{jn\omega_1 t}, \tag{10.32}$$

where

$$\dot{U}_{nm} = U_{nm} e^{-j\psi_n}, \quad \dot{I}_{nm} = I_{nm} e^{-j(\psi_n + \varphi_n)}.$$

Consequently, the calculation of nonsinusoidal currents in exponential form can be reduced to finding the complex amplitudes of the currents $\dot{I}_{nm}$ which correspond to the given complex amplitudes of the voltages $\dot{U}_{nm}$ for different values of $n$. Then, in view of the fact that the separate components of the currents have different frequencies, the complex amplitudes of the currents are not added directly but, in accordance with (32), vectors rotating with different angular velocities ($n\omega_1$) are added, i.e. the instantaneous values of the separate current harmonics are added.

EXAMPLE 6 (Ref. 90). In the Fig. 13*a* circuit consisting of a resistance $r = 125\,\Omega$, an inductance $L = 0{\cdot}125$ H and a capacitance $C = 2 \times 10^{-6}$ F connected in series, a nonsinusoidal periodic e.m.f. $u(t)$ operates (Fig. 13*b*). To determine the current in the circuit

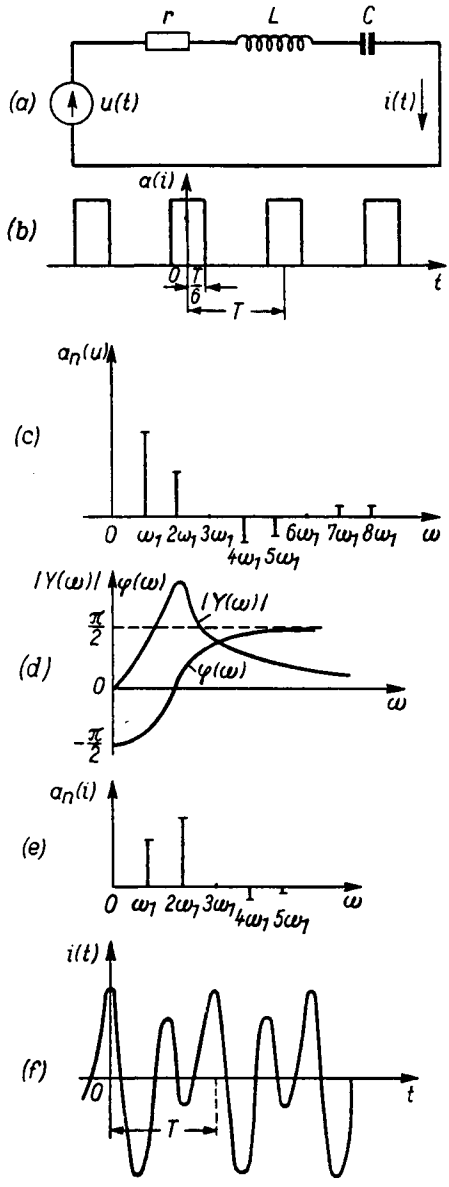



FIG. 13. Example 6.

on the condition that $u(0) = 100$ V;

$$T = 0{\cdot}00628 \text{ sec}; \quad \omega_1 = \frac{2\pi}{T} = \frac{6{\cdot}28}{0{\cdot}00628} = 10^3 \frac{1}{\text{sec}}.$$

The given periodic e.m.f. is symmetrical with respect to the ordinates axis, and consequently there are no sine curves in the expansion into a Fourier series ($b_n = 0$). On the basis of (11)

$$a_n(u) = \frac{4}{T}\int_0^{T/2} u(t)\cos n\omega_1 t\, dt = \frac{400}{T}\int_0^{T/6} \cos n\omega_1 t\, dt$$

$$= \frac{200}{\pi n}\sin\frac{\pi n}{3} \text{ V}, \quad a_0 = \frac{200}{3} \text{ V};$$

$$u(t) = 200\left[\frac{1}{6} + \sum_{n=1}^{\infty} \frac{\sin\frac{\pi n}{3}}{n\pi}\cos n\omega_1 t\right].$$

For all values of $n$ which are multiples of three, $\sin \pi n/3 = 0$. Figure 13*c* shows the factors $a_n(u)$ of the e.m.f. $u(t)$ applied to the circuit.

The relationship between the frequency and the modulus of the circuit's admittance

$$|Y(\omega)| = \frac{1}{\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}$$

and the angle

$$\varphi(\omega) = \text{arc tan}\frac{\omega L - \frac{1}{\omega C}}{r}$$

are given in Fig. 13*d*.

As a result of multiplying $a_n(u)$ by the admittance $Y(\omega)$, the factors $a_n(i)$ of current $i(t)$ are obtained. Figure 13*e* and *f* shows $a_n(i)$ and $a_n(t)$.

The required current

$$i(t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{|Y(n\omega_1)|\sin\frac{\pi n}{3}}{n}\cos[n\omega_1 t - \varphi(n\omega_1)],$$

or, substituting the values of $\omega_1$, $L$, $C$ and $r$:

$$i(t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{\pi n}{3}\cos\left(n10^3 t - \text{arc tan}\frac{n^2-4}{n}\right)}{\sqrt{[125^2 n^2 + (125n^2 - 5 \cdot 10^2)^2]}}.$$

It is to be noted that in simple cases (simple wave shapes applied to simple circuits) steady-state response to a periodic nonsinusoidal excitation function might be obtained not only in the form of an infinite series of harmonics, but in the more convenient form of a mathematical expression — in closed form, as the sum of just a few terms, valid within an appropriate interval of time, say within a half-cycle. The solution is based on the concept that, if a circuit is in the steady state, the response in one cycle has the

same form as the response in the next. For the interval in question we must write a circuit equation which is to be solved by the so-called classic method (Chapter XI) or the Laplace transformation (Chapter XIII).

This can be illustrated by the following example. Let it be supposed that the square wave of the e.m.f. of Fig.7 is acting in an $rL$ circuit. The following equation is written for time within the first half-cycle, that is, for $0 < t < T/2$

$$ri + L\frac{\mathrm{d}i}{\mathrm{d}t} = E.$$

The solution of this equation is

$$i = \frac{E}{r} + Ae^{-\frac{r}{L}t}.$$

The current at $t = 0$ starts at some $i(0)$ value.
Hence

$$A = i(0) - \frac{E}{r}.$$

It is clear that the current response must have a half-cycle symmetry like the given e.m.f. Since the inductance prevents the current from jumping (Chapter XI), the current at $t = T/2$ is of the value $-i(0)$. This gives

$$i\left(\frac{T}{2}\right) = \frac{E}{r} + \left[i(0) - \frac{E}{r}\right]e^{-\frac{r}{L}\frac{T}{2}} = i(0),$$

from which $i(0)$ is

$$i(0) = \frac{E(e^{-\alpha} - 1)}{r(e^{-\alpha} + 1)},$$

where

$$\alpha = \frac{r}{L} \cdot \frac{T}{2}.$$

Upon substitution, the equation for $i$ inside the interval from $0 < t < T/2$ is

$$i = \frac{E}{r}\left[1 - \frac{2}{1 + e^{-\alpha}}e^{-\frac{r}{L}T}\right].$$

In the next interval from $T/2 < t < T$ the current is the negative of this and thus can be written

$$i = -\frac{E}{r}\left[1 - \frac{2}{1 + e^{-\alpha}}e^{-\frac{r}{L}\left(t - \frac{T}{2}\right)}\right].$$

CHAPTER XI

# TRANSIENT ANALYSIS (CLASSIC METHOD)

## 1. Transients in Linear Systems

Up to now we have been considering the steady state in linear networks, i.e. conditions in which voltages and currents are either invariable in time (d.c. circuits) or are periodic time functions (a.c. circuits).

As a rule, the steady state, which is different from the circuit's initial operating conditions, is preceded by a *transient* in which voltages and currents change nonperiodically.

The transition from one set of operating conditions to another can be effected by switching circuit elements in or out, changing the circuit's parameters, or changing the circuit itself.

In theory we can say that a circuit is commutated instantaneously, i.e. no time is wasted in closing or opening a switch. Nevertheless the transition from the circuit's initial operating conditions to the subsequent steady state does not take place instantaneously, but after a certain length of time.

This can be explained by the fact that a certain reserve of energy in the electrical and magnetic fields corresponds to each state of the circuit. A change to new conditions entails an increase or decrease in the energy of these fields. The energy $W_L = Li^2/2$, stored in the magnetic field of the inductance $L$ and the energy $W_C = Cu^2/2$ stored in the electrical field of the capacitance $C$ cannot alter instantaneously. The energy can only change gradually, without jumping, since the power equal to the time derivative of energy would otherwise attain infinite values, which is physically impossible. If an arm with an inductance is disconnected, therefore, it is impossible to interrupt the current in the inductance instantaneously. An electrical arc through air is inevitably produced between the switch points as they separate. This arc lasts a very short time, but it allows a finite interval in which the current can fall without violating the laws of electricity. Analogously, if the terminals of a capacitor which has been previously charged are short-circuited, an electrical arc, in which the energy stored in the capacitance is dissipated, appears at the point where the circuit is shorted.

If we ignore the cases where the inductance is disconnected and the capacitance short-circuited and consider circuits in which energy stored in the magnetic or electrical field can be dispersed in the form of heat in the resistances, then, assuming that the commutation occurs instantaneously, the electrical arc can be disregarded.

Theoretically it requires an infinitely long time for transients to be completed and the steady state to commence. In practice, however, the time taken for transients is measured in small intervals in the order of several milliseconds or microseconds during which time the currents and voltages approximate so closely to the steady-state values that the difference is practically imperceptible. The more intensively the energy is dispersed in the resistances, the sooner the transients finish.

If an electrical circuit contained only resistances, with no inductances or capacitances, the change-over from one set of conditions to another would be instantaneous, and no time would be wasted. Such a circuit, however, is physically impracticable. By using special circuits and selecting the appropriate parameters, transients can be accelerated or delayed. It should then be noted that in circuits with ideal lumped parameters, the following can vary intermittently: (1) the currents in the resistances and capacitances; and (2) the voltages in the resistances and inductances.

In certain cases transients in electrical circuits are both undesirable and dangerous (e.g. in short circuits). In other cases transients are the natural and normal operating conditions of the circuit (e.g. in radio circuits and automation systems).

There are different methods of analysing transients in linear electrical circuits. This chapter is devoted to the so-called *classic method*, i.e. solving differential equations for the transients under examination.

It should be noted that for examining transients in linear electrical circuits the *Fourier integral method* (Chapter XII) and, particularly, the *Laplace transformation method* (Chapter XIII) are widely used in practice.

Other methods of analysing transients are also used. Thus, in order to find the envelope of high-frequency oscillations occurring in resonance circuits when the carrier wave of the source is close to the natural frequency of the system, the *slowly changing amplitude method* (Refs. 21 and 25) is used. When the voltage applied to an electrical circuit is given graphically, e.g. in the form of a curve from an oscillograph, and the transient response in the circuit under examination is described by a linear differential equation of the first order, the so-called *constant subtangent method* may be used (Ref. 25).

The so-called *method of reducing to steady-state conditions* in which the transients are observed for a given time (Refs. 1, 23 and 48) is based on replacing an aperiodic function by a nonsinusoidal function and an expansion into a Fourier series.

There are also some approximate numerical methods of calculating

transients, which are primarily intended for non-linear circuits (Refs. 57 *et al.*). Although these methods are also applicable to linear circuits in principle, in practice transients in linear circuits are mainly calculated by the analytic method or by computers.

## 2. Commutation Laws and Initial Conditions

The above-mentioned statement that the energy of a magnetic or electrical field cannot change abruptly, but only gradually, expresses the law of the *conservation* of the flux linkage of the inductance and the electrical charge of the capacitance.

The impossibility of a sudden change in the flux linkage is a result of the fact that an infinitely high voltage $u = \mathrm{d}\Psi/\mathrm{d}t = \infty$ would otherwise appear on the inductance, which is nonsensical from the physical point of view.

In view of the equality $\Psi = Li$ *the law of conservation of flux linkage means that with L constant current i cannot change instantaneously.* At the initial instant after commutation the current in the inductance remains the same as it was immediately before commutation, and then changes smoothly.

Analogously, the impossibility of a sudden change in the electrical charge $q$ is a result of the fact that an infinitely large current $i = \mathrm{d}q/\mathrm{d}t$ would otherwise pass through the capacitance, which is similarly nonsensical. In view of the equality $q = Cu$ *the law of conservation of electric charge means that with C constant the voltage across the capacitance cannot change instantaneously.* At the initial instant after commutation the voltage across the capacitance remains the same as it was immediately before commutation, and then changes smoothly.

The essence of the so-called *commutation laws* is that the current in the inductance and the voltage across the capacitance remain constant at the instant of commutation.

The values of the current in the inductance and the voltage across the capacitance at the instant of commutation are the *initial conditions.*

If the commutation takes place at an instant $t = 0$ the current in the inductance and the voltage across the capacitance at the instant immediately before commutation are conventionally denoted by $i_L(0-)$ and $u_C(0-)$ and at the initial instant of the transient after commutation by $i_L(0+)$ and $u_C(0+)$.

On the basis of the commutation laws

$$i_L(0-) = i_L(0+), \quad u_C(0-) = u_C(0+). \tag{11.1}$$

These equations express the initial conditions in which commutation occurs in a circuit.

At zero initial conditions, i.e. when $i_L(0-) = 0$ and $u_C(0-) = 0$ (the circuit is initially at rest), the inductance at the initial instant after commutation is equivalent to an interrupted circuit and the capacitance is equivalent to a short circuit (Fig. 1 *a*). In other words if, at an instant $t = 0$, there is no current in the inductance (i.e. the flux linkage is zero), connecting this inductance to an electrical energy source does not instantaneously produce a current in it when $t = 0+$.

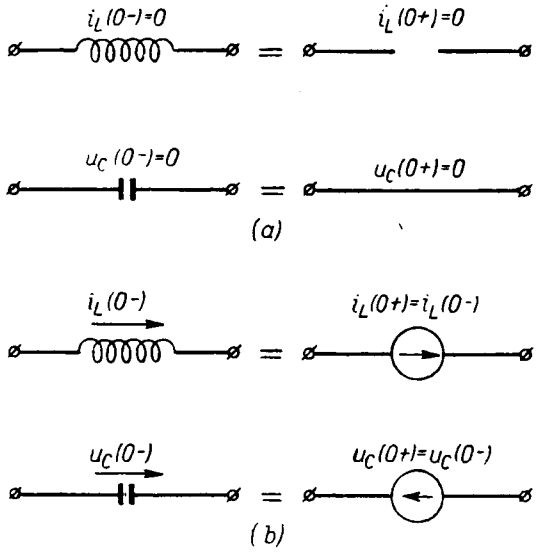



FIG. 1. Equivalent circuit of inductance and capacitance for an instant $t = 0$: (*a*) zero initial conditions; (*b*) non-zero initial conditions.

In exactly the same way if at an instant $t = 0-$ there is no voltage across the capacitance (i.e. the charge is zero), connecting this capacitance to an electrical energy source of finite power does not instantaneously produce a voltage on it when $t = 0+$. Such circuits are said to be initially relaxed, or quiescent.

In the case of initial conditions other than zero, i.e. when $i_L(0-) \neq 0$ and $u_C(0-) \neq 0$, when $t = 0$ the inductance is equivalent to the current source $i_L(0-)$ and the capacitance to the e.m.f. source $u_C(0-) = q_c(0-)/C$ (Fig. 1 *b*).

The initial conditions characterize the energy of the magnetic and electrical fields stored up to the instant of commutation. In order to analyse transients, therefore, it is essential to know the initial conditions, no matter how these conditions were produced in the circuit*.

* Initial conditions just after switching an inductor into an inductance branch passing current, or just after connecting charged capacitors in parallel, obey the laws of conserva-

In the above account we have ignored the arcing and the consequent dissipation of energy at the instant of commutation. Incidentally, if an inductance $L_2$, which is short-circuited and de-energized up to the instant of commutation, is connected in series to an inductance $L_1$, supplied from a voltage source, then at the instant when $L_2$ is switched in arcing occurs.

If the arcing ceases rapidly, the commutation process can be considered instantaneous for practical purposes, although energy is dissipated in heat and radiation. In this case the initial value of the current $i_2(0+)$ can be determined after commutation from the constant condition of the total flux linkage of the inductances:

$$L_1 i_1(0-) = (L_1 + L_2)\, i_2(0+).$$

Similarly, if capacitors $C_1$ and $C_2$ with initial charges $q_1(0-)$ and $q_2(0-)$ are connected in parallel, the initial value of the voltage across the capacitors after commutation will be determined after commutation from the constant condition of the total charge:

$$(c_1 + c_2)\, u(0+) = q_1(0-) + q_2(0-).$$

By using Kirchhoff's equation and initial conditions it is possible to calculate the currents and voltages in all sections of the circuit for an instant $t = 0+$ and find the values of the derivatives of the currents and voltages corresponding to the instant $t = 0+$. These quantities are required subsequently for finding the integration constants.

EXAMPLE 1. A direct current was flowing in the Fig. 2*a* circuit until the instant $t = 0$. When $t = 0$, contact $K$ closed. It is required to find the value $i_1(0+)$, $i_2(0+)$, $i_3(0+)$,

$$\frac{\mathrm{d}i_1}{\mathrm{d}t}(0+), \quad \frac{\mathrm{d}i_2}{\mathrm{d}t}(0+), \quad \frac{\mathrm{d}i_3}{\mathrm{d}t}(0+).$$

Before commutation the current in resistances $r_1$ and $r_2$ and in the inductance $L$ is

$$i_{r_1}(0-) = i_{r_2}(0-) = i_L(0-) = \frac{U}{r_1 + r_2}.$$

Before commutation the voltage on capacitance $C$ equals the voltage on resistance $r_2$:

$$u_C(0-) = \frac{r_2}{r_1 + r_2} U.$$

The equivalent circuit for the instant $t = 0+$ is shown in Fig. 2*b*.

On the basis of Kirchhoff's voltage law written for the Fig. 2*b* circuit containing an e.m.f. source, $r_3$ and $C$ equal

$$U - u_C(0+) = \frac{r_1}{r_1 + r_2} U = r_3 i_3(0+),$$

---

tion of *total* flux linkage of the inductances involved, or *total* charge on capacitors before and after the application of circuit change (Refs. 63 and 105). Magnetic or, correspondingly, electrical energy is not conserved in these hypothetical cases (since the electrical arc is disregarded), conservation of momentum (i.e. total flux linkage or total charge) being the only valid approach.

from which

$$i_3(0+) = \frac{Ur_1}{(r_1 + r_2)r_3}.$$

Taking into account that

$$i_2(0+) = i_L(0-) = \frac{U}{r_1 + r_2},$$

using Kirchhoff's current law we find

$$i_1(0+) = i_2(0+) + i_3(0+) = \frac{r_1 + r_3}{(r_1 + r_2)r_3} U.$$

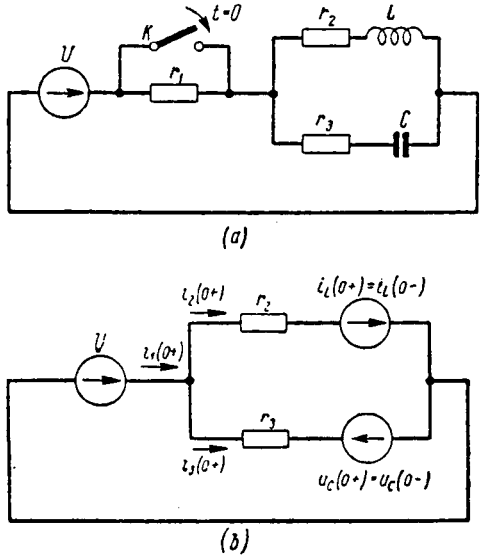



FIG. 2. Example 1: (*a*) initial circuit; (*b*) equivalent circuit for the instant $t = 0$.

In order to find the derivatives of the currents we can write differential equations of the Fig. 2*b* circuit which are true generally and in particular when $t = 0+$:

$$U = r_2 i_2 + L \frac{di_2}{dt},$$

$$U = r_3 i_3 + \frac{1}{C} \int i_3 \, dt,$$

$$i_1 = i_2 + i_3.$$

It follows from the first equation that

$$\frac{di_2}{dt} = \frac{U - r_2 i_2}{L},$$

from which, when $t = 0+$,

$$\frac{di_2}{dt}(0+) = \frac{U - \dfrac{r_2}{r_1 + r_2} U}{L} = \frac{r_1 U}{(r_1 + r_2)L}.$$

By differentiating the second equation we find

$$0 = r_3 \frac{di_3}{dt} + \frac{i_3}{C},$$

from which, when $t = 0+$,

$$\frac{di_3}{dt}(0+) = -\frac{i_3(0+)}{r_3 C} = -\frac{r_1 U}{(r_1 + r_2) r_3^2 C}.$$

Finally, differentiating the third equation we get

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt},$$

from which, when $t = 0+$,

$$\frac{di_1}{dt}(0+) = \left[\frac{r_1}{(r_1 + r_2) L} - \frac{r_1}{(r_1 + r_2) r_3^2 C}\right] U.$$

EXAMPLE 2 (Ref. 114). In the circuit shown in Fig. 3, steady-state conditions obtained before the instant $t = 0$. When $t = 0$, contact $K$ closes. It is required to find the initial values of the loop currents $i_1$, $i_2$ and $i_3$.

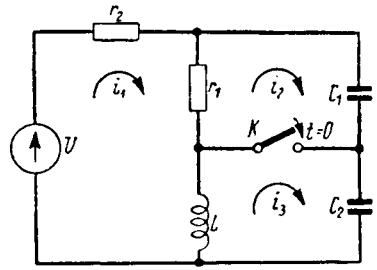



FIG. 3. Example 2.

Before commutation the current in the resistances $r_2$ and $r_1$ and in the inductance $L$ is

$$i_{r_1}(0-) = i_L(0-) = \frac{U}{r_1 + r_2}.$$

Before commutation the total voltage on capacitances $C_1$ and $C_2$ equals the voltage on resistance $r_1$, i.e.

$$U_{C_1} = U_{C_2} = \frac{r_1}{r_1 + r_2} U.$$

When capacitances $C_1$ and $C_2$ are connected in series their charges are equal: $q_1 = q_2$, whence $C_1 U_{C1} = C_2 U_{C2}$, or

$$\frac{U_{C_1}}{U_{C_2}} = \frac{C_2}{C_1}.$$

From this

$$U_{C_1} = \frac{r_1}{r_1 + r_2} \cdot \frac{C_2}{C_1 + C_2} U, \quad U_{C_2} = \frac{r_1}{r_1 + r_2} \frac{C_1}{C_1 + C_2} U.$$

According to Kirchhoff's voltage law written for the exterior circuit of Fig. 3

$$r_2 i_1 = U - U_{C_1} - U_{C_2} = \frac{r_2}{r_1 + r_2} U,$$

from which, when $t = 0+$,

$$i_1(0+) = \frac{U}{r_1 + r_2}.$$

Further, $i_1(0+) - i_3(0+) = i_L(0+) = U/r_1 + r_2$ ; consequently,

$$i_3(0+) = 0.$$

In view of the fact that at the instant of commutation the voltages on the capacitances do not change

$$r_1[i_1(0+) - i_2(0+)] = U_{C_1}$$

or

$$i_2(0+) = i_1(0+) - \frac{U_{C_1}}{r_1},$$

from which

$$i_2(0+) = \frac{U}{r_1 + r_2} - \frac{C_2}{C_1 + C_2} \cdot \frac{U}{r_1 + r_2} = \frac{C_1}{C_1 + C_2} \frac{U}{r_1 + r_2}.$$

## 3. Forced and Natural (Free) Responses

In general the analysis of transients in a linear electrical circuit is reduced to solving non-homogeneous differential equations which express Kirchhoff's laws. These equations are *linear* since the voltages and currents and their derivatives appear in the equation in the first degree, there being no terms containing the product of current and voltage. In such cases the *superposition method* can be used, and if there are several sources in the circuit the over-all response of the circuit is the sum of the responses corresponding to the effect of each source separately. We shall therefore confine our examination to transients with only one source.

As is generally known, the complete solution of a non-homogeneous linear differential equation with constant coefficients is equal to the sum of the *particular* solution of the given equation and the *complementary* solution of the corresponding homogeneous (or complementary) equation.

The *particular* solution of a non-homogeneous equation expresses *forced responses* produced by the source. If the excitation function is given in the generalized form $Ue^{j\varphi}e^{pt}$ (Chapter VI), the forced current is

$$I_{fo}(t) = \frac{Ue^{j\varphi}e^{pt}}{Z(p)}. \tag{11.2}$$

Here $Z(p)$ is the *generalized input* or *transfer impedance* (depending on the formulation of the problem), assuming that the inductive reactance is expressed by $pL$ and the capacitance reactance by $1/pC$.

In general $p$ is a complex number ($p = c + j\omega$).

If the e.m.f. is a harmonic function ($\omega \neq 0$) with an increasing ($c > 0$) or decreasing ($c < 0$) amplitude, the forced current is

$$I_{fo}(t) = \frac{Ue^{ct}e^{j(\omega t+\psi)}}{U(c + j\omega)}. \tag{11.3}$$

The instantaneous values of the current are determined by the imaginary (or real) part of $I_{fo}(t)$.

In the case of a sinusoidal e.m.f. ($c = 0$, $p = j\omega$) the forced current is equal to a steady sinusoidal current.

If the e.m.f. is an exponential function ($\psi = 0$, $p = c$), the forced current also changes according to an exponential law

$$i_{fo}(t) = \frac{Ue^{ct}}{U(c)}, \tag{11.4}$$

where $Z(c) = Z(p)_{p=c}$.

When the e.m.f. is constant ($\psi = 0$, $p = 0$), the forced current is equal to a direct current

$$i_{fo}(t) = \frac{U}{Z(0)}. \tag{11.5}$$

A method of finding a forced current when the applied voltage is given as a polynomial in the form

$$u(t) = \sum_{k=0}^{m} U_k t^k$$

has also been described in print (Ref. 90, p. 499).

Replacing the excitation function (in the case the e.m.f. $u(t)$) in the original non-homogeneous differential equation by zero, we get a homogeneous linear differential equation.

The *complementary* solution of the differential equation named the natural (or free) response, physically determines the behaviour of a circuit with no external electrical energy sources for given initial conditions. In the course of time the electrical quantities determined by the complementary equation disappear. These are called *natural (or transient) components.*

Substituting $i(t) = Ae^{pt}$ in a complementary equation gives

$$Z(p)\, A = 0.$$

Here the expression $Z(p)$ is the same as in (2).

Discarding the trivial solution corresponding to the zero value of constant integration ($A = 0$), we get the characteristic equation

$$Z(p) = 0. \tag{11.6}$$

Denoting the roots of the characteristic equation by $p_k$ we get the complementary solution in the form

$$i_{fr}(t) = \sum_{k=1}^{n} A_k e p_k t, \tag{11.7}$$

where $A_k$ are integration constants, the number of which is determined by the degree of the characteristic equation, i.e. equals the order of the differential equation.

The current found as a result of solving a non-homogeneous linear differential equation is equal to the sum of the forced and natural components of the current

$$i(t) = i_{fo}(t) + i_{fr}(t). \tag{11.8}$$

Analogously, the voltage on any section of the circuit during transients is composed of the voltages of the forced and natural responses

$$u(t) = u_{fo}(t) + u_{fr}(t). \tag{11.9}$$

In order to calculate the integration constants contained in (7), it is necessary to know the initial conditions.

In accordance with (1), (8) and (9)

$$i_L(0-) = i_{L_{fo}}(0+) + i_{L_{fr}}(0+),$$

$$u_C(0-) = u_{C_{fo}}(0+) + u_{C_{fr}}(0+),$$

from which

$$\left.\begin{aligned} i_{L_{fr}}(0+) &= i_L(0-) - i_{L_{fo}}(0+), \\ u_{C_{fr}}(0+) &= u_C(0-) - u_{C_{fo}}(0+). \end{aligned}\right\} \tag{11.10}$$

In the particular case, when the initial conditions are zero

$$\left.\begin{aligned} i_{L_{fr}}(0+) &= -i_{L_{fo}}(0+), \\ u_{C_{fr}}(0+) &= -u_{C_{fo}}(0+). \end{aligned}\right\} \tag{11.11}$$

Consequently, with zero initial conditions the natural components of the current in the inductance and the voltage across the capacitance at the initial instant of the transients are equal to the corresponding components of the forced responses for the same instant ($t = 0+$), with the opposite sign.

Equations (10) and (11) are used to find the integration constants $A_k$, related to $i_{fr}(0+)$ by the equation $i_{fr}(0+) = \sum_{k=1}^{n} A_k$, on the basis of (7). The given equations show that the integration constants depend on the disparity which exists between the initial conditions of the network and the electrical quantities of the forced responses when $t = 0$.

Taking into account the order of the differential equations describing the transients under examination, we can distinguish between networks of the first and second order and higher.

In networks of the first order, energy is stored in one element only, $L$ or $C$, in the form of a magnetic energy ($rL$ networks, Section 4) or electrical energy ($rC$ networks, Section 5). A single-loop network containing elements which store energy of both kinds, magnetic and electrical, is a second-order network ($rLC$ network, Section 6). Multi-loop networks can be of a higher order (Section 7).

In accordance with the above, the analysis of a transient by the classic method (by solving differential equations) is, in the end, reduced to the following:

(1) compiling a differential equation to describe the process under examination from Kirchhoff's laws;
(2) finding a general solution of a homogeneous differential equation (i.e. with the excitation function equal to zero) which corresponds to the free state of the network (complementary solution or natural response);
(3) finding the components of the forced state (particular solution, forced or steady-state response);
(4) determining the integration constants contained in the general solution of a homogeneous differential equation in accordance with initial conditions;
(5) adding together the electrical quantities of the natural and forced responses.

## 4. Transient Response of an $rL$ Circuit

Let us suppose that at an instant $t = 0$ a circuit consisting of a resistance $r$ and an inductance $L$ connected in series (Fig. 4) are connected to an e.m.f. source $u$.

The differential equation for the time $t \geqq 0$ is written in the form

$$u = ri + L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

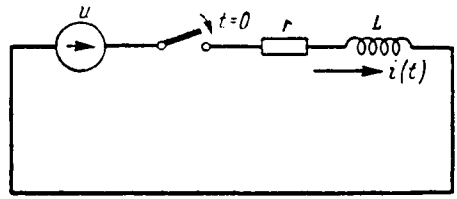



FIG. 4. Series $rL$ circuit excited by an arbitrary voltage function.

The characteristic equation has the form $r - Lp = 0$ and, correspondingly, the root of the equation is $p = -r/L$.

From this the transient portion of the current

$$i_{\text{fr}} = Ae^{pt} = Ae^{-\frac{r}{L}t}.$$

The complete current in the circuit will be determined by the sum of the forced and natural components of the current

$$i = i_{\text{fo}} + Ae^{-\frac{r}{L}t}. \tag{11.12}$$

The forced current can be found if the e.m.f. $u$ is given.

We shall examine three cases:

(a) exciting an $rL$ circuit by a constant voltage $U$;

(b) short-circuiting an $rL$ circuit;

(c) exciting an $rL$ circuit by a sinusoidal voltage

$$u = U_m \sin(\omega t + \psi).$$

### (a) $rL$ Circuit Response to a Constant Voltage

When an $rL$ circuit is connected to a constant voltage $U$, the forced current is $U/r$. Therefore, in accordance with (12),

$$i = \frac{U}{r} + Ae^{-\frac{r}{L}t}. \tag{11.13}$$

The integration constant $A$ can be found from the initial condition

$$i(0+) = i(0-) = 0.$$

According to equation (13), when $t = 0$,

$$0 = \frac{U}{r} + A,$$

from which

$$A = -\frac{U}{r}.$$

Consequently,

$$i = \frac{U}{r}\left(1 - e^{-\frac{r}{L}t}\right) = I\left(1 - e^{-\frac{t}{\tau}}\right). \tag{11.14}$$

Here $I = U/r$ is the limit value to which current $i$ tends as $t$ increases (steady-state current). At the initial instant $t = 0+$ the e.m.f. of the self-inductance

$$e_L(0+) = \left[-L\frac{\mathrm{d}i}{\mathrm{d}t}\right]_{t=0+} = -U$$

completely compensates the applied voltage and therefore the current $i(0+)$ is zero.

In the course of time the e.m.f. of the self-inductance decreases and the current in the circuit grows, asymptotically approaching the steady-state value.

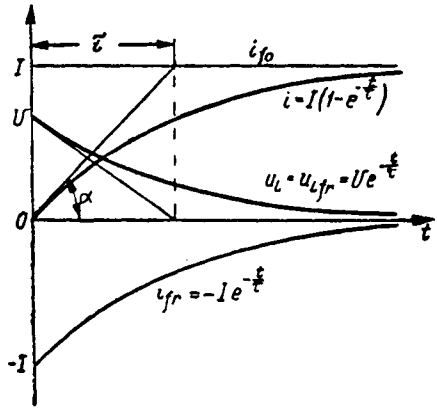



FIG. 5. Transient in an $rL$ circuit energized by a constant voltage.

Figure 5 shows the curves of the forced, natural and complete currents. The curve of the voltage across the inductance

$$u_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = Ue^{-\frac{t}{\tau}}$$

is given in the same figure.

At the initial instant of the transient the time derivative of current is

$$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0+} = \left(\frac{I}{\tau}e^{-\frac{t}{\tau}}\right)_{t=0} = \frac{I}{\tau} = \tan\alpha. \qquad (11.15)$$

The quantity $\tau = L/r$ is known as the *time constant.* The time constant is measured in seconds $[L/r] = \mathrm{H}/\Omega = \Omega \times \mathrm{sec}/\Omega = \mathrm{sec}$.

Equation (15) shows that the time constant can be determined graphically as the magnitude of the section cut off by the tangent to the current curve when $t = 0+$ on the asymptote of this curve.

The smaller the time constant and, consequently, the more rapidly the e.m.f. of the self-inductance decreases, the more rapidly the current grows. At various instants the current in the circuit, expressed as a percentage of the finite (steady) value, is:

| $t =$ | $\tau$, | $2\tau$, | $3\tau$, | $4\tau$ ... | $\infty$; |
|---|---|---|---|---|---|
| $\frac{i}{I}100\% =$ | 63·2, | 86·5, | 95·0, | 98·2, ... | 100. |

Consequently, *the time constant of an rL circuit is equal to the interval of time throughout which the natural component of the current decreases by e times and, correspondingly, the current in this circuit, connected to a constant voltage, reaches* 63·2 % *of its steady-state value.*

As Fig. 5 and the table given above show, in theory a transient continues for an infinite length of time. In practice it can be regarded as being complete when $t \geqslant 4\tau$.

### (b) Short-circuiting an *rL* Circuit

Let us suppose that an $rL$ circuit connected to a d.c. or a.c. voltage source is short-circuited when $t = 0$ (Fig. 6a). In the $rL$ loop thus formed the current does not disappear instantaneously owing to the magnetic field of the induction coil: the e.m.f. of the self-inductance, produced by the decrease in the magnetic flux, tends to maintain the current in the loop on account of the energy of the disappearing magnetic field. As the energy of the magnetic field gradually disperses, turning to heat in resistance $r$, the current in the loop approaches zero.

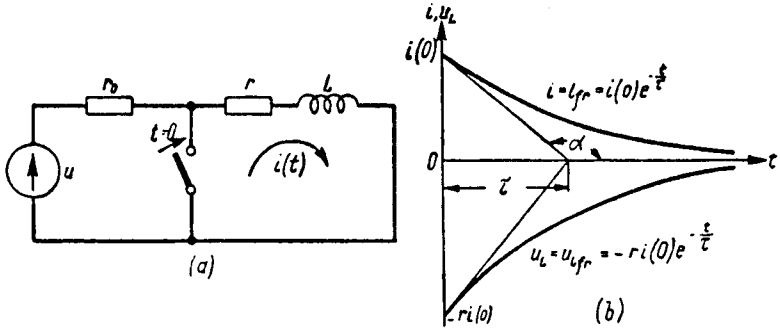



FIG. 6. Short-circuiting of an $rL$ circuit, with non-zero initial current: (*a*) calculation diagram; (*b*) voltage and current curves.

The response taking place in the short-circuited $rL$ loop is natural; in this case the forced current is zero.

Supposing that $i_{to} = 0$ in (12) we get

$$i = Ae^{-\frac{r}{L}t}.$$

The integration constant $A$ is found from the initial condition

$$i(0+) = i(0-) = i(0),$$

from which

$$i(t) = i(0)\, e^{-\frac{r}{L}t} = i(0)\, e^{-\frac{t}{\tau}};$$

here $i(0)$ is the value of the current in the inductance at the instant immediately before the short-circuiting.

If a steady state obtained before the transient under examination then, when $u = U$, $i(0) = U/(r_0 + r)$, and when

$$u = U_m \sin(\omega t + \psi)$$

$$i(0) = \frac{U_m \sin(\psi - \varphi)}{\sqrt{[(r_0 + r)^2 + (\omega L)^2]}},$$

where $\varphi = \arctan \omega L/(r_0 + r)$.

Figure 6*b* shows the curve of the current in the short-circuited loop and the curve of the voltage across the inductance

$$u_L = L \frac{\mathrm{d}i}{\mathrm{d}t} = -ri(0)\, e^{-\frac{t}{\tau}}.$$

The time constant of the loop $\tau = L/r$ can be found graphically from the tangent of curve $i$ at any point, e.g. at time $t = 0+$, when

$$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0+} = -\frac{i(0)}{\tau} = \tan\alpha = -\tan(180° - \alpha).$$

The following values for the current in the circuit, expressed as a percentage of the initial value, are obtained for various instants:

$$\frac{i}{i(0)}\,100\% = 36{\cdot}8;\quad 13{\cdot}5;\quad 5{\cdot}0;\quad 1{\cdot}8 \ldots 0.$$

$$t = \quad \tau, \quad 2\tau, \quad 3\tau, \quad 4\tau \ldots \infty;$$

Consequently *the time constant of the rL loop is equal to the interval of time during which the current decreases by e times when the loop is short-circuited, i.e. reaches a value equal to* 36·8 % *of the initial value of the current.*

Theoretically, the transient in a short-circuited loop is completed when $t = \infty$. At the end of this time the energy

$$\int_0^\infty i^2 r\, \mathrm{d}t = [i(0)]^2 r \int_0^\infty e^{-2\frac{r}{L}t}\, \mathrm{d}t = \frac{L[i(0)]^2}{2},$$

i.e. all the energy stored in the magnetic field of the coil before commutation, is released in resistance $r$ in the form of heat.

As in the previous case, the transient can in practice be regarded as complete when $t \geq 4\tau$.

### (c) $rL$ Circuit Response to a Sinusoidal Voltage

When an $rL$ circuit is excited by a sinusoidal voltage $u = U_m \sin(\omega t + \psi)$, the forced current $i_{fo} = I_m \sin(\omega t + \psi - \varphi)$, where

$$I_m = \frac{U_m}{\sqrt{[r^2 + (\omega L)^2]}}, \qquad \varphi = \arctan\frac{\omega L}{r}.$$

On the basis of (12)

$$i = I_m \sin(\omega t + \psi + \varphi) + Ae^{-\frac{t}{\tau}},$$

where

$$\tau = \frac{L}{r}.$$

The integration constant is determined from the initial condition

$$i(0+) = i(0-) = 0.$$

Consequently,

$$0 = I_m \sin(\psi - \varphi) + A,$$

from which

$$A = -I_m \sin(\psi - \varphi).$$

Thus, the unknown current

$$i = I_m\left[\sin(\omega t + \psi - \varphi) - \sin(\psi - \varphi)\, e^{-\frac{t}{\tau}}\right].$$

Figure 7*a* shows the curves of $i_{fo}$, $i_{fr}$ and $i$. The initial ordinates $i_{fo}(0+)$ and $i_{fr}(0+)$ are absolutely identical and opposite in sign. The current, therefore, equals zero at the initial instant. The natural component of the current decreases according to an exponential law. After a time $t = \tau$ the free current has decreased by $e$ times compared with its initial value $i_{fr}(0)$. The time constant is directly proportional to the $Q$ factor of the loop and inversely proportional to the frequency $\omega$

$$\tau = \frac{\omega L}{\omega r} = \frac{Q}{\omega}.$$

*If, at the instant of commutation* ($t = 0$), *the current* $i_{fo}$ *passes through zero, i.e. the condition* $\psi = \varphi$ *or* $\psi - \varphi = \pm\pi$, *obtains, then no natural current is produced* and a forced steady state commences immediately in the circuit, without any transient.

*And if the commutation takes place when* $\psi - \varphi = \pm\pi/2$, *the natural initial current is at its maximum* (Fig. 7*b*), i.e. $i_{fr}(0+) = \mp I_m$, and the

transient current reaches its maximum at the end of the first half-period. But even in the extreme case when $r = 0$ and, consequently, $\tau = L/r = \infty$, the current cannot be more than double the amplitude of the steady state (Fig. 7*b*).

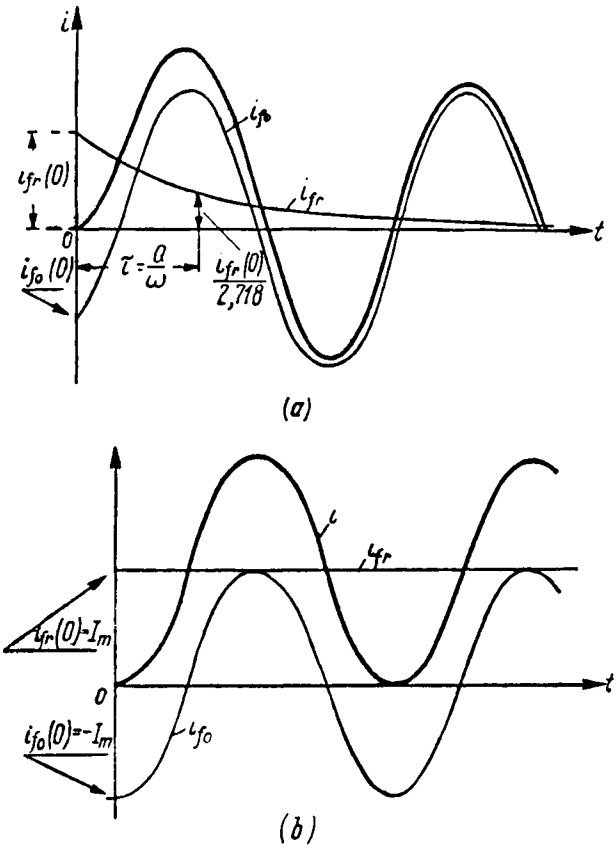



FIG. 7. Current and its components when an *rL* circuit is excited by a sinusoidal voltage: (*a*) general case; (*b*) the case (60).

## 5. Transient Response of an *rC* Circuit

Let us suppose that, at the instant $t = 0$, a circuit consisting of a resistance $r$ and capacitance $C$ in series is connected to an e.m.f. source $u$ (Fig. 8).

On the basis of Kirchhoff's voltage law the equation for time $t \geqq 0$ takes the form

$$u = ri + u_C,$$

where $u_C$ is the voltage across the capacitance.

Taking into account the fact that $i = C\,\mathrm{d}u_C/\mathrm{d}t$ we get

$$u = rC\frac{\mathrm{d}u_C}{\mathrm{d}t} + u_C;$$

here the voltage across the capacitance is the unknown quantity.

The characteristic equation has the form $1 + rCp = 0$ and, correspondingly, the root of the equation $p = -1/rC$. Consequently, the natural component of the voltage across the capacitance is

$$u_{C_{\mathrm{fr}}} = Ae^{-\frac{t}{rC}} = Ae^{-\frac{t}{\tau}},$$

where $\tau = rC$ is the time constant of the $rC$ loop.

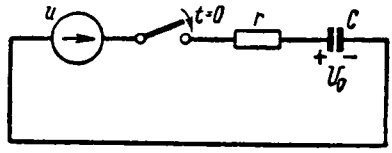



FIG. 8. Series $rC$ circuit excited by an arbitrary voltage function.

The complete voltage of the capacitance is equal to the sum of the forced and natural components of the voltage

$$u_C = u_{C_{\mathrm{fo}}} + Ae^{-\frac{t}{\tau}}. \tag{11.16}$$

In turn, the current in the loop

$$i = C\frac{\mathrm{d}u_C}{\mathrm{d}t} = C\frac{\mathrm{d}u_{C_{\mathrm{fo}}}}{\mathrm{d}t} - \frac{A}{r}e^{-\frac{t}{\tau}}. \tag{11.17}$$

We shall consider three cases:
(a) exciting an $rC$ circuit by a constant voltage $U$;
(b) short-circuiting an $rC$ circuit;
(c) exciting an $rC$ circuit by a sinusoidal voltage

$$u = U_m \sin(\omega t + \psi).$$

### (a) An $rC$ Circuit Response to a Constant Voltage

We shall connect a resistance $r$ and an initially charged capacitance $C$ (the polarity is shown in Fig. 8 by the signs + and −) in series with a constant e.m.f. $U$. The initial voltage of the capacitance will be denoted by $U_0$.

The forced voltage across the capacitance equals the e.m.f. of the source. Therefore, in accordance with (16),

$$u_C = U + Ae^{-\frac{t}{\tau}}. \tag{11.18}$$

The integration constant $A$ contained in (18) is found from the initial condition

$$u_C(0+) = U_0.$$

On the basis of equation (18), when $t = 0$, $U_0 = U + A$, from which

$$A = U_0 - U.$$

Consequently,

$$u_C = U - [U - U_0]\, e^{-\frac{t}{\tau}}. \tag{11.19}$$

According to (17) the loop current

$$i = \frac{U - U_0}{r} e^{-\frac{t}{\tau}}. \tag{11.20}$$

The quantity $\tau = rC$ is also known as a *time constant* and is measured in seconds: $[rC] = \Omega \times \mathrm{F} = \Omega(a\ \mathrm{sec})/v = \mathrm{sec}$.

In the course of time the voltage on the capacitance increases, tending to the steady value of $U$ and the current decreases, tending to zero at the extreme.

Figure 9 shows curves of the increment of $u_C$ and drop in $i$.

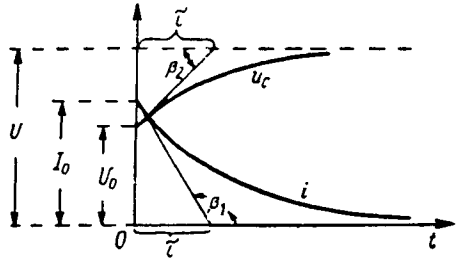



FIG. 9. Transient in an $rC$ circuit connected to a constant voltage.

The larger the time constant, the slower $u_C$ increases and $i$ decreases.

The time constant $\tau = rC$ can be found graphically as the subtangent to curve $i$ at any point, e.g. at time $t = 0+$, when

$$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0+} = -\frac{U - U_0}{r\tau} = \tan\beta_1 = -\tan(180° - \beta_1).$$

The time constant $\tau$ is also determined by the magnitude of the section cut off by the tangent to curve $u_C$ (at the point $t = 0+$) on the asymptote of this curve; in this case

$$\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right)_{t=0+} = \frac{U - U_0}{\tau} = \tan\beta_2.$$

In addition, equation (20) shows that *the time constant of an rC circuit equals the interval of time during which the current in this circuit, which is connected to a constant voltage, decreases by e times, i.e. drops to* 36·8% *of its initial value* $(U - U_0)/r$. When the initial condition is zero ($U_0 = 0$), the time constant $\tau$ can be defined as the interval of time during which $u_C$ attains 63·2% of its steady-state value $U$. The law of the change in the voltage on the capacitance is then analogous to the law of the change in the current in an $rL$ loop connected to a constant voltage.

### (b) Short-circuiting an $rC$ Circuit

Short-circuiting a circuit consisting of $r$ and $C$ connected in series is equivalent to connecting it to a source with a zero e.m.f. Let us suppose that the capacitance $C$ is initially charged to the voltage $U_0$.

Assuming an e.m.f. equal to zero in (19) and (20) we get

$$u_C = U_0 e^{-\frac{t}{\tau}}, \tag{11.21}$$

$$i = -I_0 e^{-\frac{t}{\tau}}, \tag{11.22}$$

where

$$I_0 = \frac{U_0}{r}.$$

When an $rC$ circuit is short-circuited an electric current flows from the plus terminal to the minus terminal. Consequently, when the polarity of the capacitance is as in Fig. 8, the current flows through the resistance $r$ opposite to the reference direction of Fig. 8 (in accordance with the direction of the e.m.f.). In conjunction with this a minus sign occurs in equation (22). Curves of the drop in $u_C$ and $i$ are given in Fig. 10.

In contrast with the voltage across the capacitance, which changes smoothly, the current in the $rC$ loop, which is proportional to the rate of change in $u_C$, suddenly jumps when $t = 0$. It should be noted that this result assumes that there is no inductance in the loop. Since an actual circuit would have inductance, the current would be zero at the initial instant. With a small inductance, however, it increases extremely rapidly.

The energy dissipated in the resistance in the course of the transient is equal to the energy stored in the electrical field:

$$\int_0^\infty i^2 r \, dt = \frac{U_0^2}{r} \int_0^\infty e^{-\frac{2t}{rC}} \, dt = \frac{CU_0^2}{2}.$$

As in the previous case, the transient response can be considered to have finished when $t = 4\tau$ since the capacitance discharges 98·2% and the voltage of the capacitance is reduced to 1·8% of the initial voltage in this time.

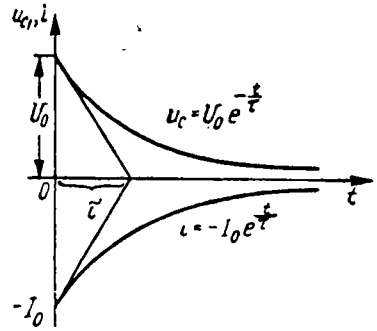



FIG. 10. Plots of the voltage across the capacitance and the current through it when an $rC$ circuit with non-zero initial voltage is short-circuited.

### (c) *rC* Circuit Response to a Sinusoidal Voltage

When an $rC$ circuit is energized by a sinusoidal voltage $u = U_m \sin(\omega t + \psi)$ the forced voltage across the capacitance

$$u_{C_{fo}} = \frac{I_m}{\omega C} \sin\left(\omega t + \psi - \varphi - \frac{\pi}{2}\right),$$

where

$$I_m = \frac{U_m}{\sqrt{\left[r^2 + \left(\frac{1}{\omega C}\right)^2\right]}}, \quad \varphi = \arctan\left(\frac{-\frac{1}{\omega C}}{r}\right).$$

From (16)

$$u_C = -\frac{I_m}{\omega C} \cos(\omega t + \psi - \varphi) + Ae^{-\frac{t}{\tau}}.$$

If we suppose that the capacitor was uncharged, then the integration constant is determined from the initial condition $u_C(0+) = 0$

$$0 = -\frac{I_m}{\omega C} \cos(\psi - \varphi) + A,$$

from which

$$A = \frac{I_m}{\omega C} \cos(\psi - \varphi).$$

Thus, the unknown voltage of the capacitance

$$u_C = -\frac{I_m}{\omega C}\left[\cos(\omega t + \psi - \varphi) - \cos(\psi - \varphi)\, e^{-\frac{t}{\tau}}\right], \tag{11.23}$$

and the current in the circuit

$$i = C\frac{du_C}{dt} = I_m\left[\sin(\omega t + \psi - \varphi) - \frac{1}{\omega rC}\cos(\psi - \varphi)\, e^{-\frac{t}{\tau}}\right]. \tag{11.24}$$

If the $rC$ circuit is switched in at the instant the forced current reaches a maximum (positive or negative), the forced voltage of the capacitance is zero, i.e. if commutation takes place when $\psi - \varphi = \pm \pi/2$, then the natural component of the voltage does not occur and a forced steady state is established without any transient.

## 6. Transient Response of an $rLC$ Circuit

When an $rLC$ circuit is excited by a voltage $u$ (Fig. 11), the transient response can be studied with the aid of a differential equation

$$\frac{du}{dt} = L\frac{d^2i}{dt^2} + r\frac{di}{dt} + \frac{i}{C}.$$

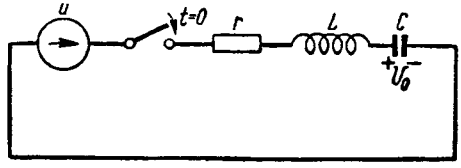



FIG. 11. Series $rLC$ circuit excited by an arbitrary voltage function.

The characteristic equation has the form

$$Lp^2 + rp + \frac{1}{C} = 0,$$

from which

$$p_{1,2} = -\frac{r}{2L} \pm \sqrt{\left[\left(\frac{r}{2L}\right)^2 - \frac{1}{LC}\right]}$$

$$= -c \pm \sqrt{(c^2 - \omega_0^2)}, \tag{11.25}$$

where

$$c = \frac{r}{2L}, \quad \omega_0 = \frac{1}{\sqrt{(LC)}}. \tag{11.26}$$

The natural component

$$i_{fr} = A_1 e^{p_1 t} + A_2 e^{p_2 t}.$$

The complete current in the circuit can be determined by adding the forced and natural components

$$i = i_{fo} + A_1 e^{p_1 t} + A_2 e^{p_2 t}. \tag{11.27}$$

The forced current is found in accordance with the given e.m.f. $u$. As for the natural component of the current, its character depends on the sign of the radicand of equation (25).

Let us examine the case of exciting an $rLC$ circuit by constant and sinusoidal voltages.

### *rLC* Circuit Response to a Constant Voltage

Taking the e.m.f. of the source to be constant, $u = U$, we begin with the assumption that the capacitance has an initial charge. We shall denote the initial voltage on the capacitance by $U_0$.

Let us examine three possible cases.

*(a) The case of $c > \omega_0$, i.e. $r > 2\sqrt{(L/C)}$ (aperiodic response)*

In view of the inductance, the current in the circuit is zero when $t = 0+$. The initial conditions in the circuit under examination are thus as follows:

$$i(0+) = 0, \quad u_C(0+) = U_0.$$

The original equation $U = ri + L\,\mathrm{d}i/\mathrm{d}t + U_C$ for the initial instant is written in the form

$$U = \left(L \frac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0+} + u_C(0+),$$

from which

$$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0+} = \frac{U - U_0}{L}. \tag{11.28}$$

In the steady state the current will be zero.

Differentiating (27) and taking into account that $i_{fo} = 0$ we get

$$\frac{\mathrm{d}i}{\mathrm{d}t} A_1 p_1 e^{p_1 t} + A_2 p_2 e^{p_2 t}. \tag{11.29}$$

Substituting $t = 0$ in (27) and (29) and using (28) we get

$$0 = A_1 + A_2, \quad \frac{U - U_0}{L} = A_1 p_1 + A_2 p_2 .$$

It follows from these equations that

$$A_1 = -A_2 = \frac{U - U_0}{L(p_1 - p_2)} = \frac{U - U_0}{2L\sqrt{(c^2 - \omega_0^2)}}.$$

Thus

$$i = \frac{U - U_0}{2L\sqrt{(c^2 - \omega_0^2)}} (e^{p_1 t} - e^{p_2 t}). \tag{11.30}$$

According to (25) factors $p_1$ and $p_2$ have negative real values. If the subscript 1 corresponds to the upper sign in front of the radical (25), then $|p_1| < |p_2|$ and therefore the curve $e^{p_1 t}$ droops more slowly than $e^{p_2 t}$.

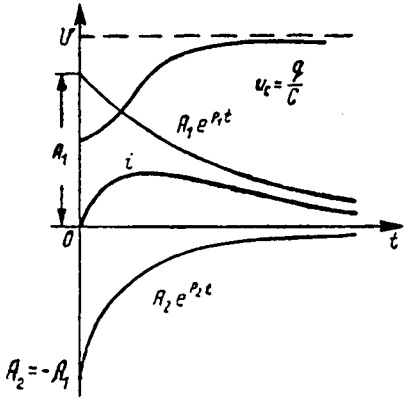



FIG. 12. Aperiodic response in a series $rLC$ circuit.

Figure 12 shows the curve $i$ constructed from equation (30). The same figure shows the curve of the voltage on the capacitance $u_C$ which also expresses, on a different, known scale, the dependence of the charge $q$ on time.

For high values of $C$, curve $i$ approximates to the curve of the current in an $rL$ circuit (Fig. 5) and for small values of $L$ it approximates to the curve of the current in an $rC$ circuit (Fig. 9).

Equation (30) can be converted into hyperbolic form

$$i = \frac{U - U_0}{L\sqrt{(c^2 - \omega_0^2)}} e^{-ct} \sinh \sqrt{(c^2 - \omega_0^2)}\, t. \tag{11.31}$$

It should be noted that when an $rLC$ circuit is shorted, i.e. when $U = 0$, current is produced in the circuit by the capacitance discharging.

*(b) The case of* $c = \omega_0$, *i.e.* $r = 2\sqrt{(L/C)}$ *(critical case)*

In accordance with (25), the roots of the characteristic equation are identical

$$p_1 = p_2 = -\frac{r}{2L} = -c.$$

Equation (30) thus results in an indeterminate quantity of the form 0/0; we can find this quantity by L'Hôpital's rule

$$i = \lim_{p_1 \to p_2} \frac{U - U_0}{L(p_1 - p_2)} (e^{p_1 t} - e^{p_2 t}) = \frac{U - U_0}{L} te^{-ct}. \qquad (11.32)$$

The current curve is the same as that shown in Fig. 12.

*(c) The case of* $C < \omega_0$, *i.e.* $r < 2\sqrt{(L/C)}$ *(oscillatory response)*

The roots of the characteristic equation are complex and conjugate

$$\left.\begin{aligned} p_1 &= -c + j\omega_c, \\ p_2 &= -c - j\omega_c, \end{aligned}\right\} \qquad (11.33)$$

$$\omega_c = \sqrt{(\omega_0^2 - c^2)}. \qquad (11.34)$$

According to (34)

$$\omega_0 = \sqrt{(c^2 + \omega_c^2)}.$$

The roots of the characteristic equation are located in the left halfplane, on a semicircle whose centre lies at the origin and which has a radius $\omega_0 = 1/\sqrt{(LC)}$ (Fig. 13).

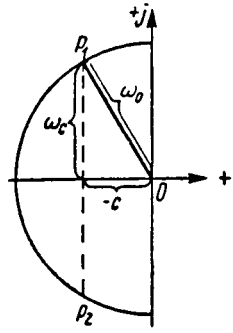



FIG. 13. Distribution of complex conjugate roots of a characteristic equation.

The quantity $\omega_c$ is called the angular velocity of the natural oscillations of the *rLC* circuit, and $T_c = 2\pi/\omega_c$ is the natural oscillation period.

According to (31) the current in the circuit

$$i = \frac{U - U_0}{jL\omega_c} e^{-ct} \sinh j\omega_c t = \frac{U - U_0}{\omega_c L} e^{-ct} \sin \omega_c t. \qquad (11.35)$$

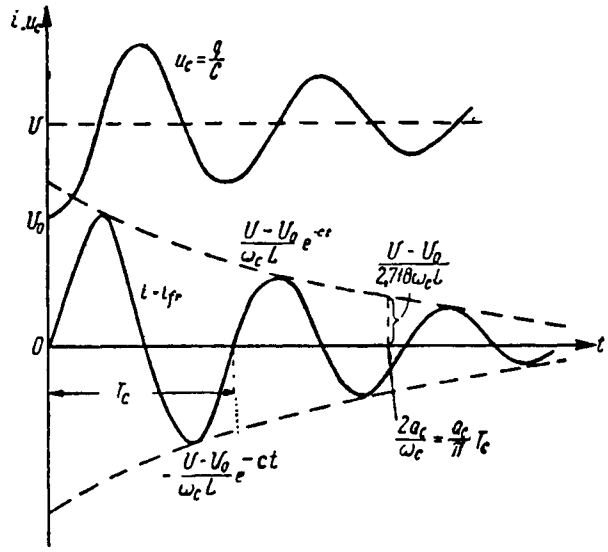



FIG. 14. Oscillatory response when a series $rLC$ circuit is excited by a constant voltage.

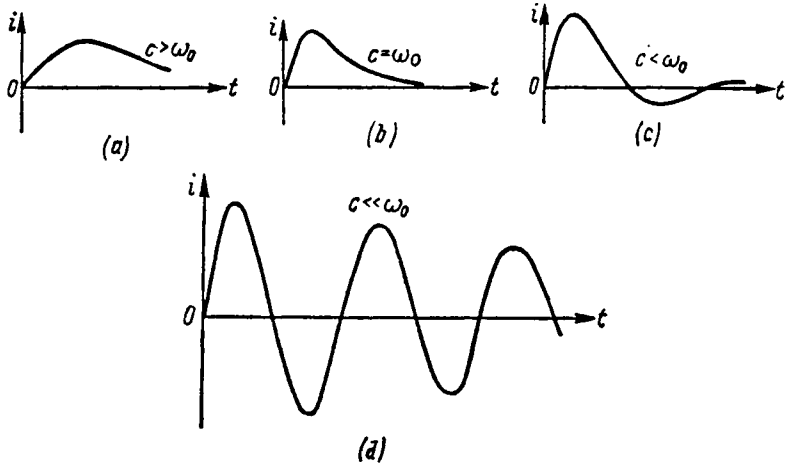



FIG. 15. Change in the character of the transient response as $c$ decreases: (*a*) aperiodic; (*b*) critical; (*c*) and (*d*) oscillatory.

The equation obtained shows that when an *rLC* circuit is connected to a constant voltage, and $c < \omega_0$, *damped sinusoidal oscillations occur in the circuit*, curves $\pm U - U_0/\omega_c L\, e^{-ct}$ (Fig. 14) being the envelopes of the transient current. When $t = 1/c$ the ordinate of the envelope is $e$ times smaller than the initial value of the envelope. If, by analogy with the $Q$-factor of an inductive coil, we denote $Q_c = \omega_c L/\gamma$, then we get $t = 1/c = 2Q_c/\omega_c = Q_c/\pi T_c$. Consequently, after $Q_c/\pi$ periods have elapsed, the envelope decreases by $e$ times.

Figure 14 also shows the curve of the voltage across the capacitance which, on a known scale, also expresses the dependence of the electrical charge $q$ on time. The functions $u_C$ and $i$ have the same attenuation factor.

When the initial conditions are zero ($U_0 = 0$), the curve $u_C$ starts at zero.

As (33) and Fig. 13 show, the angular frequency of the oscillations can be determined by the magnitude of the imaginary part of the root of the characteristic equation which is always less than the resonant frequency $\omega_0$ when $c = 0$.

The smaller $c$ is, compared with $\omega_0$, the more slowly the oscillations are damped and the nearer the frequency of natural oscillations in the *rLC* circuit approaches to the resonant frequency $\omega_0$. Within the limit $c = 0$, $\omega_c = \omega_0$, and the oscillations are not damped: the oscillation of the energy between the electrical and magnetic fields occurs without heat losses.

As an example, Fig. 15 shows in sequence the change in the nature of the transient response as the magnitude of $c$ decreases.

### *rLC* Circuit Response to a Sinusoidal Voltage

If an *rLC* circuit is energized by a sinusoidal voltage $u = U_m \sin(\omega t + \psi)$ then the forced current is

$$i_{fo} = I_m \sin(\omega t + \psi - \varphi),$$

where

$$I_m = \frac{U_m}{\sqrt{\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}}, \quad \varphi = \arctan \frac{\omega L - \frac{1}{\omega C}}{r}.$$

According to (27) the complete current is

$$i = I_m \sin(\omega t + \psi - \varphi) + A_1 e^{p_1 t} + A_2 e^{p_2 t}. \tag{11.36}$$

When initial conditions are zero, i.e. when $i(0-) = 0$ and $u_c(0-) = 0$, for the instant $t = 0+$ we get

$$U_m \sin \psi = \left(L \frac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0+} \tag{11.37}$$

On the basis of (36) and (37)

$$U_m \sin \psi = L[\omega I_m \cos(\psi - \varphi) + A_1 p_1 + A_2 p_2]. \tag{11.38}$$

Substituting $t = 0$ in (36) gives

$$0 = I_m \sin(\psi - \varphi) + A_1 + A_2. \tag{11.39}$$

The integration constants $A_1$ and $A_2$ can be found from the equations (38) and (39) (Ref. 113)

$$\left.\begin{aligned} A_1 &= \frac{U'}{2L\sqrt{(c^2 - \omega_0^2)}} - \frac{I_m}{2}\sin(\psi - \varphi), \\ A_2 &= -\frac{U'}{2L\sqrt{(c^2 - \omega_0^2)}} - \frac{I_m}{2}\sin(\psi - \varphi), \end{aligned}\right\} \tag{11.40}$$

where

$$U' = U_m \sin\psi - \omega L I_m \cos(\psi - \varphi) - \frac{rI_m}{2}\sin(\psi - \varphi).$$

From (36) and (40)

$$i = I_m \sin(\omega t + \psi - \varphi) + \frac{U' e^{-ct}}{L\sqrt{(c^2 - \omega_0^2)}} \cdot \frac{e^{\sqrt{(c^2-\omega_0^2)}t} - e^{-\sqrt{(c^2-\omega_0^2)}t}}{2}$$

$$- I_m \sin(\psi - \varphi)\, e^{-ct} \cdot \frac{e^{\sqrt{(c^2-\omega_0^2)}t} - e^{-\sqrt{(c_2-\omega_0^2)}t}}{2}.$$

In the case of an aperiodic response ($c > \omega_0$)

$$i = I_m \sin(\omega t + \psi - \varphi) + \frac{U' e^{-ct}}{L\sqrt{(c^2 - \omega_0^2)}} \cdot \sinh\sqrt{(c^2 - \omega_0^2)}\,t$$

$$- I_m \sin(\psi - \varphi)\, e^{-ct} \cosh\sqrt{(c^2 - \omega_0^2)}\,t.$$

In the case of an oscillatory response ($c < \omega_0$)

$$i = I_m \sin(\omega t + \psi - \varphi) + \frac{U' e^{-ct}}{\omega_c L}\sin\omega_c t$$

$$- I_m \sin(\psi - \varphi)\, e^{-ct}\cos\omega_c t.$$

Replacing the sum of the sine and cosine curves by one expression, we get

$$i = I_m \sin(\omega t + \psi - \varphi) + I' e^{-ct}\sin(\omega_c t - \gamma), \tag{11.41}$$

where

$$I' = \sqrt{\left\{\left(\frac{U'}{\omega_c L}\right)^2 + [I_m \sin(\psi - \varphi)]^2\right\}},$$

$$\gamma = \arctan \frac{\omega_c L I_m \sin(\psi - \varphi)}{U'}.$$

The curves of the forced, natural and complete currents during aperiodic and oscillatory responses are given as an example in Fig. 16.

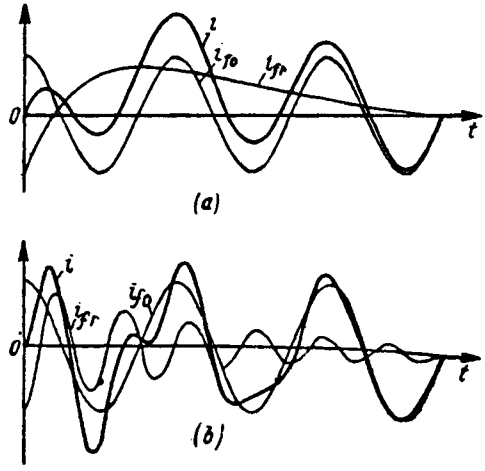



FIG. 16. (*a*) Aperiodic and (*b*) oscillatory responses when an *rLC* circuit is excited by a sinusoidal voltage.

The frequency of the forced current is equal to the frequency of the sinusoidal voltage source; and when $c < \omega_0$ the current changes with the natural frequency of the circuit $\omega_c$. Depending on the parameters $r$, $L$ and $C$ the frequency $\omega_c$ may be either equal to, less than, or greater than the applied frequency $\omega$.

The natural oscillations are superimposed on the forced current and damped in proportion to the damping factor $e^{-ct}$. As the natural component of the current is damped the curve of the complete current approaches the curve of the forced current.

That one of the two components of the complete current whose frequency is lower acts as a curvilinear axis for the other component which oscillates about it (Fig. 16*b*). When frequencies $\omega$ and $\omega_c$ coincide closely *beating* occurs in the circuit.

## 7. Transient Analysis of Multi-loop Networks

The transient response in a multi-loop network is described by a system of linear differential equations with constant coefficients.

The forced current and voltage components are calculated as shown in Section 3. If, for example, the applied e.m.f.s are constant or are periodic time functions, the forced components are calculated in exactly the same way as for steady-state conditions.

The natural components are expressed by the general solution of a system of homogeneous linear differential equations. When compiling equations in accordance with Kirchhoff's voltage law it is possible to choose independent loops in such a way that the order of the differential equations is the lowest. A set of equations compiled in this way can be reduced to a differential equation with respect to one unknown function, its order, and consequently the degree of the characteristic equation, being equal to the sum of the orders of the system's initial differential equations*.

The roots of the characteristic equation can be either real or complex. If the roots are complex they always form complex, conjugate pairs. In conjunction with this the characteristic equations of an odd degree always have at least one real root, the remainder being either real or complex conjugate. Characteristic equations of an even degree have real or complex conjugate roots. The real parts of all the roots of a characteristic equation are always negative, which is realized physically by the attenuation of the free components in passive circuits, over a duration of time (Section 3).

If a characteristic equation is of the $n$-th degree, then the unknowns are the $n$ integration constants ($A_1, A_2, \ldots, A_n$) contained in equation (7). The integration constants are found as a result of solving a set of $n$ equations corresponding to the instant $t = 0+$. This set of equations is obtained by differentiating equation (7) $(n-1)$-times

$$\left.\begin{aligned} i_{\mathrm{fr}}(0+) &= i(0+) - i_{\mathrm{fo}}(0+) = \sum_{k=1}^{n} A_k, \\ i'_{\mathrm{fr}}(0+) &= i'(0+) - i'_{\mathrm{fo}}(0+) = \sum_{k=1}^{n} p_k A_k, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ i^{(n-1)}_{\mathrm{fr}}(0+) &= i^{(n-1)}(0+) - i^{(n-1)}_{\mathrm{fo}}(0+) = \sum_{k=1}^{n} p_k^{n-1} A_k. \end{aligned}\right\} \quad (11.42)$$

* It should be emphasized that the choice of independent loops does not affect the degree of the characteristic equation.

When $t = 0+$, the values of the current contained in (42), and also their derivatives, can be found beforehand from the commutation laws (Section 2).

As an example let us examine the two-loop network shown in Fig. 17.

The first loop contains resistances $r_1$ and $r_2$ and an inductance $L$. The second loop contains a resistance $r_1$ and a capacitance $C$. The order of the differential equation for each of these loops is therefore equal to unity:

$$r_1(i_1 + i_2) + r_2 i_1 + L\frac{di_1}{dt} = u,$$

$$r_1(i_1 + i_2) + \frac{1}{C}\int i_2\,dt = u.$$

Correspondingly the degree of the characteristic equation is two.

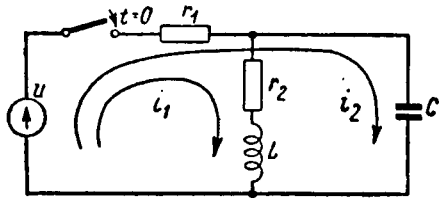



FIG. 17. Two-loop network.

The following procedure is used to obtain the characteristic equation. The set of homogeneous differential equations for the natural current components of the current

$$r_1(i_{1_{fr}} + i_{2_{fr}}) + r_2 i_{1_{fr}} + L\frac{di_{1_{fr}}}{dt} = 0,$$

$$r_1(i_{1_{fr}} + i_{2_{fr}}) + \frac{1}{C}\int i_{2_{fr}}\,dt = 0$$

is written in a symbolic algebraic form in which $p$ denotes the operation of differentiation and $1/p$ denotes the operation of integration:

$$(r_1 + r_2 + pL)\,i_{1_{fr}} + r_1 i_{2_{fr}} = 0,$$

$$r_1 i_{1_{fr}} + \left(r_1 + \frac{1}{pC}\right) i_{2_{fr}} = 0.$$

The given set of equations has a solution other than zero if the determinant is zero, i.e.

$$\Delta(p) = \begin{vmatrix} r_1 + r_2 + pL & r_1 \\ r_1 & r_1 + \dfrac{1}{pC} \end{vmatrix} = 0$$

or

$$(r_1 + r_2 + pL)\left(r_1 + \frac{1}{pC}\right) - r_1^2$$

$$= \frac{1}{pC}\left[r_1 LCp^2 + (r_1 r_2 C + L)\, p + r_1 + r_2\right] = 0.$$

A characteristic equation of the second degree is thus obtained:

$$r_1 LCp^2 + (r_1 r_2 C + L)\, p + r_1 + r_2 = 0.$$

In view of the direct proportionality between the input or transfer impedance of the network and the determinant (Chapter IV, Section 5), the characteristic equation is obtained by a calculation from formula (6)

$$Z(p) = 0.$$

The unknown current, e.g. $i_1$, is equal to the sum of the forced and natural components

$$i_1 = i_{1_{fo}} + A_1 e^{p_1 t} + A_2 e^{p_2 t},$$

where $p_1$ and $p_2$ are roots of the characteristic equation.

The integration constants $A_1$ and $A_2$ are determined from the initial conditions.

The analysis of the transient response of the Fig. 17 circuit is illustrated by the numerical Example 3.

EXAMPLE 3. To determine the current $i_1$ in the Fig. 17 circuit, if it is known that $u = 200$ V, $r_1 = 100\Omega$, $r_2 = 10\Omega$, $L = 0\,01$ H, $C = 10\ \mu$F and $i_1(0) = 0$.

Substituting the given values in the characteristic equation given above gives

$$10^{-5}p^2 + 0{\cdot}02p + 110 = 0,$$

from which

$$p_{1,2} = -1000 \pm j3160.$$

The unknown current $i_1 = i_{1_{fo}} + i_{1_{fr}}$.

The forced current is

$$i_{1_{fo}} = \frac{200}{110} \cong 1{\cdot}82a.$$

The natural component of the current is

$$i_{1_{fr}} = e^{-1000t}(A_1 e^{j3160t} + A_2 e^{-j3160t}) = e^{-1000t}(B_1 \cos 3160t + B_2 \sin 3160t).$$

At the initial instant $i_1(0) = 0$. Consequently $1{\cdot}82 - B_1 = 0$, from which $B_1 = -1{\cdot}82$. The time derivative of the current is

$$\frac{di_1}{dt} = e^{-1000t}(-3160B_1 \sin 3160t + 3160B_2 \cos 3160t)$$
$$- 1000e^{-1000t}(B_1 \cos 3160t + B_2 \sin 3160t).$$

At the initial instant the voltage across capacitance $C$ and, correspondingly, on the branch $r_2L$ parallel to it, is zero

$$u_C(0+) = r_2 i_1(0+) + L\frac{di_1}{dt}(0+) = 0.$$

Taking into account that $i_1(0+) = 0$, the initial value of the derivative $di_1/dt(0+)$ can be obtained as zero. Consequently,

$$3160B_2 - 1000B_1 = 0,$$

from which

$$B_2 = -0{\cdot}576.$$

Thus,

$$i_1 = 1{\cdot}82 - e^{-1000t}(1{\cdot}82 \cos 3160t + 0{\cdot}576 \sin 3160t).$$

CHAPTER XII

# THE FOURIER INTEGRAL

## 1. The Fourier Integral in Complex Form

Under the effect of a nonperiodic e.m.f. in a network, a nonperiodic transient response takes place and can be examined by harmonic analysis. In this case, instead of a nonsinusoidal nonperiodic wave being represented in the form of a discrete series of harmonics (Chapter X), a nonperiodic function is represented in the form of a so-called integral transformation or the *Fourier integral*, expressing the sum of the harmonic oscillations of an infinite and continuous spectrum of frequencies. The Fourier integral is written in a complex or trigonometric form.

If a function $f(t)$ is fixed on the entire axis of real values, is piecewise continuous* for each finite interval $(-T/2, T/2)$, and the integral $\int_{-\infty}^{+\infty} |f(t)|\, dt$ exists, then for all values of $t$ function $f(t)$ can be represented by the Fourier integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t}\, d\omega, \tag{12.1}$$

where

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt. \tag{12.2}$$

Let the nonperiodic function $f(t)$ be given by the curve shown in Fig. 1*a*. In the interval of time $(-T/2, T/2)$ the curve $f(t)$ coincides with the periodic function in Fig. 1*b* and can therefore be represented by a Fourier series.

Taking into account that

$$\frac{1}{T} = \frac{\omega_1}{2\pi},$$

* A function is considered to be piecewise or sectionally continuous for a given interval if this interval can be divided up into a finite number of intervals in each of which the function is continuous and has a continuous derivative, and if at the end of these intervals both the function and its derivative have finite limit values to the left and right. A sectionally continuous function can have a finite number of finite discontinuities.

we get, from equations (18) and (19) in Chapter X,

$$f(t) = \frac{1}{2}\sum_{-\infty}^{+\infty} F_n e^{jn\omega_1 t} = \frac{1}{2\pi}\sum_{-\infty}^{+\infty} e^{jn\omega_1 t}\omega_1 \int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)e^{-jn\omega_1 t}\,dt. \qquad (12.3)$$

The longer the period $T$ selected, the greater the part of the given nonperiodic function $f(t)$ which will be determined by equation (3). Thus, in order that the function $f(t)$ should be determined by equation (3) in as long an interval of time as possible, period $T$ must be increased.

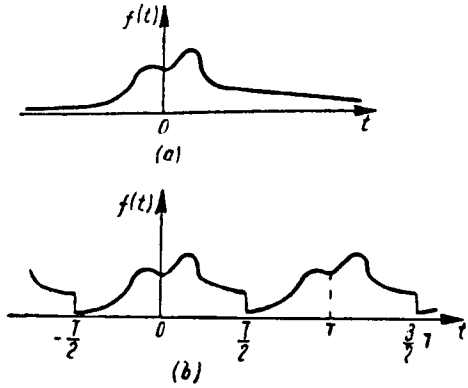



FIG. 1. Nonperiodic and periodic time functions.

As period $T$ increases, the frequency $\omega_1 = 2\pi/T$ may become as small a quantity as is required.

Finding the limit of equation (3) when $T \to \infty$ or, the same thing, when $\omega_1 \to 0$, is a complex mathematical problem since it is then also necessary to take into account the limit $n\omega_1$ when $\omega_1 \to 0$ and $n \to \infty$.

As $\omega_1$ tends to zero it can be replaced by $d\omega$.

For any value of $\omega_1$ however small, the number $n$ assumes all integer values and, consequently, the product $n\omega_1$ changes from $-\infty$ to $+\infty$. It can be replaced by a variable $\omega$, i.e. it can be considered that $n\omega_1 \to \omega$ (current frequency) when $\omega_1 \to 0$ and $n \to \infty$.

For every finite value of $\omega_1$ the magnitude of $n\omega_1$, depending on $n$, changes discretely, but as $\omega_1$ tends to zero the change in $n\omega_1$ becomes continuous. The limits of the integration extend from $-\infty$ to $+\infty$ and the integrand becomes $f(t)\,e^{-j\omega t}$. The summating operation thus becomes an integrating operation with respect to a variable $\omega$ within infinite limits

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t}\,d\omega \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt. \qquad (12.4)$$

Formula (4) is written in the form of formulae (1) and (2), known as the *two-sided Fourier transformation* (in complex form).

A *symmetrical form* of writing formula (4) is also used in literature:

$$\left.\begin{aligned} f(t) &= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}\,d\omega, \\ F(j\omega) &= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} f(t)\,e^{-j\omega t}\,dt. \end{aligned}\right\}$$

However, it is more commonly written in the form of (1) and (2).

If the function is zero when $t < 0$, the Fourier transformation can be written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}\,d\omega, \tag{12.5}$$

where

$$F(j\omega) = \int_{0}^{\infty} f(t)e^{-j\omega t}\,dt. \tag{12.6}$$

The conversion expressed by these formulae is known as the one-sided *Fourier transformation.*

Formula (4) can be rewritten as follows:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\tau)e^{j\omega(t-\tau)}\,d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)\,d\tau \\ &\quad + j\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\tau)\sin\omega(t-\tau)\,d\tau. \end{aligned}$$

The second double integral is zero since the integrand is an odd function of $\omega$. Consequently,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)\,d\tau$$

or, in view of the fact that the integrand is even with respect to $\omega$,

$$f(t) = \frac{1}{\pi}\int_{0}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)\,d\tau. \tag{12.7}$$

The resulting formula is a Fourier integral in *trigonometric form.*

As with the previous expansion of a periodic function into a Fourier series, the value of the function at the discontinuity point, according to the Fourier integral, is equal to the arithmetical mean of the limit values on the left and right of the discontinuity.

## 2. Frequency Spectrum of Nonperiodic Function (Fourier Transform)

The formulae for the Fourier transformation are interrelated integral equations, each being a solution of the other. These formulae make it possible to convert a time function $f(t)$ into a frequency function $F(j\omega)$ and vice versa.

In order to express the relationship between $F(j\omega)$ and $f(t)$ imposed by the Fourier transformation on these functions, the conventional abbreviated form $F(j\omega) \sim f(t)$ can be used. Here the sign $\sim$ denotes the *correspondence.*

The frequency function $F(j\omega)$ is known as the *frequency spectrum* or Fourier transform of the given nonperiodic function $f(t)$. This spectrum is a function of the current frequency $\omega$.

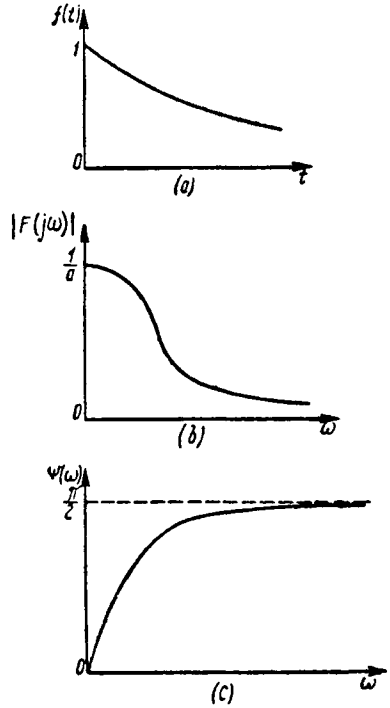



FIG. 2. Decreasing exponential function and its amplitude and phase spectra.

According to (1) *the given nonperiodic function $f(t)$ is the total of an infinitely large number of harmonics with infinitely small amplitudes* $(1/2\pi)\,F(j\omega)\,d\omega$ *in the whole range of frequencies from* $-\infty$ *to* $+\infty$.

In other words, representing a nonperiodic function as a Fourier integral means adding the undamped harmonic oscillations of an infinite and continuous spectrum of frequencies. The amplitudes of these oscillations, as the expression $(1/2\pi)\,F(j\omega)\,d\omega$ itself shows, depend on frequency.

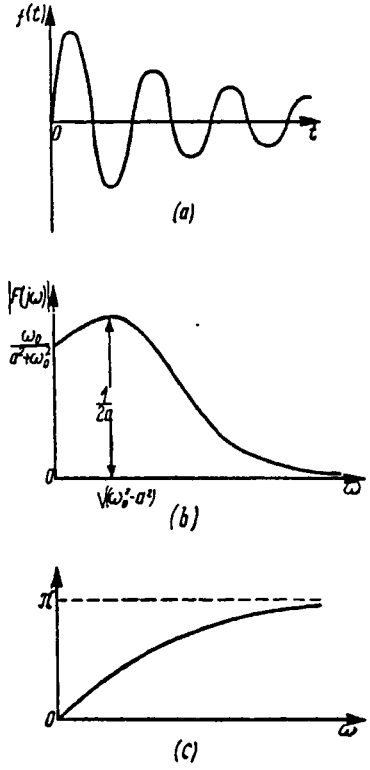



FIG. 3. Exponentially damped sinusoid and its amplitude and phase spectra.

This method of representing nonperiodic functions is somewhat similar to the method of expanding functions into a Fourier series. The Fourier integral then possesses the valuable property of reverting exactly to zero in the intervals of time in which the given nonperiodic function is zero, correctly describing the given function in the entire interval of time $(-\infty, +\infty)$.

Thus if, for example, a given nonperiodic function equals zero when $t < 0$ (a case of connecting the e.m.f. at the instant $t = 0$), representing it by a Fourier integral means that there exists an infinite number of oscillations of all possible frequencies with infinitesimally small amplitudes in

the entire interval of time from $t = -\infty$, their sum being identically zero for $t < 0$.

Representing a nonperiodic signal in the form of an aggregate of harmonic oscillations makes it possible to estimate the distribution of energy in the spectrum and the significance of the different frequency bands of the spectrum, using the spectral density. In addition, as will be shown later, a Fourier integral, in conjunction with the principle of superposition, makes it possible to analyse transient responses in linear electrical systems.

The frequency spectra of certain standard types of nonperiodic function are given below.

1. *Decreasing exponential function*

$$f(t) = \begin{cases} e^{-at} & \text{when} \quad t > 0, \\ 0 & \text{when} \quad t < 0. \end{cases}$$

$$F(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t}\, dt = \frac{1}{a + j\omega};$$

$$|F(j\omega)| = \frac{1}{\sqrt{(a^2 + \omega^2)}}; \quad \psi(\omega) = \arctan \frac{\omega}{a}.$$

The curves $f(t)$, $|F(j\omega)|$ and $\psi(\omega)$ are shown in Fig. 2.

2. *Exponentially damped sinusoid*

$$f(t) = \begin{cases} e^{-at} \sin \omega_0 t & \text{when} \quad t > 0, \\ 0 & \text{when} \quad t < 0. \end{cases}$$

$$F(j\omega) = \int_0^\infty e^{-at} \sin \omega_0 t e^{-j\omega t}\, dt = \frac{\omega_0}{(a + j\omega)^2 + \omega_0^2};$$

$$|F(j\omega)| = \frac{\omega_0}{\sqrt{[(a^2 + \omega_0)^2 + \omega^4 + 2\omega^2(a^2 - \omega_0^2)]}};$$

$$\psi(\omega) = \arctan \frac{2\omega a}{a^2 + \omega_0^2 - \omega^2}.$$

The curves $f(t)$, $|F(j\omega)|$ and $\psi(\omega)$ are shown in Fig. 3.

3. *Rectangular pulse*

$$f(t) = \begin{cases} 1 & \text{when} \quad 0 < t < \tau, \\ 0 & \text{when} \quad 0 > t > \tau. \end{cases}$$

$$F(j\omega) = \int_0^\tau e^{-j\omega t}\, dt = \frac{1}{j\omega}(1 - e^{-j\omega\tau}) = \frac{2}{\omega} \sin \frac{\omega\tau}{2} e^{-j\frac{\omega\tau}{2}},$$

$$|F(j\omega)| = \left| \frac{2}{\omega} \sin \frac{\omega\tau}{2} \right|.$$

When

$$\frac{\omega\tau}{2} = k\pi \quad (k = 1, 2, 3\ldots) \quad |F(j\omega)| = 0;$$

with

$$\frac{\omega\tau}{2} = \frac{\pi}{2}(2k + 1) \qquad |F(j\omega)| = \frac{2}{\omega} = \frac{\tau}{\frac{\pi}{2}(2k + 1)}.$$

Taking the sign of $\sin \omega\pi/2$ into account, the phase spectrum $\psi(\omega)$ is determined thus:

$$\psi(\omega) = \frac{\omega\tau}{2} \quad \text{when} \quad 0 < \omega < \frac{2\pi}{\tau};$$

$$\psi(\omega) = \frac{\omega\tau}{2} - \pi \quad \text{when} \quad \frac{2\pi}{\tau} < \omega < \frac{4\pi}{\tau}, \text{ etc.}$$

The curves $f(t)$, $|F(j\omega)|$ and $\psi(\omega)$ are shown in Fig. 4.

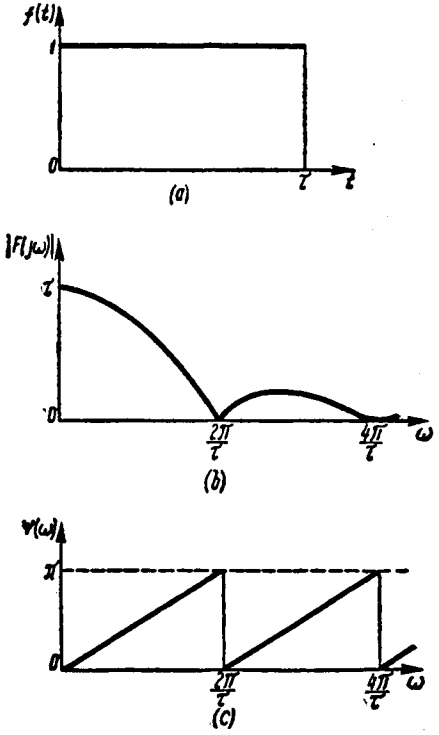



FIG. 4. Rectangular pulse and its amplitude and phase spectra.

4. *Pulse-modulated sine wave*

$$f(t) = \begin{cases} \sin \omega_0 t & \text{when } 0 < t < \tau, \\ 0 & \text{when } 0 > t > \tau. \end{cases}$$

Let

$$\tau = nT = n\frac{2\pi}{\omega_0},$$

where $T$ is the period of the sine curve and $n$ the number of periods in a length of time $\tau$.

$$F(j\omega) = \int_0^{\tau} \sin \omega_0 t e^{-j\omega t}\, dt$$

$$= \frac{-j\omega \sin \omega_0\tau - \omega_0 \cos \omega_0\tau}{\omega_0^2 - \omega^2} e^{-j\omega\tau} + \frac{\omega_0}{\omega_0^2 - \omega^2}.$$

Substituting $\tau = n\,2\pi/\omega_0$ and supposing that $n$ is an integer, we get:

$$F(j\omega) = \frac{-j\omega \sin n2\pi - \omega_0 \cos n2\pi}{\omega_0^2 - \omega^2} e^{-j\frac{\omega}{\omega_0}n2\pi} + \frac{\omega_0}{\omega_0^2 - \omega^2}$$

$$= \frac{\omega_0}{\omega_0^2 - \omega^2} - \frac{\omega_0 e^{-j\frac{\omega}{\omega_0}n2\pi}}{\omega_0^2 - \omega^2} = \omega_0 \frac{1 - e^{-j\frac{\omega}{\omega_0}n2\pi}}{\omega_0^2 - \omega^2}.$$

From this

$$|F(j\omega)| = \left| \frac{\omega_0}{\omega_0^2 - \omega^2} 2 \sin \frac{\omega}{\omega_0} n\pi \right|,$$

when $\omega = \omega_0$

$$|F(j\omega)| = \lim_{\omega \to \omega_0} \left| \frac{\omega_0 \frac{n\pi}{\omega_0} 2 \cos \frac{\omega}{\omega_0} n\pi}{-2\omega} \right| = \frac{n\pi}{\omega_0};$$

when

$$\frac{\omega}{\omega_0} n = k \quad (k = 0, 1, 2, 3, \ldots) \quad |F(j\omega)| = 0.$$

The curves $f(t)$, $|F(j\omega)|$ and $\psi(\omega)$ are shown in Fig. 5.

As $n$ increases the spectrum becomes concentrated in a narrower and narrower range of frequencies around $\omega_0$. When $n \to \infty$, the spectrum component with a frequency $\omega = \omega_0$ also tends to infinity. In physical terms this means that the energy of the signal expressed by the function under examination, is infinite and concentrated mainly about the frequency $\omega_0$. Functions determining a signal with infinite energy do not satisfy the condition of absolute convergence, and an integral of the type

in equation (2) is not applicable to functions of this kind. When analysing such functions harmonically, in order to obviate the limitations imposed by the condition of absolute convergence, it is necessary to use the artificial procedure described in Section 6.

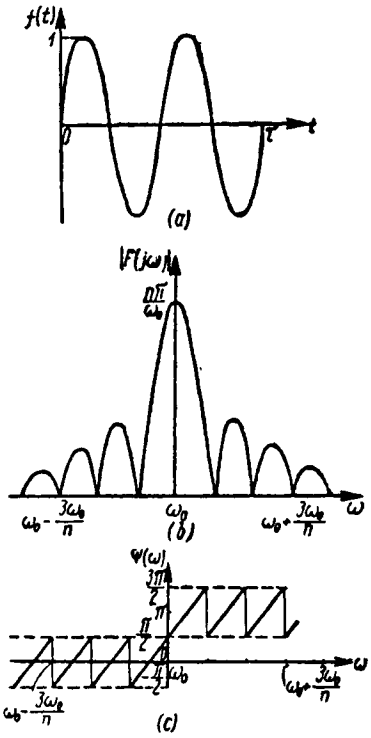



FIG. 5. Phase-modulated carrier wave and phase spectra.

## 3. Relation Between the Spectrum and the Envelope of the Factors of a Fourier Series

It is interesting to compare the formulae of an expansion into a Fourier series with an integral formula, as is shown in Table 1.

It follows from Chapter X, Section 5 that $F_n$ forms a discrete number of values which describe a broken frequency spectrum of the given periodic function.

In contrast to $F_n$ the frequency function $F(j\omega)$ is a *continuous* spectrum of the given nonperiodic function.

If the nonperiodic function $f(t)$ is given within finite limits, then

period $T$ can always be selected in such a way that the relationship

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t = \int_{-T/2}^{+T/2} f(t)\, e^{-j\omega t}\, \mathrm{d}t$$

is obtained.

The periodic function corresponding to this case will be characterized by the coefficients of an expansion into a Fourier series

$$\dot{F}_n = \frac{\omega_1}{\pi} \int_{-T/2}^{+T/2} f(t)\, e^{-jn\omega_1 t}\, \mathrm{d}t. \tag{12.8}$$

It follows from this that

$$\dot{F}_n = \frac{\omega_1}{\pi} [F(j\omega)]_{\omega = n\omega_1}. \tag{12.9}$$

TABLE 1.

| Fourier series (periodic function) | Fourier integral (nonperiodic function) |
|---|---|
| $f(t) = \frac{1}{2} \sum_{-\infty}^{+\infty} \dot{F}_n e^{jn\omega_1 t}$ | $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, \mathrm{d}\omega$ |
| $\dot{F}_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_1 t}\, \mathrm{d}t$ | $F(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t$ |
| $= \frac{\omega_1}{\pi} \int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_1 t}\, \mathrm{d}t$ | |
| $\dot{F}_n = F_n e^{-j\psi_n}$ | $F(j\omega) = \lvert F(j\omega)\rvert\, e^{-j\psi(\omega)}$ |

Thus, *the amplitude spectrum* $|F(j\omega)|$ *of a nonperiodic function given within finite limits on the scale* $\omega_1/\pi$ *shows the envelope of a discrete series of coefficients of* $F_n$ *of the expansion corresponding to the periodic repetition of the given function.* The functions $F(j\omega)$ and $\dot{F}_n$ can be real or complex.

On the basis of (2)

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, \mathrm{d}t$$

$$= \int_{-\infty}^{+\infty} f(t) \cos \omega t\, \mathrm{d}t - j \int_{-\infty}^{+\infty} f(t) \sin \omega t\, \mathrm{d}t, \tag{12.10}$$

from which the modulus (amplitude spectrum)

$$|F(j\omega)| = \sqrt{\left\{\left[\int_{-\infty}^{+\infty} f(t)\cos\omega t\,\mathrm{d}t\right]^2 + \left[\int_{-\infty}^{+\infty} f(t)\sin\omega t\,\mathrm{d}t\right]^2\right\}} \quad (12.11)$$

and the phase (phase spectrum)

$$\psi(\omega) = \arctan \frac{\int_{-\infty}^{+\infty} f(t)\sin\omega t\,\mathrm{d}t}{\int_{-\infty}^{+\infty} f(t)\cos\omega t\,\mathrm{d}t}. \quad (12.12)$$

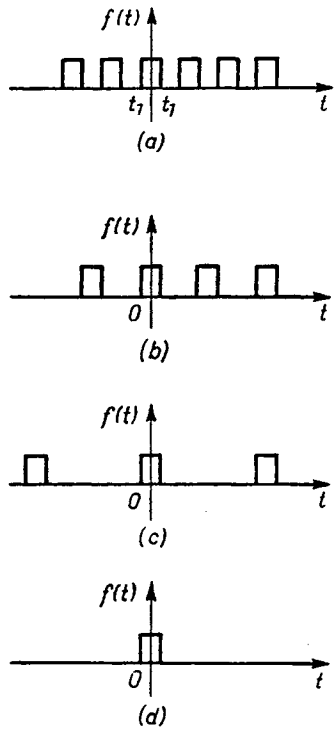



FIG. 6. Periodic time function, the period of which is increasing.

Equations (11) and (12) show that the modulus of the frequency spectrum is an even function of the frequency $\omega$ and the phase characteristic $\psi(\omega)$ is an odd function of $\omega$.

Consequently

$$\left.\begin{aligned} |F(j\omega)| &= |F(-j\omega)|, \\ \psi(\omega) &= -\psi(-\omega). \end{aligned}\right\} \quad (12.13)$$

In accordance with (8)

$$\psi_n(n\omega_1) = \arctan \frac{\int_{t_0}^{t_0+T} f(t) \sin n\omega_1 t \, dt}{\int_{t_0}^{t_0+T} f(t) \cos n\omega_1 t \, dt}. \tag{12.14}$$

It is evident from a comparison of formulae (12) and (14), that in the same limits of integration and with $\omega = n\omega_1$,

$$\psi_n(n\omega_1) = \psi(\omega). \tag{12.15}$$

This also follows from (9) in so far as the expressions contained in the left- and right-hand sides of (9) are generally complex.

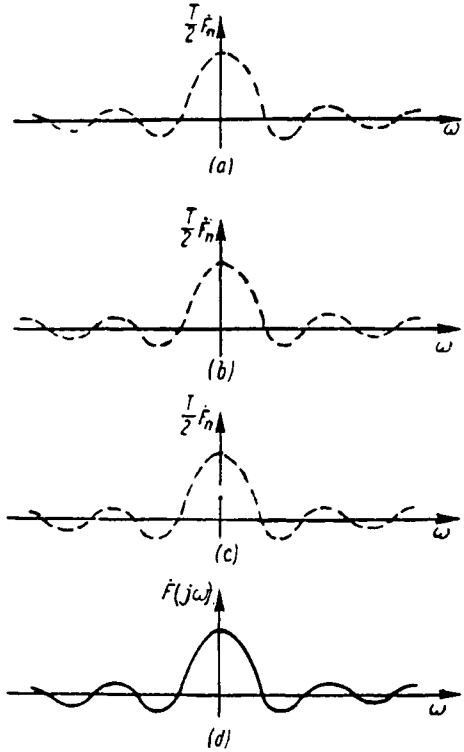



FIG. 7. Spectra of a time function corresponding to an increasing period.

As an example, Fig. 6*a*, *b* and *c* (p. 479) shows periodic functions with different periods and Fig. 6*d* shows a nonperiodic function. The corresponding frequency characteristics are given in Fig. 7 (Ref. 59). The greater the pulse-sequence period, the thicker (denser) the spectrum line, since the

frequency spacing between pulses is inversely proportional to the period. When $T = \infty$ a single pulse (a nonperiodic function) is obtained, with a corresponding continuous spectrum.

The periodic functions (Fig. 6*d*) in the interval from $-T/2$ to $T/2$ are given by the conditions

$$f(t) = \begin{cases} 1 & \text{when} \quad -t_1 < t < +t_1, \\ 0 & \text{when} \quad -t_1 > t > +t_1. \end{cases} \tag{12.16}$$

The same conditions are in force on the entire time axis $t$ for the nonperiodic function (Fig. 6*d*).

According to (8)

$$\dot{F}_n = \frac{2}{T}\int_{-t_1}^{+t_1} e^{-jn\omega_1 t}\,dt = \frac{2}{-jTn\omega_1} e^{-jn\omega_1 t}$$

$$= \frac{4}{T}\cdot\frac{e^{jn\omega_1 t_1} - e^{-jn\omega_1 t_1}}{2jn\omega_1} = \frac{4t_1}{T}\frac{\sin n\omega_1 t_1}{n\omega_1 t_1}. \tag{12.17}$$

Equation (17) is the formula for the envelope of discrete values of $\dot{F}_n$.

According to (2)

$$F(j\omega) = \int_{-t_1}^{+t_1} e^{-j\omega t}\,dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-t_1}^{+t_1} = 2t_1\frac{\sin \omega t_1}{\omega t_1}. \tag{12.18}$$

In accordance with (9) the relationship between the ordinates of the curves (Fig. 7) is

$$\frac{T}{2}\dot{F}_n = F(j\omega)_{\omega_1 = n\omega_1}.$$

The real and imaginary parts of the spectrum, like the amplitude and phase characteristics, are interdependent frequency functions.

The relationship between the real and imaginary parts of the spectrum is analogous with the relationship between the frequency characteristics of the resistance and the reactance examined in Chapter VI, Section 18.

### 4. Cases of Symmetry of a Nonperiodic Function

If it is symmetrical, a nonperiodic function $f(t)$ can be either odd or even with respect to $t$.

In the first case $f(t) = f(-t)$, and in the second case $f(t) = -f(-t)$.

According to (2)

$$F(j\omega) = F_1(\omega) - jF_2(\omega), \tag{12.19}$$

where

$$\left.\begin{aligned} F_1(\omega) &= \int_{-\infty}^{+\infty} f(t)\cos\omega t\,\mathrm{d}t, \\ F_2(\omega) &= \int_{-\infty}^{+\infty} f(t)\sin\omega t\,\mathrm{d}t. \end{aligned}\right\} \tag{12.20}$$

The product of an even function and an odd function is an odd function, and the product of two even or two odd functions is an even function.

If the integrand is odd, the principal value of the integral* within the limits $-\infty$ to $+\infty$ is zero.

In view of the fact that a cosine and sine are, respectively, the even and odd function of $t$, then from (20), for an even function $f(t)$,

$$F(j\omega) = F_1(\omega) = 2\int_0^{\infty} f(t)\cos\omega t\,\mathrm{d}t, \tag{12.21}$$

i.e. $F(j\omega)$ is real and an even function of $\omega$.

According to (1)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\cos\omega t\,\mathrm{d}\omega + j\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\sin\omega t\,\mathrm{d}\omega. \tag{12.22}$$

When $F_1(\omega)$ is even the integrand $F(j\omega)\sin\omega t$ is an odd function of $\omega$ (since the sine is an odd function) and therefore the second integral in formula (22) contributes nothing. Then

$$f(t) = \frac{1}{\pi}\int_{-0}^{+\infty} F_1(\omega)\cos\omega t\,\mathrm{d}\omega. \tag{12.23}$$

When function $f(t)$ is odd

$$F(j\omega) = -jF_2(\omega) = -j2\int_0^{\infty} f(t)\sin\omega t\,\mathrm{d}t, \tag{12.24}$$

i.e. $F(j\omega)$ is imaginary and an odd function of $\omega$.

When $F_2(\omega)$ is odd the integrand $F(j\omega)\cos\omega t$ is an odd function of $\omega$ (since the cosine is an even function) and therefore the first integral in formula (22) contributes nothing. Then

$$f(t) = \frac{1}{\pi}\int_0^{\infty} F_2(\omega)\sin\omega t\,\mathrm{d}\omega. \tag{12.25}$$

The above formulae can be applied to a function which is not symmetrical if it is represented in the form of the sum of two components, one odd and the other even.

* The value obtained by going to negative and positive infinity in a symmetrical fashion.

In fact, the even function

$$f_1(t) = \frac{f(t) + f(-t)}{2}$$

and the odd function

$$f_2(t) = \frac{f(t) - f(-t)}{2}$$

together give the given function $f(t)$

$$f_1(t) + f_2(t) = f(t).$$

Thus any non-symmetrical function can be expressed by using the trigonometric formulae (23) and (25), $F(j\omega)$ being a complex quantity in this case.

## 5. Basic Properties of Fourier Transforms

If $F(j\omega)$ is the spectral characteristic of the time function $f(t)$, then, using the abbreviated form of the expression for the correspondence between the spectral characteristic and the time function, we get

$$F(j\omega) \sim f(t).$$

With this condition as the initial data it can be shown that certain operations on $f(t)$ correspond exactly to certain operations on $F(j\omega)$, and vice versa.*

The main properties of a Fourier transform, which in practice make it considerably easier to find time functions or frequency spectra, are given below (Refs. 55 and 79). If, for example, one of the operations described below is carried out on a time function whose spectrum is known, the spectrum corresponding to the resultant time function can be found from the original spectrum by using certain rules (see Examples 1–3).

In turn, if the spectrum is subjected to some operation, the time function corresponding to the new spectrum can also be found with the said rules. These rules are used for studying transient responses.

### Interchangeability of Independent Variables

$$f(\omega) \sim \frac{1}{2\pi} F(-jt). \tag{12.26}$$

* For functions satisfying the limitations mentioned in Section 1, the improper integrals in equations (1) and (2) converge uniformly. In formula (4), therefore, it is possible to change the order of integration and to differentiate under the integral sign in (1) and (2).

For proof, let us introduce the factor $1/(2\pi)$ after the integral sign in (1) and, reversing the sign of $\omega$, obtain

$$f(t) = \int_{+\infty}^{-\infty} \frac{F(-j\omega)}{2\pi} e^{-j\omega t}(-\mathrm{d}\omega) = \int_{-\infty}^{+\infty} \frac{F(-j\omega)}{2\pi} e^{-j\omega t}\,\mathrm{d}\omega. \quad (12.27)$$

In turn, multiplying both parts of equation (2) by $1/(2\pi)$ and also reversing the sign of $\omega$, we get

$$\frac{1}{2\pi} F(-j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\, e^{j\omega t}\,\mathrm{d}t. \quad (12.28)$$

Reversing the positions of $t$ and $\omega$ in (27) and (28) we find

$$f(\omega) = \int_{-\infty}^{+\infty} \frac{F(-jt)}{2\pi} e^{-j\omega t}\,\mathrm{d}t,$$

$$\frac{1}{2\pi} F(-jt) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)\, e^{j\omega t}\,\mathrm{d}\omega.$$

The resulting equations prove the correspondence given in (26).

**Replacing a Time Function and a Frequency Spectrum by Conjugate Complex Values (Changing the Sign of $\omega$)**

$$\overset{*}{f}(t) \sim \overset{*}{F}(-j\omega). \quad (12.29)$$

If $F(t)$ and $F(j\omega)$ are complex values then, by substituting the conjugate value $\overset{*}{F}(j\omega)$ for $F(j\omega)$ in (1) and at the same time changing the sign in front of $j$ in the exponent, we find

$$\overset{*}{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overset{*}{F}(j\omega)\, e^{-j\omega t}\,\mathrm{d}\omega,$$

since a function complexly conjugate with $F(j\omega)\, e^{j\omega t}$ comes after the integral sign.

In contrast with formula (1) the exponent in the last expression has a minus sign. In order to obtain a formula for the Fourier transformation it is sufficient to reverse the sign of $\omega$, i.e.

$$\overset{*}{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overset{*}{F}(-j\omega)\, e^{j\omega t}\,\mathrm{d}\omega.$$

Consequently, the Fourier transformation is satisfied for complex values of $\overset{*}{F}(-j\omega)$ and $\overset{*}{f}(t)$.

**Linearity of the Fourier Transform**

$$\sum_{i=1}^{n} a_i f_i(t) \sim \sum_{i=1}^{n} a_i F_i(j\omega). \tag{12.30}$$

On the basis of (1)

$$\int_{-\infty}^{+\infty} \sum_{i=1}^{n} a_i f_i(t)\, e^{-j\omega t}\, dt = \sum_{i=1}^{n} a_i \int_{-\infty}^{+\infty} f_i(t)\, e^{-j\omega t}\, dt = \sum_{i=1}^{n} a_i F_i(j\omega),$$

which is equivalent to (30). The property of linearity confirms the applicability of the superposition method to frequency spectra.

**Differentiating a Time Function**

For a one-sided Fourier transformation,

$$\frac{d}{dt} f(t) \sim j\omega F(j\omega) - f(0). \tag{12.31}$$

This formula is given in Chapter XIII, Section 7.
For a two-sided Fourier transformation,

$$\frac{d}{dt} f(t) \sim j\omega F(j\omega).$$

**Integrating a Time Function**

For a one-sided Fourier transformation,

$$\int_0^t f(t)\, dt \sim \frac{F(j\omega)}{j\omega}. \tag{12.32}$$

Let us denote the unknown spectrum of the function $\int_0^t f(t)\, dt$ by $\Phi(j\omega)$.

On the basis of (31), the function $\frac{d}{dt}\int_0^t f(t)\, dt = f(t)$ has a spectrum $j\omega\Phi(j\omega)$, since

$$\left[\int_0^t f(t)\, dt\right]_{t=0} = 0.$$

According to the data, function $f(t)$ has a spectrum $F(j\omega)$. Consequently, $F(j\omega) = j\omega\Phi(j\omega)$ from which

$$\Phi(j\omega) = \frac{F(j\omega)}{j\omega}.$$

In the case of a two-sided Fourier transformation,

$$\int_{-\infty}^{t} f(t)\,dt \sim \frac{F(j\omega)}{j\omega}.$$

**Differentiating a Frequency Spectrum**

$$\frac{d}{d\omega} F(j\omega) \sim (-jt) f(t). \tag{12.33}$$

Differentiating both parts of equation (2) with respect to $\omega$, we get

$$\frac{dF(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} (-jt) f(t)\, e^{-j\omega t}\,dt,$$

which is equivalent to (33).

**Integrating a Frequency Spectrum**

$$\int_{-\infty}^{\omega} F(j\omega)\,d\omega \sim \frac{f(t)}{-jt}. \tag{12.34}$$

Let us suppose that $\int_{-\infty}^{\infty} F(j\omega)\,d\omega$ is the spectrum of the function $\Phi(t)$. Then

$$\frac{d}{d\omega} \int_{-\infty}^{\omega} F(j\omega)\,d\omega = F(j\omega)$$

is the spectrum of the function $(-jt)\,\Phi(t)$.

According to the data, $F(j\omega)$ is the spectrum of the function $f(t)$. Consequently, $(-jt)\,\Phi(t) = f(t)$, from which

$$\Phi(t) = \frac{f(t)}{-jt}.$$

**Changing the Scale of an Independent Variable**

$$f(\alpha t) \sim \frac{1}{\alpha} F\left(j\frac{\omega}{\alpha}\right). \tag{12.35}$$

Substituting the function $f(\alpha t)$ for $f(t)$ in the integral of (2) yields

$$\int_{-\infty}^{+\infty} f(\alpha t)\, e^{-j\omega t}\,dt = \frac{1}{\alpha}\int_{-\infty}^{+\infty} f(\alpha t)\, e^{-j\frac{\omega}{\alpha}\alpha t}\,d\alpha t = \frac{1}{\alpha}\int_{-\infty}^{+\infty} f(\tau)\, e^{-j\frac{\omega}{\alpha}\tau}\,d\tau$$

$$= \frac{1}{\alpha} F\left(j\frac{\omega}{\alpha}\right),$$

which is proved by (35).

**Shifting a Time Function**

$$f(t \pm t_0) \sim e^{\pm j\omega t_0} F(j\omega). \tag{12.36}$$

Substituting the function $F(j\omega)\, e^{\pm j\omega t_0}$ for $F(j\omega)$ in the integral of (1) gives

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\, e^{\pm j\omega t_0} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega(t \pm t_0)}\, d\omega = f(t \pm t_0).$$

Thus, multiplying the spectral density by $e^{\pm j\omega t_0}$ shifts the time function to the left (to make it lead) or right (to make it lag) by $t_0$ (Fig. 8).

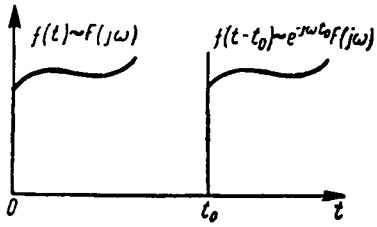



FIG. 8. Multiplication of a spectrum by $e^{-j\omega t_0}$ (shifting the time function by $t_0$).

The reverse proposition is also self-evident: on the transition from $f(t)$ to $f(t \pm t_0)$, i.e. when the time function $f(t)$ is shifted to lead or lag by the magnitude $t_0$, all the components of the spectrum $F(j\omega)$ shift by an angle $\pm\omega t_0$ directly proportional to the frequency $\omega$. The amplitude spectrum remains constant, because

$$|e^{\pm j\omega t_0} F(j\omega)| = |F(j\omega)|,$$

and the phase spectrum changes by $\pm\omega t_0$.

**Shifting a Frequency Spectrum**

$$f(t)e^{\pm j\omega_0 t} \sim F(j\omega \mp j\omega_0). \tag{12.37}$$

Substituting the complex quantity $f(t)\, e^{\pm j\omega_0 t}$ for the function $f(t)$ in the integral of (2) gives

$$\int_{-\infty}^{+\infty} f(t)e^{\pm j\omega_0 t} e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} f(t)\, e^{-j(\omega \mp \omega_0)t}\, dt = F(j\omega \mp j\omega_0).$$

Thus when the function is multiplied by $e^{\pm j\omega_0 t}$ the spectrum shifts to the right or left by the value $\omega_0$ over the scale of frequencies (Fig. 9).

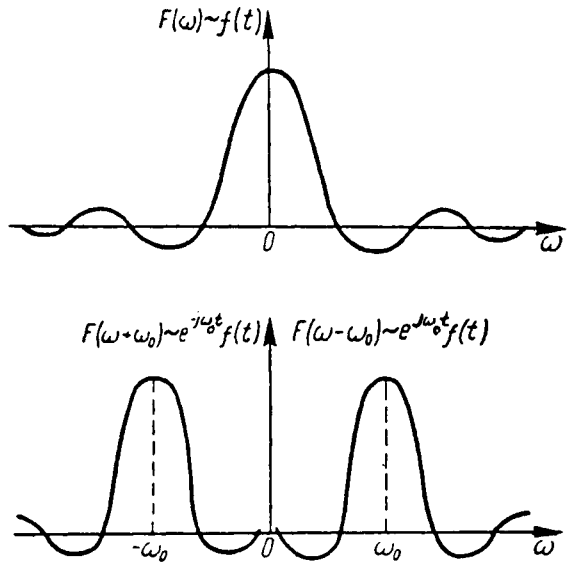



FIG. 9. Multiplication of a time function by $e^{\pm j\omega_0 t}$ (shifting the spectrum by $\pm\omega_0$).

**Multiplying a Time Function by a Cosine or Sine**

$$\left.\begin{aligned} f(t)\cos\omega_0 t &\sim \frac{1}{2}\,[F(j\omega - j\omega_0) + F(j\omega + j\omega_0)], \\ f(t)\sin\omega_0 t &\sim \frac{1}{2j}\,[F(j\omega - j\omega_0) - F(j\omega + j\omega_0)]. \end{aligned}\right\} \tag{12.38}$$

Cosine and sine oscillations with an envelope $f(t)$ are expressed by the exponential functions

$$f(t)\cos\omega_0 t = \frac{1}{2}\,[f(t)\,e^{j\omega_0 t} + f(t)\,e^{-j\omega_0 t}],$$

$$f(t)\sin\omega_0 t = \frac{1}{2j}\,[f(t)\,e^{j\omega_0 t} - f(t)\,e^{-j\omega_0 t}].$$

Consequently, formulae (38) can be obtained from (37).

**Multiplying Frequency Spectra**

For a one-sided Fourier transformation

$$F_1(j\omega)\,F_2(j\omega) \sim \int_0^t f_1(\tau) f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau) f_2(\tau)\,d\tau. \tag{12.39}$$

If

$$F_1(j\omega) = \int_{-\infty}^{+\infty} f_1(\tau)\, e^{-j\omega\tau}\, \mathrm{d}\tau, \quad F_2(j\omega) = \int_{-\infty}^{+\infty} f_2(\theta)\, e^{-j\omega\theta} \mathrm{d}\theta,$$

then

$$F_1(j\omega)\, F_2(j\omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(\tau) f_2(\theta)\, e^{-j\omega(\tau+\theta)}\, \mathrm{d}\tau\, \mathrm{d}\theta,$$

where $\tau$ and $\theta$ are independent variables.

Let us substitute $\theta = t - \tau$, $\mathrm{d}\theta = \mathrm{d}t$. Consequently,

$$F_1(j\omega)\, F_2(j\omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(\tau) f_2(t - \tau)\, e^{-j\omega t}\, \mathrm{d}\tau\, \mathrm{d}t,$$

which is equivalent to (39) for a one-sided Fourier transformation.

The variables $\tau$ and $\theta$ are interchangeable and therefore the subscripts 1 and 2 can be reversed.

**Multiplying Time Functions**

$$f_1(t) f_2(t) \sim \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_1(j\Omega)\, F_2(j\omega - j\Omega)\, \mathrm{d}\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_1(j\omega - j\Omega)\, F_2(j\Omega)\, \mathrm{d}\Omega. \tag{12.40}$$

If

$$f_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_1(j\Omega)\, e^{j\Omega t}\, \mathrm{d}\Omega$$

and

$$f_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_2(j\xi)\, e^{j\xi t} \mathrm{d}\xi,$$

then

$$f_1(t) f_2(t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(j\Omega)\, F_2(j\xi)\, e^{j(\Omega+\xi)t}\, \mathrm{d}\Omega\, \mathrm{d}\xi,$$

where $\Omega$ and $\xi$ are independent variables.

Let us substitute $\xi = \omega - \Omega$, $\mathrm{d}\xi = \mathrm{d}\omega$. Consequently,

$$f_1(t) f_2(t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(j\Omega)\, F_2(j\omega - j\Omega)\, e^{j\omega t}\, \mathrm{d}\Omega\, \mathrm{d}\omega,$$

which is equivalent to (40). The variables $\Omega$ and $\xi$ are interchangeable and therefore the subscripts 1 and 2 may be reversed.

**Integrating the Product of Time Functions**

$$\int_{-\infty}^{+\infty} f_1(t)\overset{*}{f_2}(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(j\omega)\overset{*}{F_2}(j\omega)\,d\omega. \tag{12.41}$$

The property is evident from the following steps:

$$\begin{aligned}\int_{-\infty}^{+\infty} f_1(t)\overset{*}{f_2}(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_1(t)\,dt\int_{-\infty}^{+\infty}\overset{*}{F_2}(j\omega)\,e^{-j\omega t}\,d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\overset{*}{F_2}(j\omega)\left[\int_{-\infty}^{+\infty} f_1(t)\,e^{-j\omega t}\,dt\right]d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(j\omega)\overset{*}{F_2}(j\omega)\,d\omega.\end{aligned}$$

**Parseval's Relation***

$$\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(j\omega)|^2\,d\omega. \tag{12.42}$$

This equality follows from (41) on the assumption that

$$f_1(t) = f_2(t) = f(t) \quad \text{and} \quad F_1(j\omega) = F_2(j\omega) = F(j\omega).$$

In cases when $f(t)$ is a real value, the sign of the absolute value in the left-hand side of (42) is dropped.

If $f(t)$ denotes the current in a network, the integral in the left-hand side of (42) is proportional to the energy released by the current in this network over an infinite interval of time. Consequently the right-hand side of (42) is also proportional to the energy released by the current. In conjunction with this a density of energy proportional to $|F(j\omega)|^2$ can be ascribed to each section of the frequency spectrum.

**Finding the Frequency Spectrum of a Section of a Periodic Function**

$$F_1(j\omega) = (1 - e^{-j\omega\tau})\,F(j\omega), \tag{12.43}$$

where $F(j\omega)$ is the frequency spectrum of the function $f(t)$ which is equal to zero when $t < 0$ and periodic when $t > 0$ (the spectrum of such a function is obtained by using the generalized Fourier integral—see Section 6); $\tau$ is the time, a multiple of a whole number of periods $f(t)$.

$F_1(j\omega)$ is the frequency spectrum of the function $f_1(t)$ which coincides with $f(t)$ in the interval of time from 0 to $\tau$ and is equal to zero outside

* Also referred to in literature as the Rayleigh theorem.

this interval. In view of the fact that a whole number of periods $f(t)$ is contained in the said interval of time, the function $f_1(t)$ can be obtained by the superposition method—as the difference between two functions: $f(t)$ and the function obtained by shifting $f(t)$ to the right of the section $\tau$, i.e.

$$f_1(t) = f(t) - f(t - \tau).$$

On the strength of (30) and (36)

$$F_1(j\omega) = F(j\omega) - e^{-j\omega\tau} F(j\omega).$$

**Finding a Time Function from a Frequency Spectrum (Switching Formulae)**

$$\frac{F_1(j\omega)}{F_2(j\omega)} \sim \sum \operatorname{Res} \frac{F_1(p)e^{pt}}{F_2(p)} = j \sum_{k=1}^{n} \frac{F_1(j\omega_k)e^{j\omega_k t}}{\left[\frac{\mathrm{d}}{\mathrm{d}\omega} F_2(j\omega)\right]_{\omega=\omega_k}}, \quad (12.44\text{a})$$

$$\frac{F_1(j\omega)}{j\omega F_3(j\omega)} \sim \sum \operatorname{Res} \frac{F_1(p)\, e^{pt}}{pF_3(p)}$$

$$= \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n-1} \frac{F_1(j\omega_k)e^{j\omega_k t}}{\omega_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega} F_3(j\omega)\right]_{\omega=\omega_k}}. \quad (12.44\text{b})$$

Here $\omega_k$ denotes the roots of the equation $F_2(j\omega) = 0$ or $F_3(j\omega) = 0$.

The switching formulae (44a) and (44b) are given for one-sided Fourier transformation in Chapter XIII, Section 17.

The main properties of the Fourier transformation which have been described above are given in a table in Appendix II.

EXAMPLE 1. To use the formula for differentiating a time function (31) in order to find the frequency spectrum of a function (pulse-modulated cosine wave)

$$f_1(t) = \begin{cases} \cos \omega_0 t & \text{when } 0 < t < \tau, \\ 0 & \text{when } 0 > t > \tau, \end{cases}$$

where $\tau = nT = n2\pi/\omega_0$, $T$ is the period of the cosine curve, and $n$ is the number of periods during the time $\tau$.

According to Section 2 (subsection 4) the function $f(t)$ shown in Fig. 5*a* has a spectrum

$$F(j\omega) = \omega_0 \frac{1 - e^{-j\frac{\omega}{\omega_0} n2\pi}}{\omega_0^2 - \omega^2}.$$

In view of the fact that $\cos \omega_0 t = (1/\omega_0)\, \mathrm{d}/\mathrm{d}t \sin \omega_0 t$, let us use formula (31). In this case $f_1(t) = (1/\omega_0) f'(t)$ since $f(0) = \sin 0 = 0$:

$$\frac{1}{\omega_0} f'(t) \sim j\omega \frac{1 - e^{-j\frac{\omega}{\omega_0} n2\pi}}{\omega_0^2 - \omega^2} = \frac{\omega}{\omega_0^2 - \omega^2} 2 \sin \frac{\omega n\pi}{\omega_0} e^{-j\left(\frac{\omega}{\omega_0} n\pi - \pi\right)}.$$

The amplitude spectrum of the function

$$|F_1(j\omega)| = \left| \frac{\omega}{\omega_0^2 - \omega^2} 2 \sin \frac{\omega}{\omega_0} n\pi \right|.$$

The curves $f_1(t)$, $|F(j\omega)|$ and $\psi(\omega)$ are shown in Fig. 10.

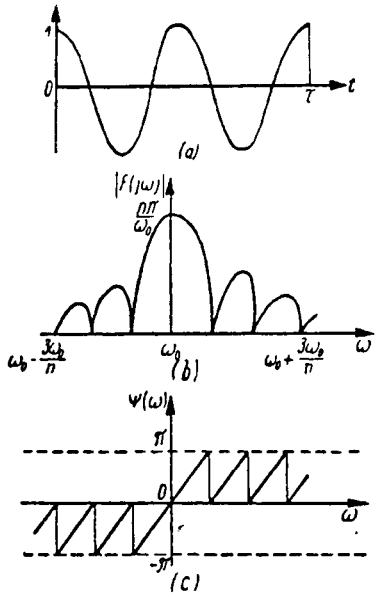



FIG. 10. Function $f_1(t)$ and its amplitude and phase spectra (Example 1).

The resulting correlations are confirmed by formula (43), taking into account that in this case (see Section 6) $F(j\omega) = j\omega/(\omega_0^2 - \omega^2)$.

EXAMPLE 2. To use the formula for the displacement of a time function (36) in order to find the spectrum of a function

$$f_1(t) = \begin{cases} e^{-a(t-t_0)} & \text{when } t > t_0, \\ 0 & \text{when } t < t_0. \end{cases}$$

In accordance with Section 2 (subsection 1), the function $f(t)$ shown in Fig. 2*a* has a frequency spectrum

$$F(j\omega) = \frac{1}{a + j\omega}.$$

If the function $f(t)$ is shifted to the lagging side by a length of time $t_0$, the given function $f_1(t)$ is obtained. Using formula (36) we get

$$F_1(j\omega) = e^{-j\omega t_0} F(j\omega) = \frac{e^{-j\omega t_0}}{a + j\omega} = \frac{1}{\sqrt{(a^2 + \omega^2)}} e^{-j\left(\arctan \frac{\omega}{a} + \omega t_0\right)}.$$

As should be expected, the functions $f(t)$ and $f_1(t)$ have an identical amplitude spectrum and the phase spectrum of the function $f_1(t)$ is shifted with respect to the phase spectrum of the function $f(t)$ by an angle $\omega t_0$, which is directly proportional to the frequency.

EXAMPLE 3. To use formula (38) for multiplying a time function by a sine in order to find the frequency spectrum of the function

$$f_1(t) = \begin{cases} \sin \omega_0 t & \text{when } 0 < t < \tau, \\ 0 & \text{when } 0 > t > \tau. \end{cases}$$

The given time function is the sine wave with an envelope $f(t)$ shown in Fig. 4*a*:

$$f_1(t) = f(t) \sin \omega_0 t.$$

According to Section 2 (subsection 3), function $f(t)$ has the frequency spectrum

$$F(j\omega) = \frac{1 - e^{-j\omega\tau}}{j\omega}.$$

Consequently, on the basis of (38), the spectrum of the function $f_1(t)$ has the form

$$F_1(j\omega) = \frac{1}{2j} [F(j\omega - j\omega_0) - F(j\omega + j\omega_0)]$$

$$= \frac{1}{2j} \left[ \frac{1 - e^{-j(\omega - \omega_0)\tau}}{j(\omega - \omega_0)} - \frac{1 - e^{-j(\omega + \omega_0)\tau}}{j(\omega + \omega_0)} \right]$$

$$= \frac{\omega_0 - e^{-j\omega\tau} (\omega_0 \cos \omega_0 \tau + j\omega \sin \omega_0 \tau)}{\omega_0^2 - \omega^2}.$$

When $\omega_0 \tau = n2\pi$, $n$ being an integer,

$$F_1(j\omega) = \frac{\omega_0 (1 - e^{-j\omega\tau})}{\omega_0^2 - \omega^2},$$

which agrees with the result of subsection 4 in Section 2.

The resulting equation can also be found from formula (43) taking into account that $F(j\omega) = \omega_0/(\omega_0^2 - \omega^2)$ in the case being studied (see Section 6).

## 6. Generalized Form of the Fourier Integral

Among the conditions imposed on a nonperiodic function $f(t)$ in Section 1 we mentioned that the function should be absolutely convergent within infinite limits. This condition obtains for time functions of voltages, currents, etc., which occur in actual networks. But sometimes functions to which the double Fourier integral cannot be applied directly, in the form in

which it was given above, are used in the theoretical analysis of transients. For example, these include step functions of the kind

$$f(t) = \begin{cases} 1 & \text{when} \quad t > 0 \\ 0 & \text{when} \quad t < 0 \end{cases} \tag{12.45}$$

$$f(t) = \begin{cases} \sin \omega_0 t & \text{when} \quad t > 0, \\ 0 & \text{when} \quad t < 0, \end{cases} \tag{12.46}$$

which are illustrated in Fig. 11.

The following artificial procedure is adopted for such functions.

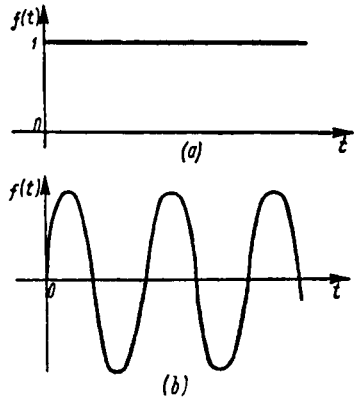



FIG. 11. Step functions.

Let us suppose we are given a nonperiodic function $f(t)$ which satisfies the Dirichlet conditions in any finite interval when $t > 0$, equals zero for $t < 0$ and does not satisfy the condition of absolute convergence within the interval from 0 to $\infty$.

Let us suppose that there is a positive number $c$ which makes the integral $\int_{-\infty}^{+\infty} |e^{-ct} f(t)|\, dt$ converge. In this case

$$e^{-ct} f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t}\, d\omega \int_{0}^{\infty} e^{-ct} f(t)\, e^{-j\omega t}\, dt \tag{12.47}$$

or

$$e^{-ct} f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(c + j\omega) e^{j\omega t}\, d\omega,$$

where

$$F(c + j\omega) = \int_{0}^{\infty} f(t) e^{-(c+j\omega)t}\, dt. \tag{12.48}$$

The given function $f(t)$ can be regarded as the limit of equation (47) with $c$ tending to zero, i.e. with the attenuation factor $e^{-ct}$ tending to unity

$$f(t) = \lim_{c \to 0} e^{-ct} f(t) = \lim_{c \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(c + j\omega) e^{j\omega t} \, d\omega. \qquad (12.49)$$

Equation (49) is one of the generalized forms of the Fourier integral.

In its turn the frequency spectrum can be found as the limit of the function $F(c + j\omega)$ with $c$ tending to zero:

$$F(j\omega) = \lim_{c \to 0} F(c + j\omega). \qquad (12.50)$$

As an example, let us examine step functions in the form of (45) and (46).

With respect to the unit step function (45) we get, from (49),

$$\begin{aligned} f(t) &= \lim_{c \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} \, d\omega \int_{0}^{\infty} e^{-(c + j\omega)t} \, dt \\ &= \lim_{c \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{j\omega t}}{c + j\omega} \, d\omega \\ &= \lim_{c \to 0} \frac{1}{2\pi} \left[ \int_{-\infty}^{+\infty} \frac{\cos \omega t}{c + j\omega} \, d\omega + j \int_{-\infty}^{+\infty} \frac{\sin \omega t}{c + j\omega} \, d\omega \right] \\ &= \lim_{c \to 0} \frac{1}{2\pi} \left[ \pi e^{-ct} + j \int_{-\infty}^{+\infty} \frac{\sin \omega t}{c + j\omega} \, d\omega \right]. \end{aligned}$$

Proceeding to the limit under the integral sign we obtain an equation for the unit step function (45) in integral form

$$f(t) = \tfrac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \omega t}{\omega} \, d\omega = \begin{cases} 1 & \text{when } t > 0, \\ 0 & \text{when } t < 0. \end{cases} \qquad (12.51)$$

In accordance with (50) the spectrum of this function is

$$F(j\omega) = \lim_{c \to 0} \frac{1}{c + j\omega} = \frac{1}{j\omega}.$$

Curves of the modulus $|F(j\omega)| = 1/|\omega|$ and phase $\psi(\omega) = \pi/2$ of the spectrum are given in Fig. 12.

Thus, the spectrum of a unit step function (45) contains any frequency, the energy of the low-frequency components predominating. When $\omega \to 0$, the modulus $|F(j\omega)|$ tends to infinity* and the phase $\psi$ is the same for all frequencies.

* If the function does not satisfy the condition of absolute convergence within finite limits, the amplitude spectrum $|F(j\omega)|$ will equal infinity, if only at one point.

Applying the transformation (49) to a sinusoidal step function (step-modulated carrier-wave), we get

$$f(t) = \lim_{c\to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t}\, d\omega \int_0^{\infty} \sin \omega_0 t e^{-(c+j\omega)t}\, dt$$

$$= \lim_{c\to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\omega_0 e^{j\omega t}}{(c+j\omega)^2 + \omega_0^2}\, d\omega$$

$$= \lim_{c\to 0} \frac{1}{4\pi j} \left[ \pi e^{-(c-j\omega_0)t} - \pi e^{-(c+j\omega_0)t} + 2j \int_{-\infty}^{+\infty} \frac{\omega_0 \cos \omega t}{(c+j\omega)^2 + \omega_0^2}\, d\omega \right]. \tag{12.52}$$

Proceeding to the limit under the integral sign we get a sinusoidal step function (46) in integral form

$$f(t) = \frac{\sin \omega_0 t}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\omega_0 \cos \omega t}{\omega_0^2 - \omega^2}\, d\omega = \begin{cases} \sin \omega_0 t & \text{when } t > 0, \\ 0 & \text{when } t < 0. \end{cases} \tag{12.53}$$

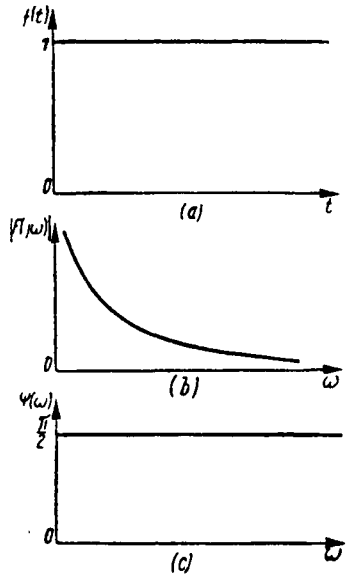



FIG. 12. Unit step function and its amplitude and phase spectra.

In accordance with (50) the frequency spectrum of this function is

$$F(j\omega) = \lim_{c\to 0} \frac{\omega_0}{(c+j\omega)^2 + \omega_0^2} = \frac{\omega_0}{\omega_0^2 - \omega^2}. \tag{12.54}$$

The curves $F|(j\omega)|$ and $\psi(\omega)$ are shown in Fig. 13; the phase angle $\psi(\omega)$ equals zero when $\omega < \omega_0$ and equals $\pi$ when $\omega > \omega_0$.

The integral form of a cosinusoidal step function can be obtained similarly:

$$f(t) = \lim_{c \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t}\, d\omega \int_0^{\infty} \cos \omega_0 t e^{-(c+j\omega)t}\, dt$$

$$= \lim_{c \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(c + j\omega)e^{j\omega t}\, d\omega}{(c + j\omega)^2 + \omega_0^2},$$

from which

$$f(t) = \frac{\cos \omega_0 t}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\omega \sin \omega t}{\omega_0^2 - \omega^2}\, d\omega = \begin{cases} \cos \omega_0 t & \text{when } t > 0, \\ 0 & \text{when } t < 0. \end{cases} \quad (12.55)$$

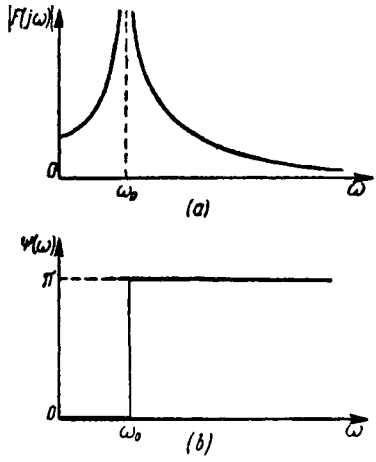



FIG. 13. Amplitude and phase spectra of a sinusoidal step function.

From (50) the spectrum is found to be

$$F(j\omega) = j \frac{\omega}{\omega_0^2 - \omega^2}. \quad (12.56)$$

The curves of the modulus $|F(j\omega)|$ and the phase $\psi(\omega)$ of this characteristic are shown in Fig. 14.

The spectrum (56) can also be determined from (54) with the aid of (31), taking it into account that

$$\cos \omega_0 t = \frac{1}{\omega_0} \frac{d}{dt} \sin \omega_0 t.$$

An analogous procedure was used in Example 1.

A sinusoidal step function with an arbitrary initial phase $\alpha$ can be written in integral form with the aid of equations (53) and (55) since

$$\sin(\omega_0 t + \alpha) = \cos\alpha \sin\omega_0 t + \sin\alpha \cos\omega_0 t.$$

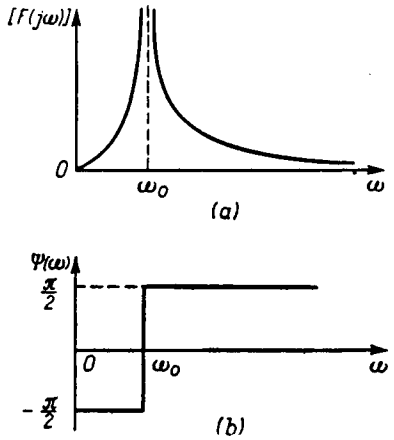



FIG. 14. Amplitude and phase spectra of a cosinusoidal step function.

## 7. Transient Response of a Two-terminal Network

Let us suppose that a nonperiodic voltage $u(t)$ is applied to a linear two-terminal network with an impedance $Z(j\omega)$. It is required to determine the current $i(t)$ in the network (Fig. 15).

The applied voltage can be represented, with the aid of the integral (1) in the form of the aggregate of an infinite number of harmonics with infinitely small complex amplitudes $(1/2\pi)\, U(j\omega)\, d\omega$. In view of the linearity

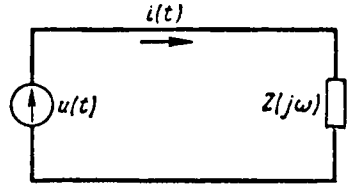



FIG. 15. Effect of a nonperiodic e.m.f. on a linear two-terminal network.

of the given network, the current $i(t)$ can be obtained by adding together an infinite number of harmonics of currents with infinitesimal complex amplitudes

$$\frac{1}{2\pi}\frac{U(j\omega)}{Z(j\omega)}\, d\omega = \frac{1}{2\pi} I(j\omega)\, d\omega.$$

The summation is carried out within the range of frequencies from $-\infty$ to $+\infty$. In accordance with (1)

$$i(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} I(j\omega)e^{j\omega t}\,d\omega, \tag{12.57}$$

where

$$I(j\omega) = Y(j\omega)U(j\omega), \tag{12.58}$$

$$U(j\omega) = \int_{-\infty}^{+\infty} u(t)\,e^{-j\omega t}\,dt, \tag{12.59}$$

$Y(j\omega) = 1/Z(j\omega)$ being the admittance of the network.

Thus the following is a possible order for determining $i(t)$:

(a) finding the frequency spectrum $U(j\omega)$ for the given function $u(t)$ from formula (59);

(b) determining the admittance $Y(j\omega)$ in steady-state conditions for the entire spectrum of frequencies;

(c) determining the current spectrum from formula (58);

(d) calculating the current from formula (57) or (44).

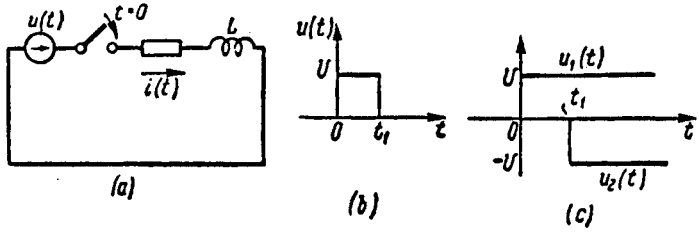



FIG. 16. Example 4.

EXAMPLE 4. It is required to find the current in a network consisting of $r$ and $L$ elements connected in series if the excitating e.m.f. $u(t)$ has the form of a rectangular pulse $t_1$ in length (Fig. 16).

Using the superposition method let us examine the given function $u(t)$ as the sum of the two step functions shown in Fig. 16c, i.e.

$$u(t) = u_1(t) + u_2(t).$$

The function $u_1(t)$ has a spectrum $U/(j\omega)$ (see Section 6).

The admittance of the network is $Y(j\omega) = 1/(r + j\omega L)$. Consequently, the frequency spectrum of the current $i_1(t)$ is

$$I_1(j\omega) = \frac{U}{j\omega(r + j\omega L)}.$$

In this case it is convenient to use the switching formula (44b), taking

$$F_1(j\omega) = U, \quad F_2(j\omega) = r + j\omega L.$$

Consequently,

$$\omega_k = j\frac{r}{L}, \quad \frac{d}{d\omega}F_2(j\omega) = jL,$$

and the current $i_1(t)$, produced by the e.m.f. $u_1(t)$, equals

$$i_1(t) = \frac{U}{r}\left(1 - e^{-\frac{r}{L}t}\right).$$

The resultant current equals the required current $i(t)$ in the interval of time from 0 to $t_1$.

An e.m.f. $u_2(t)$ begins to act from the instant $t_1$, producing a current

$$i_2(t) = -\frac{U}{r}\left[1 - e^{-\frac{r}{L}(t-t_1)}\right].$$

Therefore, when $t \geqq t_1$, the unknown current can be found from the equation

$$i(t) = i_1(t) + i_2(t) = \frac{U}{r}\left[\left(1 - e^{-\frac{r}{L}t}\right) - \left(1 - e^{-\frac{r}{L}(t-t_1)}\right)\right]$$

$$= \frac{U}{r}\left[e^{-\frac{r}{L}(t-t_1)} - e^{-\frac{r}{L}t}\right]. \quad (t > t_1)$$

Besides the above-mentioned order for calculating the current, there are also other methods of analysing the transient response of a two-terminal network.

Thus it is evident from equation (58) that the spectrum of the sought-for current is equal to the product of two functions: $Y(j\omega)$ and $U(j\omega)$. Regarding function $Y(j\omega)$ as the spectrum of a certain time function $y(t)$ and bearing in mind that $U(j\omega)$ corresponds to $u(t)$, we can use the formula (39):

$$i(t) = \int_0^t u(\tau)y(t-\tau)\,d\tau = \int_0^t u(t-\tau)y(\tau)\,d\tau. \qquad (12.60)$$

Function $y(t)$ is the response of the network to a so-called *pulse function* of the first order (Chapter XIII, Sections 14 and 17).

EXAMPLE 5. To work the previous example, using formula (60).

The time function $y(t) = (1/L)\,e^{-rt/L}$ corresponds to the function $Y(j\omega) = 1/(r + j\omega L)$, in accordance with Section 2 (subsection 1).

An e.m.f. $u_1(t) = U$ is active within the interval from 0 to $t_1$. Taking into account the fact that $u_1(t-\tau) = U$, we find

$$i_1(t) = \int_0^t \frac{U}{L} e^{-\frac{r}{L}\tau}\,d\tau = \frac{U}{r}\left(1 - e^{\frac{r}{L}t}\right).$$

The calculation continues as for Example 4.

When the e.m.f. is given in the form of a step function (45) or (46) and the frequency characteristic of the conductance or susceptance of the two-terminal network is known, it is possible to use the step function in integral form.

**Response to a Constant Voltage**

Let us consider the connexion of a constant e.m.f. source, at the instant $t = 0$, to a two-terminal network with an admittance

$$Y(j\omega) = g(\omega) - jb(\omega).$$

Here $g(\omega)$ and $b(\omega)$ are the frequency characteristics of the conductance and susceptance.

Using formula (51) for the unit step function, we can find the current in the network

$$i(t) = \frac{U}{2Z(0)} + \frac{U}{\pi}\int_0^\infty \frac{\sin(\omega t - \varphi)}{\omega\,|Z(j\omega)|}\,d\omega$$

$$= \frac{U}{2Z(0)} + \frac{U}{\pi}\int_0^\infty \frac{\cos\varphi \sin\omega t}{\omega\,|Z(j\omega)|}\,d\omega - \frac{U}{\pi}\int_0^\infty \frac{\sin\varphi\cos\omega t}{\omega\,|Z(j\omega)|}\,d\omega.$$

Introducing the conductance and susceptance into the integrand

$$g(\omega) = \frac{\cos\varphi}{|Z(j\omega)|} \quad \text{and} \quad b(\omega) = \frac{\sin\varphi}{|Z(j\omega)|},$$

we get

$$i(t) = \frac{Ug(0)}{2} + \frac{U}{\pi}\int_0^\infty g(\omega)\frac{\sin\omega t}{\omega}\,d\omega - \frac{U}{\pi}\int_0^\infty b(\omega)\frac{\cos\omega t}{\omega}\,d\omega. \quad (12.61)$$

Equation (61), which is true for any instant, equals zero when $t < 0$. In other words, when $t > 0$, $i(-t) = 0$ or, the same thing,

$$\frac{Ug(0)}{2} - \frac{U}{\pi}\int_0^\infty g(\omega)\frac{\sin\omega t}{\omega}\,d\omega - \frac{U}{\pi}\int_0^\infty b(\omega)\frac{\cos\omega t}{\omega}\,d\omega = 0. \quad (12.62)$$

Adding (61) and (62) we get (Ref. 26)

$$i(t) = i(t) + i(-t)$$

$$= Ug(0) - \frac{2U}{\pi}\int_0^\infty b(\omega)\frac{\cos\omega t}{\omega}\,d\omega \quad \text{when} \quad t > 0. \quad (12.63)$$

In turn, subtracting (62) from (61), we find

$$i(t) = i(t) - i(-t)$$

$$= \frac{2U}{\pi}\int_0^\infty g(\omega)\frac{\sin\omega t}{\omega}\,d\omega \quad \text{when} \quad t > 0. \quad (12.64)$$

**Response to a Sinusoidal Voltage**

Let us now consider the case of a sinusoidal or cosinusoidal e.m.f. applied to a two-terminal network with an admittance

$$Y(j\omega) = g(\omega) - jb(\omega)$$

at an instant $t = 0$.

Using formula (53) for a sinusoidal step function, we can find the current in the network

$$i(t) = \frac{U_m}{2} \frac{\sin(\omega_0 t - \varphi_0)}{|Z(j\omega_0)|} + \frac{U_m}{\pi} \int_0^\infty \frac{\omega_0 \cos(\omega t - \varphi)}{(\omega_0^2 - \omega^2)\,|Z(j\omega)|}\,d\omega$$

$$= \frac{U_m}{2} g(\omega_0) \sin \omega_0 t - \frac{U_m}{2} b(\omega_0) \cos \omega_0 t$$

$$+ \frac{U_m}{\pi} \int_0^\infty \frac{\omega_0 g(\omega) \cos \omega t}{\omega_0^2 - \omega^2}\,d\omega + \frac{U_m}{\pi} \int_0^\infty \frac{\omega_0 b(\omega) \sin \omega t}{\omega_0^2 - \omega^2}\,d\omega. \quad (12.65)$$

Continuing the calculation as for the above example we can obtain calculation formulae* for $t > 0$, i.e.

$$i(t) = i(t) + i(-t)$$

$$= -U_m b(\omega_0) \cos \omega_0 t + \frac{2U_m}{\pi} \int_0^\infty \frac{\omega_0 g(\omega) \cos \omega t}{\omega_0^2 - \omega^2}\,d\omega, \quad (12.66)$$

$$i(t) = i(t) - i(-t)$$

$$= U_m g(\omega_0) \sin \omega_0 t + \frac{2U_m}{\pi} \int_0^\infty \frac{\omega_0 b(\omega) \sin \omega t}{\omega_0^2 - \omega^2}\,d\omega. \quad (12.67)$$

When a two-terminal network is excited by a cosinusoidal voltage source the current can be found by using formula (55) for a cosinusoidal step function:

$$i(t) = \frac{U_m}{2} \frac{\cos(\omega_0 t - \varphi_0)}{|Z(j\omega_0)|} - \frac{U_m}{\pi} \int_0^\infty \frac{\omega \sin(\omega t - \varphi)}{(\omega_0^2 - \omega^2)\,|Z(j\omega)|}\,d\omega$$

$$= \frac{U_m}{2} g(\omega_0) \cos \omega_0 t + \frac{U_m}{2} b(\omega_0) \sin \omega_0 t$$

$$- \frac{U_m}{\pi} \int_0^\infty \frac{\omega g(\omega) \sin \omega t}{\omega_0^2 - \omega^2}\,d\omega + \frac{U_m}{\pi} \int_0^\infty \frac{\omega b(\omega) \cos \omega t}{\omega_0^2 - \omega^2}\,d\omega. \quad (12.68)$$

* These formulae were educed by the author in 1943 and published in his paper '*A method of analysing the transient response at the input of an electrical filter*'. Transactions of the Leningrad Electrotechnical Communications Institute, 2, 1947.

By analogy with the above we obtain the calculation formulae for $t > 0$

$$i(t) = i(t) + i(-t)$$
$$= U_m g(\omega_0) \cos \omega_0 t + \frac{2U_m}{\pi} \int_0^\infty \frac{\omega b(\omega) \cos \omega t}{\omega_0^2 - \omega^2} \, d\omega. \qquad (12.69)$$

$$i(t) = i(t) - i(-t)$$
$$= U_m b(\omega_0) \sin \omega_0 t - \frac{2U_m}{\pi} \int_0^\infty \frac{\omega g(\omega) \sin \omega t}{\omega_0^2 - \omega^2} \, d\omega. \qquad (12.70)$$

EXAMPLE 6. A two-terminal network having a conductance with an idealized frequency characteristic

$$g(\omega) = \begin{cases} g = \text{const} & \text{when} \quad 0 \leqq \omega \leqq \omega_0 \\ 0 & \text{when} \quad \omega > \omega_0, \end{cases}$$

is connected to a constant voltage source $U$. It is required to determine the time $t$ taken by the current in the network to attain 90% of the steady-state value.

Using formula (64) we get

$$i(t) = \frac{2U}{\pi} \int_0^{\omega_0} g \frac{\sin \omega t}{\omega} \, d\omega = \frac{2Ug}{\pi} \operatorname{Si}(\omega_0 t),$$

i.e. the current in the network increases in accordance with the sine-integral law.

The steady-state current is $i_{st} = gU$.

Consequently, in accordance with the data

$$\frac{2}{\pi} \operatorname{Si}(\omega_0 t) = 0{\cdot}9, \quad \text{from which} \quad t = \frac{1{\cdot}67}{\omega_0}.$$

EXAMPLE 7. A two-terminal network having a conductance with an idealized frequency characteristic

$$g(\omega) = \begin{cases} g = \text{const} & \text{when} \quad \omega_1 \leqq \omega \leqq \omega_2 \\ 0 & \text{when} \quad \omega_1 > \omega > \omega_2, \end{cases}$$

is connected to a sinusoidal voltage source

$$u = U_m \sin \omega_0 t.$$

It is required to find the law of the current increment in the network if the frequency of the source $\omega_0$ satisfies the condition $b(\omega_0) = 0$.

From (66)

$$i(t) = \frac{2U_m}{\pi} \int_{\omega_1}^{\omega_2} \frac{g\omega_0}{\omega_0^2 - \omega^2} \cos \omega t \, d\omega = \frac{gU}{\pi} \int_{\omega_1}^{\omega_2} \left( \frac{\cos \omega t}{\omega + \omega_0} - \frac{\cos \omega t}{\omega - \omega_0} \right) d\omega$$

$$= \frac{gU_m}{\pi} \left[ \int_{\omega_1}^{\omega_2} \frac{\cos (\omega + \omega_0) t \cos \omega_0 t}{\omega + \omega_0} \, d\omega + \int_{\omega_1}^{\omega_2} \frac{\sin (\omega + \omega_0) t \sin \omega_0 t}{\omega + \omega_0} \, d\omega \right.$$

$$\left. - \int_{\omega_1}^{\omega_2} \frac{\cos (\omega - \omega_0) t \cos \omega_0 t}{\omega - \omega_0} \, d\omega + \int_{\omega_1}^{\omega_2} \frac{\sin (\omega - \omega_0) t \sin \omega_0 t}{\omega - \omega_0} \, d\omega \right]$$

$$= \frac{gU_m}{\pi} \left[ \cos \omega_0 t \int_{(\omega_1 + \omega_0)t}^{(\omega_2 + \omega_0)t} \frac{\cos x}{x} \, dx + \sin \omega_0 t \int_{(\omega_1 + \omega_0)t}^{(\omega_2 + \omega_0)t} \frac{\sin x}{x} \, dx \right.$$

$$\left. - \cos \omega_0 t \int_{(\omega_1 - \omega_0)t}^{(\omega_2 - \omega_0)t} \frac{\cos x}{x} \, dx + \sin \omega_0 t \int_{(\omega_1 - \omega_0)t}^{(\omega_2 - \omega_0)t} \frac{\sin x}{x} \, dx \right].$$

$$i(t) = \frac{gU_m}{\pi}\{\cos\omega_0 t[\mathrm{Ci}(\omega_2+\omega_0)t - \mathrm{Ci}(\omega_1+\omega_0)t$$
$$+ \sin\omega_0 t[\mathrm{Si}(\omega_2+\omega_0)t - \mathrm{Si}(\omega_1+\omega_0)t]$$
$$- \cos\omega_0 t[\mathrm{Ci}(\omega_2-\omega_0)t - \mathrm{Ci}(\omega_1-\omega_0)t]$$
$$+ \sin\omega_0 t[\mathrm{Si}(\omega_2-\omega_0)t - \mathrm{Si}(\omega_1-\omega_0)t]\}.$$

By using formula (95) in Chapter VI it is possible to show that the condition $b(\omega_0)=0$ obtains when $\omega_0 = \sqrt{\omega_1\omega_2}$. In this case, if $\omega_1$ and $\omega_2$ are sufficiently great, it can be considered that, approximately,

$$\omega_2 - \omega_0 \cong \omega_0 - \omega_1 \cong \tfrac{1}{2}(\omega_2 - \omega_1),$$
$$\omega_2 + \omega_0 \cong \omega_1 + \omega_0.$$

Bearing in mind that $\mathrm{Ci}(\omega_1-\omega_0)t = \mathrm{Ci}(\omega_0-\omega_1)t$, and $\mathrm{Si}(\omega_1-\omega_0)t = -\mathrm{Si}(\omega_0-\omega_1)t$, the current equation obtained above can be simplified, taking this form:

$$i(t) = \frac{2gU_m}{\pi}\,\mathrm{Si}\,\frac{\omega_2-\omega_1}{2}\,t\,.\sin\omega_0 t.$$

Formulae (61)–(70) can also be used to determine the current in coupled circuits. In this case, when the current is being determined in a loop without an e.m.f., the real and imaginary components of the transfer admittance of the loops should be understood by the quantities $g(\omega)$ and $b(\omega)$.

The advantage of calculation formulae (64), (66) and (70) over (61), (65) and (68) is that the one frequency characteristic of the conductance $g(\omega)$ is sufficient for calculating a transient response, without any limitations being imposed upon the frequency characteristic of the susceptance $b(\omega)$, apart from the value $b(\omega_0)$ contained in (66) and (70). This is a result of the fact that the conductance and the susceptance of the network are interdependent (Chapter VI, Section 18).

## 8. Transient Response of a Four-terminal Network

Let a voltage $u_1(t)$, the frequency spectrum of which is denoted by $U_1(j\omega)$, be applied to a passive linear four-terminal network.

If the transfer function for steady-state sinusoidal conditions is $K(j\omega)$, the spectra of voltage and current at the output of the network can be obtained from the formulae

$$\left.\begin{aligned} U_2(j\omega) &= K(j\omega)U_1(j\omega), \\ I_2(j\omega) = \frac{U_2(j\omega)}{Z_2(j\omega)} &= \frac{K(j\omega)}{Z_2(j\omega)}\,U_1(j\omega) = Y_{21}(j\omega)U_1(j\omega), \end{aligned}\right\} \qquad (12.71)$$

where $Y_{21}$ is the transfer admittance.

Correspondingly, the voltage and current at the output of the network can be determined from the formulae

$$\left.\begin{aligned} u_2(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} K(j\omega)U_1(j\omega)e^{j\omega t}\,d\omega, \\ i_2(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y_{21}(j\omega)U_1(j\omega)e^{j\omega t}\,d\omega, \end{aligned}\right\} \tag{12.72}$$

or directly from the switching formulae (44).

When
$$K(j\omega) = |K(j\omega)|\, e^{-j\theta(\omega)},$$
where
$$|K(j\omega)| = K = \text{const}, \tag{12.73}$$
$$\theta(\omega) = \omega t_3,$$

$$u_2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} KU_1(j\omega)e^{j\omega(t-t_3)}\,d\omega = Ku_1(t - t_3). \tag{12.74}$$

Thus, if the amplitude characteristic of a four-terminal network does not depend on frequency ($K$ being constant) and the phase characteristic is linear, the voltage curve at the output of the network is analogous to that at the input multiplied by $K$ and shifted along the time axis by a quantity $t_3$, which is the delay time, determined by the slope of the phase characteristic

$$t_3 = \frac{d\theta(\omega)}{d\omega} = \frac{\theta(\omega)}{\omega}.$$

In other words equations (73) describe the absence of distortions when any signals are being transmitted through a four-terminal network.

Formula (60) can also be used for calculating the transient response at the output of a four-terminal network, taking $u(t)$ as the given voltage at the input and $y(t)$ as a time function corresponding to the spectrum $K(j\omega)$ or $Y_{21}(j\omega)$.

In addition, as we mentioned in Section 7, formulae (61)–(70) are also applicable in this case, if we take $g(\omega)$ and $b(\omega)$ to stand for the real and imaginary components of the transfer admittance.

The equations given in this section show that if the transfer function of the network for all $\omega$ in steady-state is known, the transient response for given excitation conditions is completely described. If a pair of four-terminal networks of different design possess identical frequency characteristics throughout the spectrum, then they both have the same transient response to any signal applied to them.

The reverse is also true: i.e. an electrical circuit possessing certain frequency characteristics in the steady state, i.e. a certain amplitude-phase characteristic, is required in order to obtain a given transient response at the output of a four-terminal network.

Just as the frequency characteristics of a circuit define the transient process simply, so the frequency characteristics can be defined in their turn by the circuit's "response".

The application of the Fourier transformation to network transient analysis makes it possible to extend many of the concepts and properties of networks examined above for the steady state to transient conditions. For example, in view of the equality of the transfer admittances $Y_{12}(j\omega)$ and $Y_{21}(j\omega)$ which is a feature of reciprocal networks, the reciprocity theorem is also valid for a transient response.

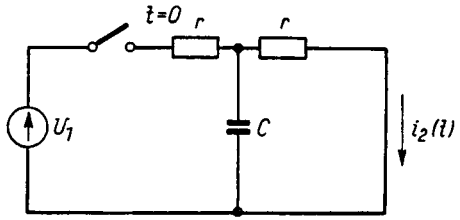



FIG. 17. Example 8.

EXAMPLE 8. A symmetrical T-network consisting of two resistances $r$ and a capacitance $C$ is connected to a constant voltage source $U_1$, at an instant $t=0$ (Fig. 17). It is required to determine the current $i_2(t)$ in the short-circuited output terminals of the network, using the switching formula.

The spectrum of the unknown current is the product of the spectrum of the voltage applied to the network and the transfer admittance of the input and output loops

$$I_2(j\omega) = Y_{21}(j\omega) \times U_1(j\omega).$$

According to Section 6 the spectrum of a step voltage applied to the input of the network is

$$U_1(j\omega) = \frac{U_1}{j\omega}.$$

The transfer admittance of the input and output loops of the given network is numerically equal to the current sinor in the output loop when the voltage sinor at the input equals unity:

$$Y_{21}(j\omega) = \frac{1}{r + \dfrac{r\dfrac{1}{j\omega C}}{r + \dfrac{1}{j\omega C}}} \cdot \frac{\dfrac{1}{j\omega C}}{r + \dfrac{1}{j\omega C}} = \frac{1}{2r + jr^2\omega C}.$$

Thus,

$$I_2(j\omega) = \frac{U_1}{j\omega(2r + jr^2\omega C)}.$$

The equation obtained corresponds in form to the left-hand side of the switching formula (44b), and so, in this case

$$F_1(j\omega) = U_1, \quad F_3(j\omega) = 2r + jr^2\omega C$$

and

$$\frac{dF_3(j\omega)}{d\omega} = jr^2C.$$

The solution of the equation $2r + jr^2\omega C = 0$ is $\omega_1 = 2j/(rC)$. Consequently, from (44b),

$$i_2(t) = \frac{U_1}{2r} + \frac{U_1 e^{-\frac{2}{rC}t}}{j\frac{2}{rC}jr^2C} = \frac{U_1}{2r}\left(1 - e^{-\frac{2}{rC}t}\right).$$

EXAMPLE 9. To calculate the transient voltage at the output of the four-terminal network shown in Fig. 18 if a voltage is applied at the input.

$$u_1(t) = Ue^{-at}.$$

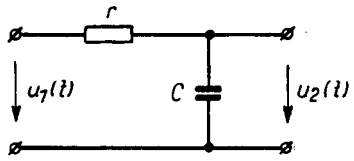



FIG. 18. Example 9.

The voltage-ratio transfer constant

$$K(j\omega) = \frac{\frac{1}{j\omega C}}{r + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega rC}.$$

The applied voltage has the spectrum (see Section 2, subsection 1)

$$U_1(j\omega) = \frac{U}{a + j\omega}.$$

Consequently, the spectrum of the output voltage is

$$U_2(j\omega) = \frac{U}{(a + j\omega)(1 + j\omega rC)}.$$

According to the switching formula (44a)

$$u_2(t) = \frac{U}{1 - arC}\left(e^{-at} - e^{-\frac{1}{rC}t}\right).$$

The same result is obtained from formula (60) if it is remembered that the time function

$$K(j\omega) = \frac{1}{1 + j\omega rC}$$

corresponds to the function

$$y(t) = \frac{1}{rC} e^{-\frac{1}{rC}t},$$

$$u_2(t) = \int_0^t \frac{U}{rC} e^{-at} e^{-\frac{1}{rC}(t-\tau)} \, d\tau = \frac{U}{1 - arC} \left( e^{-at} - e^{-\frac{1}{rC}t} \right).$$

EXAMPLE 10. To prove that the real part of the voltage-ratio transfer function is an even and the imaginary part an odd frequency function, using the frequency spectra of the voltages at the input and the output of a four-terminal network.

The voltage-ratio transfer function is defined as the ratio of the voltage spectra at the output and input of the network

$$K_U(j\omega) = \frac{U_2(j\omega)}{U_1(j\omega)}.$$

The spectrum of the voltage at the output is

$$U_2(j\omega) = \int_{-\infty}^{+\infty} u_2(t) e^{-j\omega t} \, dt = \int_{-\infty}^{+\infty} u_2(t) \cos \omega t \, dt - j \int_{-\infty}^{+\infty} u_2(t) \sin \omega t \, dt$$

$$= U_2'(\omega) - jU_2''(\omega).$$

Analogously,

$$U_1(j\omega) = U_1'(\omega) - jU_1''(\omega).$$

Thus,

$$K_U(j\omega) = \frac{U_2' - jU_2''}{U_1' - jU_1''} = \frac{U_1'U_2' - U_1''U_2''}{(U_1')^2 + (U_1'')^2} - j \frac{U_1'U_2'' - U_1''U_2'}{(U_1')^2 + (U_1'')^2}.$$

The proof of the required proposition follows from the fact that $U_1'(\omega)$ and $U_2'(\omega)$ are even and $U_1''(\omega)$ and $U_2''(\omega)$ odd.

CHAPTER XIII

# THE LAPLACE TRANSFORMATION

## 1. General Information

In practice the solution of linear differential equations (in ordinary or partial derivatives) describing transients in linear networks often presents a long and difficult problem, which becomes more difficult as the order of the equations becomes higher.

The *Laplace transformation* or operational calculus is a combination of methods of mathematical analysis which considerably simplify and speed up the process of calculating, by "algebraicizing" differential equations and subsequently "interpreting" the results obtained. As distinct from the classic theory of differential equations, the Laplace transformation makes it possible to find the integral of a linear differential equation for given initial conditions without it being necessary to determine the integration constants separately.

The Laplace transformation originated with the commencement of mathematical analysis, when the analogies between algebraic rules and the processes of differentiation and integration were first discovered. Its next stage was characterized by research, particularly in the application of the Laplace transformation (symbolic methods) to the solution of linear differential equations.

In the middle of the nineteenth century a considerable amount of work was devoted to these methods, among which the monograph *Symbolic Calculus and its Application to the Integration of Linear Differential Equations* by M. Ye. Vashchenko–Zakharchenko (Kiev, 1862) holds an important place. In particular, this monograph contains expansion formulae for simple and multiple roots of characteristic equations.

At the end of the nineteenth century the eminent English scientist Oliver Heaviside used his own transformation method to solve electrical engineering problems. In his work he used two main methods of solving problems: using power series and expansion formulae.

As a result of further research by many scientists methods involving operators have now received a sound mathematical basis and have been developed in accordance with general functional transform theory and, in particular, Fourier and Laplace transforms.

In their fundamental works, Soviet and foreign scientists like A. M. Efros and A. M. Danilevskii (Ref. 61), A. I. Lur'ye (Ref. 38), D. R. Carson (Ref. 26) and K. W. Wagner (Ref. 115) have developed the transformation method of solving many different kinds of mechanical and electrical engineering problems.

The similarity between Fourier and Laplace transform integral makes for the close connexion between the transformation method and harmonic analysis methods, which results in a far-reaching analogy between spectrum characteristics and operator functions.

## 2. Direct and Inverse Transform Integrals

The idea of the Laplace transformation consists in transferring a solution from the region of the functions of a real variable (the time domain) to the region of the functions of a complex variable (the complex frequency domain) where the operations are simplified. After the operations have been carried out on the functions of the complex variable, the solution is converted back to the time domain.

In this respect the Laplace transformation can be compared to the use of logarithms—numbers are converted into logarithms upon which the appropriate mathematical operations are carried out, and finally the logarithmic solution is converted back into numerals.

Let us suppose that $f(t)$, the function of a real variable $t$ which is given in the region $t > 0$ and equals zero when $t < 0$, increases no faster than the exponential function, i.e. satisfies the conditions for which the function $e^{-c_0 t} f(t)$ can be represented in the form of a double Fourier integral (Chapter XII, Section 6). Here $c_0$ is a positive number.

In this case

$$e^{-c_0 t} f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} \, d\omega \int_0^{\infty} e^{-c_0 t} f(t) e^{-j\omega t} \, dt. \tag{13.1}$$

The lower limit of the inner integral is here taken to equal zero from the condition that $f(t) = 0$ for $t < 0$.

Multiplying both parts of (1) by $e^{c_0 t}$, we get

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(c_0 + j\omega)t} \, d\omega \int_0^{\infty} f(t) e^{-(c_0 + j\omega)t} \, dt. \tag{13.2}$$

Let us replace the variable by a complex variable

$$p = c_0 + j\omega. \tag{13.3}$$

Accordingly,

$$dp = j \, d\omega. \tag{13.4}$$

Substituting (3) and (4) in (2) and changing the limits, we get

$$f(t) = \frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty} e^{pt}\,dp \int_0^{\infty} f(t)e^{-pt}\,dt. \tag{13.5}$$

Denoting

$$F(p) = \int_0^{\infty} f(t)e^{-pt}\,dt, \tag{13.6}$$

we find

$$f(t) = \frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty} F(p)e^{pt}\,dp. \tag{13.7}$$

Equations (6) and (7) are *direct and inverse Laplace transforms*, respectively.

It will be known from mathematical analysis that an integral with an infinite limit in the form $\int_0^{\infty} f(t)\,e^{-pt}\,dt$ converges uniformly with respect

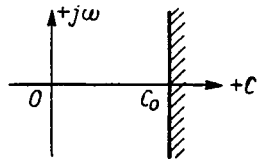



FIG. 1. Complex plane.

to $p$ and is an analytic function of a complex variable $p = c + j\omega$ in the region

$$\operatorname{Re} p = c \geqq c_0 > c_1, \tag{13.8}$$

provided that $|f(t)| < Me^{c_1 t}$.

In other words, in these conditions integral (6) is a single-valued, continuous and differentiable function of the complex variable $p$ at each point of the region to the right of the straight line $c_0$ (Fig. 1) and on the boundary of the region.

### 3. Laplace Transforms

In the integral equations (6) and (7) the function $f(t)$ is known as the *original (or time) function** and the function $F(p)$ as the *Laplace transform.*

Consequently, the original and the transform are paired functions of the real variable $t$ and the complex variable $p$ related by the Laplace transformation.

* Which we shall refer to as "the original".

In accordance with the integral equations (6) and (7) the relationship between the transform $F(p)$ and the original $f(t)$ can be written conventionally with the use of the *correspondence sign* $\sim$

$$F(p) \sim f(t)$$

or, the same thing,

$$f(t) \sim F(p). \tag{13.9}$$

Thus, the abbreviated form (9) corresponds to the integral equations (6) and (7). Other symbols are used as a shorthand notation for the direct transform:

$$L[f(t)] = F(p);$$

and for the inverse transform:

$$L^{-1}[F(p)] = f(t).$$

It follows from a comparison of the integral equation (6) with the spectrum of the nonperiodic function $f(t)$ (on the condition that $f(t) = 0$ when $t < 0$) that if the imaginary quantity $j\omega$ is substituted for the complex variable $p$ contained in the Laplace transform, an equation for the spectrum is obtained

$$F(j\omega) = [F(p)]_{p=j\omega}. \tag{13.10}$$

The close relationship between the Laplace and Fourier transformations thus introduces a physical significance into the concept of the "transform" as a generalized form of spectrum. This affinity extends to the basic properties of the Laplace and Fourier transformations.

In addition to the Laplace transform, the *Carson (or Heaviside) transform* is also used in literature:

$$F(p) = p\int_0^\infty f(t)\, e^{-pt}\, dt, \tag{13.11}$$

or, in its inverted form,

$$f(t) = \frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty} \frac{F(p)}{p}\, e^{pt}\, dp. \tag{13.12}$$

It is evident from comparing (6) and (11) that a Carson transform differs from a Laplace transform with respect to the factor $p$.

$F(p)$ (Carson) $= pF(p)$ (Laplace) or, the same thing,

$$F_C(p) = pF_L(p). \tag{13.13}$$

A Carson transform has the same dimensions as the original. This is one of the positive features of the Carson form; but this characteristic is of no essential value for practical purposes.

The Laplace transform is mainly used in present-day literature on radio engineering and automation since the relationship between the transformation method and harmonic analysis is in this case most easily apparent, and the properties of the Laplace and Fourier transformations are analogous: substituting $p = j\omega$ in Laplace transforms gives the corresponding Fourier transforms (*vide* Section 17). This is a decisive advantage possessed by the Laplace transformation method and for this reason the following explanation is based entirely on the Laplace transformation. A table comparing the basic properties of the Laplace and Carson transformations is given in Appendix III.

## 4. Laplace Transforms of Unit Step Functions and Exponential Functions

Figure 2 shows a unit step function given by the condition

$$f(t) = \begin{cases} 1 & \text{when} \quad t > 0, \\ 0 & \text{when} \quad t < 0. \end{cases}$$

In accordance with (6), when $\operatorname{Re} p > 0$

$$F(p) = \int_0^\infty e^{-pt}\,dt = \frac{e^{-pt}}{-p}\Bigg|_0^\infty = \frac{1}{p}.$$

Consequently,

$$\frac{1}{p} \sim 1 \quad \text{when} \quad (t > 0). \tag{13.14}$$

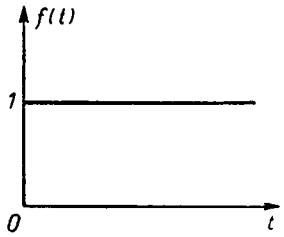



FIG. 2. Unit step function.

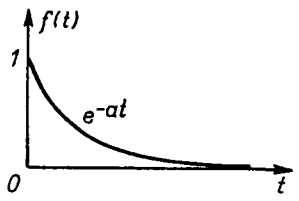



FIG. 3. Decreasing exponential function.

Figure 3 shows a decreasing exponential function given by the condition

$$f(t) = \begin{cases} e^{-at} & \text{when} \quad t > 0, \\ 0 & \text{when} \quad t < 0. \end{cases}$$

In accordance with (6), when $\operatorname{Re} p > -a$

$$F(p) = \int_0^\infty e^{-at}e^{-pt}\,dt = \frac{e^{-(p+a)t}}{-(p+a)}\Bigg|_0^\infty = \frac{1}{p+a}.$$

Consequently,

$$\frac{1}{p+a} \sim e^{-at} \quad (\text{when} \quad t > 0). \tag{13.15}$$

Finally, Fig. 4 shows an increasing exponential function in the form

$$f(t) = \begin{cases} e^{at} & \text{when} \quad t > 0, \\ 0 & \text{when} \quad t < 0. \end{cases}$$

In accordance with (6)

$$F(p) = \int_0^\infty e^{at} e^{-pt}\, dt = \frac{e^{-(p-a)t}}{-(p-a)}\Bigg|_0^\infty = \frac{1}{p-a}.$$

Consequently,

$$\frac{1}{p-a} \sim e^{at} \quad (\text{when} \quad t > 0). \tag{13.16}$$

The condition $\operatorname{Re} p > a$ must then hold.

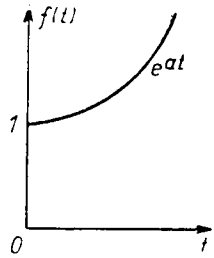



FIG. 4. Increasing exponential function.

Formulae (15) and (16) are also valid for a purely imaginary value of $a$. Thus, for example, when $\operatorname{Re} p > 0$ in accordance with (15) and (16) the transform of the function $e^{\pm j\omega_0 t}$ has the form

$$\frac{1}{p \mp j\omega_0} \sim e^{\pm j\omega_0 t}. \tag{13.17}$$

Using the rules for finding a spectrum from (10), we can obtain the spectra of the unit step function and the decreasing exponential function from (14) and (15), i.e.

$$F(j\omega) = \frac{1}{j\omega} \quad \text{and} \quad F(j\omega) = \frac{1}{a + j\omega},$$

which agree with the equations obtained above (Chapter XII, Sections 2 and 6).

## 5. Basic Properties of the Laplace Transformation

*Linearity*

*The transform of a linear combination is the same as a linear combination of transforms*

$$\sum_{i=1}^{n} a_i f_i(t) \sim \sum_{i=1}^{n} a_i F_i(p). \tag{13.18}$$

Let $f(t) = \sum_{i=1}^{n} a_i f_i(t)$, where $a_i$ are constants. Let us suppose that $f_i(t) \sim F_i(p)$. Then

$$F(p) = \int_0^\infty \sum_{i=1}^{n} a_i f_i(t) e^{-pt}\, dt = \sum_{i=1}^{n} a_i \int_0^\infty f_i(t) e^{-pt}\, dt = \sum_{i=1}^{n} a_i F_i(p).$$

EXAMPLE 1. To find the transform of the functions $\cos(\omega_0 t + \psi)$ and $\sin(\omega_0 t + \psi)$, using formula (17).

It is known that

$$\cos(\omega_0 t + \psi) = \frac{1}{2}\left[e^{j(\omega_0 t + \psi)} + e^{-j(\omega_0 t + \psi)}\right],$$

$$\sin(\omega_0 t + \psi) = \frac{1}{2j}\left[e^{j(\omega_0 t + \psi)} - e^{-j(\omega_0 t + \psi)}\right].$$

According to (17) and (18)

$$\cos(\omega_0 t + \psi) \sim \frac{1}{2}\left[\frac{e^{j\psi}}{p - j\omega_0} + \frac{e^{-j\psi}}{p + j\omega_0}\right] = \frac{p\cos\psi - \omega_0 \sin\psi}{p^2 + \omega_0^2}, \tag{13.19}$$

$$\sin(\omega_0 t + \psi) \sim \frac{1}{2j}\left[\frac{e^{j\psi}}{p - j\omega_0} - \frac{e^{-j\psi}}{p + j\omega_0}\right] = \frac{p\sin\psi + \omega_0 \cos\psi}{p^2 + \omega_0^2}. \tag{13.20}$$

EXAMPLE 2. To find the transform of the functions $\cosh at$ and $\sinh at$, using formulae (15) and (16).

We shall express the hyperbolic cosine and sine by exponential functions, i.e.

$$\cosh at = \tfrac{1}{2}(e^{at} + e^{-at}),$$

$$\sinh at = \tfrac{1}{2}(e^{at} - e^{-at}).$$

From (18)

$$\cosh at \sim \frac{1}{2}\left[\frac{1}{p - a} + \frac{1}{p + a}\right] = \frac{p}{p^2 - a^2}, \tag{13.21}$$

$$\sinh at \sim \frac{1}{2}\left[\frac{1}{p - a} - \frac{1}{p + a}\right] = \frac{a}{p^2 - a^2}. \tag{13.22}$$

EXAMPLE 3. To find the transform of the function

$$\frac{1}{a}(1 - e^{-at}).$$

From (18)

$$1 - e^{-at} \sim \frac{1}{p} - \frac{1}{p + a} = \frac{a}{p(p + a)}.$$

Consequently,

$$\frac{1}{a}(1 - e^{-at}) \sim \frac{1}{p(p + a)}. \tag{13.23}$$

**Differentiating an Original (Differentiation Theorem)**

If $f(t) \sim F(p)$, then

$$f'(t) \sim pF(p) - f(0+). \tag{13.24}$$

Here, $f(0+)$ denotes the limit to which $f(t)$ tends when $t$ decreases infinitely, while remaining positive. Henceforth we shall write $f(0)$ to signify $f(0+)$.

The transform of the derivative is

$$\int_0^\infty f'(t)e^{-pt}\,dt = \int_0^\infty e^{-pt}df(t).$$

Integrating by parts

$$\int u\,dv = uv - \int v\,du,$$

and assuming that

$$u = e^{-pt}, \quad v = f(t),$$

we get

$$\int_0^\infty e^{-pt}\,df(t) = e^{-pt}f(t)\Big|_0^\infty + p\int_0^\infty e^{-pt}f(t)\,dt = pF(p) - f(0).$$

When the original is continuous at the initial point, i.e. $f(0) = 0$, *the operation of multiplying the transform by p corresponds to differentiating the original*

$$f'(t) \sim pF(p). \tag{13.25}$$

The theorem of differentiating can be extended to include higher orders of derivative by repeating the operation expressed by formula (24):

$$f''(t) \sim p[pF(p) - f(0)] - f'(0) = p^2F(p) - pf(0) - f'(0), \quad \text{etc.},$$

from which

$$f^{(n)}(t) \sim p^nF(p) - \sum_{k=1}^{n} f^{(k-1)}(0)p^{n-k}, \tag{13.26}$$

where $f^{(k)}(t) = d^kf(t)/dt^k$, $f^{(k-1)}(0)$ is the right-hand limit value, i.e. $\lim_{+t\to 0} f^{(k-1)}t$.

If, in particular, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, then

$$f^{(n)}(t) \sim p^nF(p), \tag{13.27}$$

and, consequently, multiplication by $p$, $n$ times corresponds to differentiating the original $n$ times.

EXAMPLE 4. To find the transform of the function $\cos \omega_0 t$, using the relationship $\sin \omega_0 t \sim \omega_0/(p^2 + \omega_0^2)$.

The cosine can be expressed in the form of a derivative of the sine

$$\cos \omega_0 t = \frac{1}{\omega_0} \frac{d}{dt} \sin \omega_0 t.$$

On the basis of (18) and (25), taking into account that $f(0) = \sin 0 = 0$, we get

$$\cos \omega_0 t \sim \frac{1}{\omega_0} \frac{p\omega_0}{p^2 + \omega_0^2} = \frac{p}{p^2 + \omega_0^2}.$$

EXAMPLE 5. To find the transform of the function $\sin \omega_0 t$, using the relationship $\cos \omega_0 t \sim p/(p^2 + \omega_0^2)$.

The sine is expressed in the form of a derivative of the cosine

$$\sin \omega_0 t = -\frac{1}{\omega_0} \frac{d}{dt} \cos \omega_0 t.$$

From (18) and (24), taking into account that $f(0) = \cos 0 = 1$, we get

$$\sin \omega_0 t \sim -\frac{1}{\omega_0} \left[ \frac{p^2}{p^2 + \omega_0^2} - 1 \right] = \frac{\omega_0}{p^2 + \omega_0^2}.$$

EXAMPLE 6. To find the original of a function having a transform $p/(p + a)(p + b)$, if the following relationship is known:

$$\frac{1}{(p + a)(p + b)} \sim \frac{1}{b - a} (e^{-at} - e^{-bt}). \tag{13.28}$$

From (24), taking into account that $f(0) = 0$, we get

$$\frac{p}{(p + a)(p + b)} \sim \frac{d}{dt} \frac{e^{-at} - e^{-bt}}{b - a} = \frac{1}{b - a} (be^{-bt} - ae^{-at}). \tag{13.29}$$

## Integrating an Original (Integration Theorem)

If $f(t) \sim F(p)$, then

$$\int_0^t f(t)\, dt \sim \frac{F(p)}{p}. \tag{13.30}$$

Let us denote the transform of the integral by $\Psi(p)$. In accordance with the above

$$\frac{d}{dt} \int_0^t f(t)\, dt = f(t) \sim p\Psi(p) - \left[ \int_0^t f(t)\, dt \right]_{t=0} = p\Psi(p).$$

It follows from this that $\Psi(p) = F(p)/p$. Thus *dividing the transform by p corresponds to integrating the original.*

By repeating the operation expressed by formula (30) we get

$$\int_0^t \cdots \int_0^t f(t)\, (dt)^n \sim \frac{F(p)}{p^n}. \tag{13.31}$$

The properties of differentiating and integrating an original show that the complex variable $p$ can in a certain sense be regarded as an operator (hence the term "operator method"). In addition they explain the relationship between Heaviside's formal operation calculus and the Laplace transformation.

EXAMPLE 7. To find the original of the transform $1/p^n$.

According to (14) $1/p \sim 1$. Consequently, the unknown original is, in accordance with (31),

$$\int_0^t \cdots \int_0^t (\mathrm{d}t)^{n-1} = \frac{t^{n-1}}{(n-1)!}.$$

Thus

$$\frac{1}{p^n} \sim \frac{t^{n-1}}{(n-1)!}. \tag{13.32}$$

**Differentiating a Transform**

If $f(t) \sim F(p)$, then

$$(-t)^n f(t) \sim \frac{\mathrm{d}^n F(p)}{\mathrm{d}p^n}. \tag{13.33}$$

Differentiating the Laplace equation with respect to $p$ yields

$$\frac{\mathrm{d}F(p)}{\mathrm{d}p} = -\int_0^\infty t f(t)\, e^{-pt}\, \mathrm{d}t.$$

As the result of repeated differentiation we get

$$\frac{\mathrm{d}^n F(p)}{\mathrm{d}p^n} = (-1)^n \int_0^\infty t^n f(t) e^{-pt}\, \mathrm{d}t,$$

which is equivalent to (33).

**Integrating a Transform**

If $f(t) \sim F(p)$, then

$$\frac{f(t)}{t} \int_p^\infty F(p)\, \mathrm{d}p. \tag{13.34}$$

For proof we shall integrate both sides of the Laplace equation with respect to $p$ within the limits from $p$ to $\infty$:

$$\int_p^\infty F(p)\, \mathrm{d}p = \int_p^\infty \int_0^\infty f(t)\, e^{-pt}\, \mathrm{d}t\, \mathrm{d}p.$$

The order of the integration can be changed in view of the fact that the function $f(t)$ can be transformed by the Laplace method. Consequently,

$$\int_p^\infty F(p)\,dp = \int_0^\infty f(t)\,dt \int_p^\infty e^{-pt}\,dp = \int_0^\infty \frac{f(t)}{t}\,e^{-pt}\,dt,$$

which is equivalent to (34).

**Differentiating with Respect to a Parameter**

If $f(t, x) \sim F(p, x)$, then

$$\frac{\partial}{\partial x} f(t, x) \sim \frac{\partial}{\partial x} F(p, x). \tag{13.35}$$

Differentiation with respect to a parameter and the Laplace transformation are commutable.

Differentiating the expression $F(p, x)$

$$\frac{\partial}{\partial x} F(p, x) = \frac{\partial}{\partial x} \int_0^\infty f(t, x)\,e^{-pt}\,dt = \int_0^\infty \frac{\partial f(t, x)}{\partial x}\,e^{-pt}\,dt,$$

confirms the validity of the relationship (35).

The given formula makes it possible to apply the Laplace transformation to the equations in partial derivatives where, besides $t$, there are other independent variables (e.g. the spatial coordinate $x$). In addition formula (35) can be used to find originals from given transforms.

EXAMPLE 8. To find the original of the transform $a/(p^2 + a^2)^2$. According to (20)

$$\sin at \sim \frac{a}{p^2 + a^2} \quad \text{or} \quad \frac{\sin at}{a} \sim \frac{1}{p^2 + a^2}.$$

From (35)

$$\frac{\partial}{\partial a}\,\frac{1}{p^2 + a^2} \sim \frac{\partial}{\partial a}\,\frac{a}{\sin at},$$

from which

$$-\frac{2a}{(p^2 + a^2)^2} \sim \frac{a}{t \cos at} - \frac{\sin at}{a^2}$$

or

$$\frac{a}{(p^2 + a^2)^2} \sim \frac{\sin at}{2a^2} - \frac{t \cos at}{2a}. \tag{13.36}$$

**Integrating with Respect to a Parameter**

If $f(t, x) \sim F(p, x)$, then

$$\int_{x_0}^{x} f(t, x)\,dx \sim \int_{x_0}^{x} F(p, x)\,dx. \tag{13.37}$$

Integration with respect to a parameter and the Laplace transformation are commutable. Let us integrate the expression for $F(p, x)$ with respect to $x$

$$\int_{x_0}^{x} F(p, x)\,dx = \int_{x_0}^{x} dx \int_0^{\infty} f(t, x)\, e^{-pt}\,dt = \int_0^{\infty} e^{-pt} \left[\int_{x_0}^{x} f(t, x)\right] dt.$$

The relationship (37) is thus proved.

EXAMPLE 9. To calculate the integral

$$\int_0^{\infty} \frac{\cos x}{1 + x^2}\,dx$$

using formula (37).

The transform $p/(p^2 + x^2)$ corresponds to the original $\cos tx$, and consequently $\cos tx/(1 + x^2) \sim p/(p^2 + x^2)(1 + x^2)$.

In accordance with (37)

$$\int_0^{\infty} \frac{\cos tx}{1 + x^2}\,dx \sim \int_0^{\infty} \frac{p\,dx}{(p^2 + x^2)(1 + x^2)}$$

$$= \frac{p}{p^2 - 1} \int_0^{\infty} \left(\frac{1}{x^2 + 1} - \frac{1}{p^2 + x^2}\right) dx = \frac{\pi}{2(p + 1)}.$$

Consequently,

$$\int_0^{\infty} \frac{\cos tx}{1 + x^2}\,dx = \frac{\pi}{2} e^{-t},$$

from which, supposing $t = 1$, we obtain

$$\int_0^{\infty} \frac{\cos x}{1 + x^2}\,dx = \frac{\pi}{2e}.$$

The above example illustrates the possibilities afforded by the Laplace transformation in the calculation of integrals.

## Changing the Scale of an Independent Variable (Similarity Theorem)

If $f(t) \sim F(p)$, then

$$f(at) \sim \frac{1}{a} F\left(\frac{p}{a}\right). \tag{13.38}$$

Converting the expression for $F(p)$ in the form

$$\int_0^{\infty} f(at)\, e^{-pt}\,dt = \frac{1}{a} \int_0^{\infty} f(at)\, e^{-\frac{p}{a}at}\,d(at)$$

and substituting $\tau$ for $at$, we get

$$\frac{1}{a} \int_0^{\infty} f(\tau)\, e^{-\frac{p}{a}\tau} d\tau = \frac{1}{a} F\left(\frac{p}{a}\right),$$

since the value of this definite integral does not depend on the symbol of the integration variable.

**Displacement in the Region of a Real Variable (Real Shifting Theorem)**

If $f(t) \sim F(p)$, then

$$f(t - t_0) \sim e^{-pt_0}F(p). \tag{13.39}$$

It is supposed here that $f(t - t_0) = 0$ when $t < t_0$.

Let us denote the transform of the original $f(t - t_0)$ by $\Psi(p)$

$$\Psi(p) = \int_0^\infty f(t - t_0)\, e^{-pt}\, dt.$$

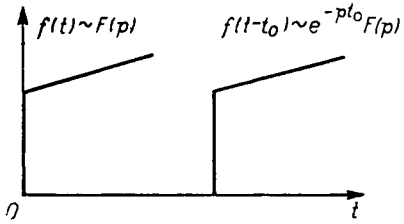



FIG. 5. Shift in the region of a real variable.

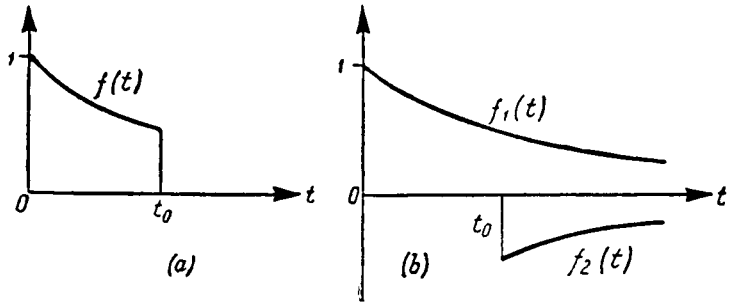



FIG. 6. Example 10.

We shall replace the variable: $t - t_0 = \tau$; $t = \tau + t_0$; $dt = d\tau$. Then

$$\Psi(p) = \int_0^\infty f(\tau)\, e^{-p(\tau + t_0)}\, d\tau = e^{-pt_0} \int_0^\infty f(\tau) e^{-p\tau}\, d\tau = e^{-pt_0}F(p),$$

since the value of this definite integral does not depend on the symbol of the integration variable.

The real shifting theorem shows that multiplying the transform by $e^{-pt_0}$ shifts the graph of the original to the right by $t_0$ (Fig. 5).

EXAMPLE 10. To find the transform of the function (Fig. 6*a*)

$$f(t) = \begin{cases} e^{-at} & \text{when } 0 < t < t_0 \\ 0 & \text{when } 0 > t > t_0. \end{cases}$$

Function $f(t)$ is equal to the sum of the two functions shown in Fig. 6*b*:

$$f(t) = f_1(t) + f_2(t).$$

Function $f_1(t)$ has the transform $1/(p + a)$.
In accordance with (39) function $f_2(t)$ has the transform

$$-\frac{e^{-at_0}\, e^{-pt_0}}{p + a}.$$

As a result of the linearity of the Laplace transformation the unknown transform of the function $f(t)$ is

$$\frac{1}{p + a}(1 - e^{-(p+a)t_0}).$$

**Displacement in the Region of a Complex Variable (Complex Shifting Theorem)**

If $f(t) \sim F(p)$, then

$$e^{\pm \lambda t} f(t) \sim F(p \mp \lambda). \tag{13.40}$$

Determining the transform $f(t)e^{\pm\lambda}$, we get

$$\int_0^\infty e^{\pm \lambda t} f(t)\, e^{-pt}\, \mathrm{d}t = \int_0^\infty e^{-(p \mp \lambda)t} f(t)\, \mathrm{d}t = F(p \mp \lambda).$$

This theorem shows the way in which the transform changes when the original is multiplied by an exponential function. It should then be borne in mind that the number $\lambda$ can be either real or complex.

If $F(p)$ is determined for $\operatorname{Re} p > c_0$, then the transform (40) is determined for $\operatorname{Re} p > c_0 + \operatorname{Re} \lambda$.

**Limit Relationships (Initial- and Final-value Theorems)**

If $f(t) \sim F(p)$, then

$$\lim_{t \to 0} f(t) = \lim_{p \to \infty} pF(p); \tag{13.41}$$

$$\lim_{t \to \infty} f(t) = \lim_{t \to 0} pF(p). \tag{13.42}$$

The proof of formulae (41) and (42) follows from the equation

$$\int_0^\infty f'(t)\, e^{-pt}\, \mathrm{d}t = pF(p) - f(0).$$

When $p \to \infty$ $e^{-pt} \to 0$; the left-hand side of the equation tends correspondingly to zero

$$0 = \lim_{p \to \infty} [pF(p) - f(0)].$$

Taking into account that $f(0) = \lim_{t\to 0} f(t)$ we get formula (41). This formula makes it possible to find the initial value of function $f(t)$ directly from the transform when $t = 0+$. It establishes the existence of the equality between two particular values: the value of function $f(t)$ at the origin (approaching from the right) and the value of function $pF(p)$ at an infinitely remote point.

When $p \to 0$

$$\lim_{p\to 0}\int_0^\infty f'(t)\, e^{-pt}\, dt = \lim_{p\to 0}[pF(p) - f(0)].$$

Since $p$ does not depend on $t$ the order in which the operations are carried out in the left-hand side can be varied

$$\int_0^\infty f'(t)\, dt = \lim_{p\to 0}[pF(p) - f(0)]. \tag{13.43}$$

The left-hand side of equation (43) can be represented in the form

$$\lim_{t\to\infty}\int_0^t f'(t)\, dt = \lim_{t\to\infty}[f(t) - f(0)]. \tag{13.44}$$

Equating the right-hand sides of (43) and (44) we get formula (42). This formula is only valid if the limit of function $f(t)$ exists when $t \to \infty$.

EXAMPLE 11. To find the initial and final values of a function having a transform $1/(p + a)$.

From (41) and (42)

$$f(0) = \lim_{p\to\infty}\frac{p}{p+a} = 1 \quad \text{and} \quad f(\infty) = \lim_{p\to 0}\frac{p}{p+a} = 0.$$

The validity of the resulting equations can be proved from the original $f(t) = e^{-at}$.

**Multiplying Transforms (Real Convolution Theorem)**

If

$$f_1(t) \sim F_1(p) \quad \text{and} \quad f_2(t) \sim F_2(p),$$

then

$$F_1(p)\, F_2(p) \sim \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau = \int_0^t f_2(\tau)\, f_1(t-\tau)\, d\tau. \tag{13.45}$$

For proof we shall introduce the parameter $\tau$ and use the transformation $F_1(p) = \int_0^\infty f_1(\tau) e^{-p\tau}\, d\tau$. Then

$$F_1(p) \cdot F_2(p) = \int_0^\infty f_1(\tau)\, F_2(p)\, e^{-p\tau}\, d\tau.$$

According to the delay theorem

$$e^{-p\tau}F_2(p) \sim f_2(t-\tau).$$

Consequently,

$$f_1(\tau)\,F_2(p)\,e^{-p\tau} \sim f_1(\tau)\,f_2(t-\tau).$$

Integrating with respect to $\tau$, in accordance with (37) we get

$$\int_0^\infty f_1(\tau)\,F_2(p)\,e^{-p\tau}\,d\tau \sim \int_0^\infty f_1(\tau)\,f_2(t-\tau)\,d\tau. \tag{13.46}$$

In view of the fact that $f_2(t-\tau) = 0$ when $t-\tau < 0$ or $\tau > t$, the upper limit in the right-hand side of relationship (46) can be replaced by $t$. In addition, as a result of the equality of functions $f_1(t)$ and $f_2(t)$, the subscripts 1 and 2 after the integral sign in the right-hand side of (46) can change places.

*Corollary.* Using the rule for differentiating an original, we get

$$pF_1(p)\,F_2(p) \sim \frac{d}{dt}\int_0^t f_1(\tau)\,f_2(t-\tau)\,d\tau = \frac{d}{dt}\int_0^t f_2(\tau)\,f_1(t-\tau)\,d\tau. \tag{13.47}$$

The operation expressed by the integrals (45) is known as a *convolution* in the real domain (functions $f_1(t)$ and $f_2(t)$ "convolve").

The theorem establishes that convolution in the real domain corresponds to multiplication in the transform domain.

If a transform is the product of more than two cofactors, the convolution theorem can be applied to the cofactors grouped in pairs (in any order).

EXAMPLE 12. To prove formula (30), using the convolution theorem.

The transform $F(p)/p$ is the product of two functions: $F_1(p) = F(p)$ and $F_2(p) = 1/p$ which correspond to the originals $f_1(t) = f(t)$ and $f_2(t) = 1$. Therefore, from (45),

$$\frac{F(p)}{p} \sim \int_0^t f(\tau)\,d\tau = \int_0^t f(t)\,dt.$$

EXAMPLE 13. To find the original from the transform $1/(p+a)(p+b)$ if it is known that $1/(p+a) \sim e^{-at}$; $1/(p+b) \sim e^{-bt}$.

From (45)

$$\int_0^t e^{-a\tau}\,e^{-b(t-\tau)}\,d\tau = e^{-bt}\,\frac{e^{-(a-b)\tau}}{-(a-b)}\Bigg|_0^t = \frac{1}{b-a}\,[e^{-at} - e^{-bt}].$$

**Duhamel's Integral**

If $f_1(t) \sim F_1(p)$ and $f_2(t) \sim F_2(p)$, then

$$pF_1(p)\,F_2(p) \sim f_1(t)\,f_2(0) + \int_0^t f_1(\tau)\,f'_{2(t)}(t-\tau)\,d\tau$$

$$= f_1(0)\,f_2(t) + \int_0^t f_2(\tau)\,f'_{1(t)}(t-\tau)\,d\tau. \tag{13.48}$$

Let us apply the formula for differentiating a definite integral with respect to parameter $t$.

It is known from the practice of analysis that if $\varphi(t) = \int_{\psi_1(t)}^{\psi_2(t)} f(t, \tau)\, d\tau$ then

$$\frac{d\varphi(t)}{dt} = \int_{\psi_1(t)}^{\psi_2(t)} f'_{(t)}(t, \tau)\, d\tau + \psi'_2(t) f[t, \psi_2(t)] - \psi'_1(t) f[t, \psi_1(t)].$$

In the present case $f(t, \tau) = f_1(\tau) f_2(t - \tau)$ or

$$f(t, \tau) = f_2(\tau) f_1(t - \tau); \quad \psi_1(t) = 0; \quad \psi_2(t) = t; \quad \psi'_2(t) = 1.$$

Consequently,

$$\frac{d}{dt}\int_0^t f_1(\tau) f_2(t - \tau)\, d\tau = \int_0^t f_1(\tau) f'_{2(t)}(t - \tau)\, d\tau + f_1(t) f_2(0)$$

or

$$\frac{d}{dt}\int_0^t f_2(\tau) f_1(t - \tau)\, d\tau = \int_0^t f_2(\tau) f'_{1(t)}(t - \tau)\, d\tau + f_2(t) f_1(0).$$

The use of formula (48) simplifies the calculation by comparison with (47) in those cases when, after function $f_1(t - \tau)$ or $f_2(t - \tau)$ has been differentiated, the integrand is simplified thus making the calculation of the integral easier.

In Section 13 we shall make a special examination of a method of network transient analysis with the aid of Duhamel's integral.

EXAMPLE 14. To find the original of the transform $p/(p + a)(p + b)$ using formula (48). It is known that $1/(p + a) \sim e^{-at}$; $1/(p + b) \sim e^{-bt}$. Consequently, from (48)

$$\frac{p}{(p + a)(p + b)} \sim e^{-at} + \int_0^t e^{-a\tau}[-be^{-b(t-\tau)}]\, d\tau$$

$$= e^{-at} - be^{-bt}\int_0^t e^{-(a-b)\tau}\, d\tau = e^{-at} - \frac{be^{-bt}\left|e^{-(a-b)\tau}\right|_0^t}{-(a - b)}$$

$$= e^{-at} - \frac{b}{b - a}[e^{-at} - e^{-bt}] = \frac{be^{-bt} - ae^{-at}}{b - a}.$$

**Multiplying Originals (Complex Convolution)**

If $f_1(t) \sim F_1(p)$ and $f_2(t) \sim F_2(p)$, then

$$f_1(t) f_2(t) \sim \frac{1}{2\pi j}\int_{c_2 - j\infty}^{c_2 + j\infty} F_2(q) F_1(p - q)\, dq. \tag{13.49}$$

Let us denote the transform which corresponds to the product of the functions $f_1(t)$ and $f_2(t)$ by $F(p)$

$$F(p) = \int_0^\infty f_1(t) f_2(t) e^{-pt}\, dt. \tag{13.50}$$

For the function $f_2(t)$ let us substitute its integral expression with function $F_2(p)$ based on formula (7)

$$f_2(t) = \frac{1}{2\pi j} \int_{c_2-j\infty}^{c_2+j\infty} F_2(q)\, e^{qt}\, dq. \tag{13.51}$$

In equation (51) the integration is carried out in a complex plane along a path parallel to the imaginary axis which is taken from the right to a distance $c_2$ from the origin.

Substituting (51) in (50) gives

$$F(p) = \int_0^\infty f_1(t) \left[ \frac{1}{2\pi j} \int_{c_2-j\infty}^{c_2+j\infty} F_2(q)\, e^{qt}\, dq \right] e^{-pt}\, dt.$$

In view of the fact that the functions can be transformed by the Laplace method, one can change their order of integration

$$F(p) = \frac{1}{2\pi j} \int_{c_2-j\infty}^{c_2+j\infty} F_2(q) \int_0^\infty f_1(t)\, e^{-(p-q)t}\, dt\, dq.$$

By analogy with (6)

$$\int_0^\infty f_1(t)\, e^{-(p-q)t}\, dt = F_1(p-q).$$

Theorem (49) stating that the operation of convolution in a complex plane corresponds to the multiplication of two time functions is thereby proved. This theorem is analogous to the theorem on the multiplication of transforms.

In view of the equality of functions $f_1(t)$ and $f_2(t)$, the positions of the subscripts 1 and 2 after the integral sign in the right-hand side of (49) can be reversed.

The necessity of integrating in a complex plane is obviated if one of the transforms, e.g. $F_1(p) = A(p)/B(p)$ is a rational algebraic fraction with $n$ simple poles. In this case the relationship

$$f_1(t) f_2(t) \sim \sum_{k=1}^{n} \frac{A(p_k)}{B'(p_k)} F_2(p - p_k) \tag{13.52}$$

holds.

In fact, as will be shown in Section 7, the original

$$f_1(t) = \sum_{k=1}^{n} \frac{A(p_k)\, e^{p_k t}}{B'(p_k)}$$

corresponds to the transform $F_1(p) = A(p)/B(p)$, $p_k$ being the root of the equation $B(p) = 0$.

Thus

$$f_1(t)\,f_2(t) = \sum_{k=1}^{n} \frac{A(p_k)}{B'(p_k)}\, e^{p_k t} f_2(t)$$

and

$$\int_0^\infty f_1(t)\,f_2(t) e^{-pt}\,dt = \int_0^\infty \sum_{k=1}^{n} \frac{A(p_k)}{B'(p_k)} f_2(t)\, e^{-(p-p_k)t}\,dt$$

$$= \sum_{k=1}^{n} \frac{A(p_k)}{B'(p_k)} F_2(p - p_k).$$

In this case it is permissible to change the order of the integration and summation owing to the linearity of the Laplace transformation. The resulting expression is equivalent to the relationship (52).

EXAMPLE 15. To find the transform of the function $\sin at \,.\, \sin bt$.
It is known that

$$\sin at \sim \frac{a}{p^2 + a^2}\,; \quad \sin bt \sim \frac{b}{p^2 + b^2}.$$

The poles of the function $F_1(p) = a/(p^2 + a^2)$ occur at the points $-ja$. Consequently, from (52)

$$f_1(t)\,.\,f_2(t) \sim \frac{a}{2(ja)} \cdot \frac{b}{(p - ja)^2 + b^2} + \frac{a}{2(-ja)} \frac{b}{(p + ja)^2 + b^2}$$

$$= \frac{2abp}{[p^2 + (a + b)^2]\,[p^2 + (a - b)^2]}. \tag{13.53}$$

## Commutability of the Laplace Transformation and Re or Im Symbol Operations

If $f(t) \sim F(p)$, then

$$\operatorname{Re} f(t) \sim \operatorname{Re} F(p)\,; \tag{13.54}$$

$$\operatorname{Im} f(t) \sim \operatorname{Im} F(p). \tag{13.55}$$

Here $f(t)$ is a complex function.

The proof follows directly from the linearity of the Laplace transformation. Supposing that $f(t) = f_1(t) + jf_2(t)$, we get

$$F(p) = F_1(p) + jF_2(p).$$

EXAMPLE 16. To find the transforms of the functions $e^{-at}\cos(\omega_0 t + \psi)$ and $e^{-at}\sin(\omega_0 t + \psi)$, using the transform of the complex function $e^{-at}e^{j(\omega_0 t + \psi)}$.
As is known

$$e^{-at}e^{j(\omega_0 t + \psi)} \sim \frac{e^{j\psi}}{p + a - j\omega_0} = \frac{e^{j\psi}(p + a + j\omega_0)}{(p + a)^2 + \omega_0^2}$$

$$= \frac{(p + a)\cos\psi - \omega_0 \sin\psi}{(p + a)^2 + \omega_0^2} + j\,\frac{\omega_0 \cos\psi + (p + a)\sin\psi}{(p + a)^2 + \omega_0^2}.$$

Thus, in accordance with (54) and (55)

$$e^{-at}\cos(\omega_0 t + \psi) \sim \frac{(p+a)\cos\psi - \omega_0 \sin\psi}{(p+a)^2 + \omega_0^2}, \tag{13.56}$$

$$e^{-at}\sin(\omega_0 t + \psi) \sim \frac{(p+a)\sin\psi + \omega_0 \cos\psi}{(p+a)^2 + \omega_0^2}. \tag{13.57}$$

**Multiplying an Original by a Cosine or Sine**

If $f(t) \sim F(p)$, then

$$\cos\omega_0 t \,.\, f(t) \sim \frac{1}{2}[F(p - j\omega_0) + F(p + j\omega_0)]; \tag{13.58}$$

$$\sin\omega_0 t \,.\, f(t) \sim \frac{1}{2j}[F(p - j\omega_0) - F(p + j\omega_0)]. \tag{13.59}$$

Here $f(t)$ is a real or a complex function.

The proof is based on the complex shifting theorem:

$$\cos\omega_0 t \,.\, f(t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]f(t) \sim \frac{1}{2}[F(p - j\omega_0) + F(p + j\omega_0)];$$

$$\sin\omega_0 t \,.\, f(t) = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]f(t) \sim \frac{1}{2j}[F(p - j\omega_0) - F(p + j\omega_0)].$$

**Integrating the Product of Originals**

If $f_1(t) \sim F_1(p)$ and $f_2(t) \sim F_2(p)$, then

$$\int_0^\infty f_1(t)\, f_2(t)\, \mathrm{d}t = \frac{1}{2\pi j}\int_{c_2 - j\infty}^{c_2 + j\infty} F_2(q)\, F_1(-q)\, \mathrm{d}q. \tag{13.60}$$

The proof follows from formula (49)

$$\int_0^\infty f_1(t)\, f_2(t) e^{-pt}\, \mathrm{d}t = \frac{1}{2\pi j}\int_{c_2 - j\infty}^{c_2 + j\infty} F_2(q) F_1(p - q)\, \mathrm{d}q.$$

Assuming that $p = 0$ we get (60).

When $f_1(t) = f_2(t) = f(t)$ and $F_1(p) = F_2(p) = F(p)$, in accordance with (60),

$$\int_0^\infty [f(t)]^2\, \mathrm{d}t = \frac{1}{2\pi j}\int_{c_2 - j\infty}^{c_2 + j\infty} F(q)\, F(-q)\, \mathrm{d}q$$

or, the same thing

$$\int_0^\infty [f(t)]^2\, \mathrm{d}t = \frac{1}{2\pi j}\int_{c_0 - j\infty}^{c_0 + j\infty} F(p)\, F(-p)\, \mathrm{d}p. \tag{13.61}$$

Formula (61) is analogous to Parseval's relation.

The basic properties of the Laplace and Carson transformations are given in Appendix III.

Formulae written as Laplace transforms can be converted into Carson formulae by using (13).

For example, the Laplace transform of $f'(t)$ is $pF_L(p) - f(0)$. Consequently, the Carson transform of $f'(t)$ is

$$p[pF_L(p) - f(0)] = p[F_C(p) - f(0)].$$

The Laplace transform of $e^{-\lambda t}f(t)$ is $F_L(p + \lambda)$. Consequently, the Carson transform of the same time function is, in accordance with (13),

$$pF_L(p + \lambda) = \frac{p}{p + \lambda} F_C(p + \lambda).$$

As will be seen from Appendix IV, many relationships remain identical for both transformations. For example, in accordance with the real shifting theorem, the Laplace transform of function $f(t - t_0)$ is $e^{-pt_0}F_L(p)$. In accordance with (13) the Carson transform of the same function is

$$p[e^{-pt_0}F_L(p)] = e^{-pt_0}F_C(p).$$

## 6. Calculation of Inverse Transforms

Formula (7) for the inverse Laplace transformation is the solution of an integral equation (6) with respect to the function $f(t)$. Therefore, if the transform $F(p)$ is known, the original $f(t)$ can be found by calculating the line integral (7)

$$f(t) = \frac{1}{2\pi j} \int_{c_0 - j\infty}^{c_0 + j\infty} F(p)\, e^{pt}\, dp.$$

Here $c_0$ is a real number which satisfies the inequality (8). The latter can be realized for various values of $c_0$. Therefore any infinite straight line parallel to the imaginary axis and at a distance $c \geqq c_0$ from the latter can be taken for integration, all the singularities of function $F(p)$ remain on the left of the path of the integration.

When formula (7) is used in practice the integration path along a straight line parallel to the imaginary axis is replaced by a closed contour which makes it possible to calculate the integral with the aid of residues (Appendix I).

Let us examine the integral (7), when $t > 0$, taking $\lim_{|p| \to \infty} F(p) = 0$. We shall construct an arc $L_1$ of radius $R$ and its centre at the origin, on the left of the straight line $c_0$ (Fig. 7).

Obviously,

$$\frac{1}{2\pi j}\oint F(p)\,e^{pt}\mathrm{d}p = \frac{1}{2\pi j}\int_{AB} F(p)\,e^{pt}\,\mathrm{d}p + \frac{1}{2\pi j}\int_{L_1} F(p)\,e^{pt}\,\mathrm{d}p. \quad (13.62)$$

Here $\oint$ is the closed-contour integral; $\int_{AB}$ is the line integral along the straight line $AB$; and $\int_{L_1}$ is the line integral along the arc $L_1$.

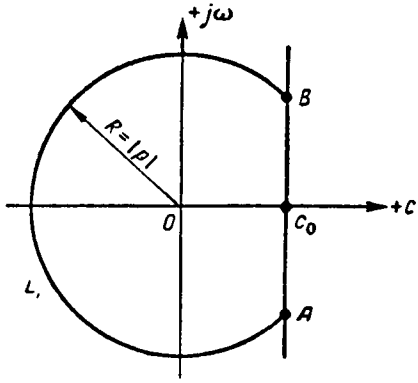



FIG. 7. Integration around a closed contour $ABL_1$.

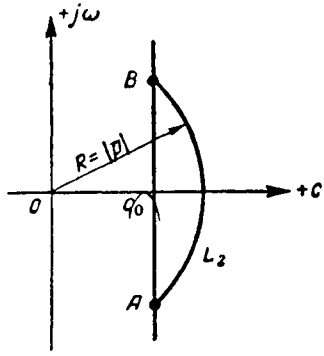



FIG. 8. Integration around a closed contour $ABL_2$.

According to Jordan's lemma (Refs. 38, 66 and 95), if function $F(p)$ tends to zero when modulus $|p|$ increases indefinitely, then when $t > 0$

$$\lim_{R\to\infty}\int_{L_1} F(p)\,e^{pt}\,\mathrm{d}p = 0. \quad (13.63)$$

In these conditions the contour integral in the left-hand side of equality (62) is equal to the sum of the residues at the poles of the integrand $F(p)e^{pt}$.

Thus

$$\frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty} F(p)\,e^{pt}\,dp = \frac{1}{2\pi j}\oint_{|p|\to\infty} F(p)\,e^{pt}\,dp$$

$$= \sum \text{Res}\, F(p)\,e^{pt}. \tag{13.64}$$

Let us now examine the integral (7) when $t < 0$. Let us construct an arc $L_2$ of radius $R$, with its centre at the origin, on the left on the straight line (Fig. 8).

As in the previous calculation

$$\frac{1}{2\pi j}\oint F(p)\,e^{pt}\,dp = \frac{1}{2\pi j}\int_{AB} F(p)\,e^{pt}\,dp$$

$$+ \frac{1}{2\pi j}\int_{L_2} F(p)\,e^{pt}\,dp. \tag{13.65}$$

Here $\int_{L_2}$ is the line integral along the arc $L_2$.

If the radius $R$ increases indefinitely when $t < 0$, the integral along $L_2$ tends to zero, according to Jordan's lemma.

In the region bounded by the contour under examination ($c \geqq c_0$) the integrand is analytic (there are no singularities). Therefore, according to Cauchy's theorem

$$\frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty} F(p)\,e^{pt}\,dp = \frac{1}{2\pi j}\oint F(p)\,e^{pt}\,dp = 0. \tag{13.66}$$

Consequently the inverse Laplace transformation yields function $f(t)=0$ when $t < 0$.

Formula (64), given above for $t > 0$, makes it possible to educe the expansion theorem (*vide* Section 7) with application to the rational function $F(p)$, i.e. when the only singularities of the integrand are poles.

## 7. Expansion Theorem

Let us suppose that a transform is given as a fraction

$$F(p) = \frac{F_1(p)}{F_2(p)}, \tag{13.67}$$

function $F(p)$ having no other singularities besides poles. The location of the poles of function $F(p)$ can be determined from the roots of the equation

$$F_2(p) = 0. \tag{13.68}$$

Let us denote the $n$ roots of equation (68) by $p_1, p_2, \ldots, p_n$. There are then two possibilities: (a) all the roots will be simple, or (b) certain or all of the roots will be multiple.

We shall consider the two cases separately.

**(a) Simple roots**

It is known from the theory of complex variable functions that the residue of function $\varphi(p)/\psi(p)$, with respect to a pole of the first order $p = p_k$, is

$$\operatorname*{Res}_{p \to p_k} \frac{\varphi(p)}{\psi(p)} = \left[\frac{\varphi(p)}{\psi'(p)}\right]_{p=p_k}. \tag{13.69}$$

If the function of the complex variable $p$ has the form $[F_1(p)/F_2(p)]\, e^{pt}$, all the roots of equation (68) being simple (poles of the first order), then it follows from (64) and (69) that

$$\frac{F_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{F_1(p_k)\, e^{p_k t}}{F_2'(p_k)}. \tag{13.70}$$

This is the general form of the expansion theorem for the case of simple roots.

EXAMPLE 17. To find the inverse transform of

$$\frac{1}{(p+a)(p+b)},$$

using the expansion theorem (70).

In the given case

$$F_1(p) = 1; \quad F_2(p) = (p+a)(p+b).$$

The roots of equation (68) equal $p_1 = -a$ and $p_2 = -b$, respectively. The derivative of the denominator is $F_2'(p) = p + a + b + p$.

From (70)

$$\frac{1}{(p+a)(p+b)} \sim \frac{e^{-at}}{-a+b} + \frac{e^{-bt}}{-b+a} = \frac{e^{-at} - e^{-bt}}{b-a}.$$

*A particular case — when one of the roots equals zero.*

Then

$$F_2(p) = pF_3(p).$$

Let us denote the root equal to zero by $p_n$. The roots $p_1, p_2, \ldots, p_{n-1}$ can then be determined from the equation $F_3(p) = 0$. The derivative of $F_2(p) = pF_3(p)$ with respect to $p$ is

$$[F_2(p)]' = F_3(p) + pF_3'(p),$$

the roots $p_1, p_2, \ldots, p_{n-1}$ converting this formula into $p_k F_3'(p_k)$. In conjunction with this, formula (70) yields

$$\frac{F_1(p)}{pF_3(p)} \sim \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n-1} \frac{F_1(p_k)e^{pk^t}}{p_k F_3'(p_k)}. \tag{13.71}$$

Formula (71) is an extended form of the expansion theorem for the case of *simple* roots, when one of the roots equals zero.

EXAMPLE 18. To find the inverse transform of

$$\frac{1}{p(p+a)(p+b)},$$

using the expansion theorem (71).

In this case

$$F_1(p) = 1, \quad F_2(p) = pF_3(p) = p(p+a)(p+b),$$

i.e

$$F_3(p) = (p+a)(p+b).$$

The roots of the equation $F_2(p) = 0$ equal, respectively,

$$p_1 = -a, \quad p_2 = -b, \quad p_3 = 0.$$

The derivative

$$F_3'(p) = p + a + p + b.$$

Consequently, from (71)

$$\frac{1}{p(p+a)(p+b)} \sim \frac{1}{ab} + \frac{e^{-at}}{(-a)(-a+b)} + \frac{e^{-bt}}{(-b)(-b+a)}$$

$$= \frac{1}{ab} - \frac{be^{-at} - ae^{-bt}}{ab(b-a)}.$$

## (b) Multiple Roots

Let us suppose that equation (68) has $n$ different roots, $p_1, p_2, \ldots, p_n$, the root $p_1$ being repeated $m_1$ times, $p_2$ being repeated $m_2$ times and $p_n$ being repeated $m_n$ times.

It is known from the theory of complex variable functions that if function $[F_1(p)/F_2(p)]\, e^{pt}$ has a pole of the order $m_k$ at point $p_k$, the residue of the function at this point will be

$$\operatorname{Res} \frac{F_1(p)}{F_2(p)} e^{pt} = \frac{1}{(m_k - 1)!} \times \left[ \frac{d^{m_k-1}}{dp^{m_k-1}} \left( \frac{F_1(p)}{F_2(p)} (p - p_k)^{m_k} e^{pt} \right) \right]_{p=p_k}.$$

Therefore, from (64)

$$\frac{F_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{1}{(m_k - 1)!} \times \left[ \frac{d^{m_k-1}}{dp^{m_k-1}} \left( \frac{F_1(p)}{F_2(p)} (p - p_k)^{m_k} e^{pt} \right) \right]_{p=p_k}. \tag{13.72}$$

The resulting formula represents the expansion theorem for *multiple* roots.

EXAMPLE 19. To determine the original from the transform

$$\frac{p+2}{(p-1)^2p^3},$$

using the expansion theorem (72).

In this case

$$F_1(p) = p + 2; \quad F_2(p) = (p-1)^2p^3.$$

The given function has a pole of the second order $p_1 = 1$ and a pole of the third order $p_2 = 0$. Consequently $m_1 = 2$ and $m_2 = 3$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}p}\frac{(p+2)e^{pt}}{p^3}\right]_{p=1} = (3t-8)e^t;$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}p^2}\frac{(p+2)e^{pt}}{(p-1)^2}\right]_{p=0} = 2t^2 + 10t + 16.$$

Therefore, in accordance with (72),

$$\frac{p+2}{(p-1)^2p^3} \sim (3t-8)e^t + t^2 + 5t + 8.$$

Formula (70) given above for simple roots follows from formula (72) if $m_1 = m_2 = \cdots = m_n = 1$, since, when $m_k = 1$, $(m-1)! = 1$ and

$$\left[\frac{\mathrm{d}^{m_k-1}}{\mathrm{d}p^{m_k-1}}\left(\frac{F_1(p)}{F_2(p)}(p-p_k)^{m_k}e^{pt}\right)\right]_{p=p_k} = \left[\frac{F_1(p)e^{pt}}{\dfrac{F_2(p)}{p-p_k}}\right]_{p=p_k} = \frac{F_1(p_k)e^{p_kt}}{F_2'(p_k)}.$$

If the characteristic equation (68) includes both simple and multiple-order roots, formulae (70) and (72) are to be used separately, for simple and multiple-order roots respectively.

It should be noted that it is most convenient to compute $F_2'(p_k)$ as $[F_2(p)/(p-p_k)]_{p=p_k}$. Common factors $p - p_k$ are first cancelled, and $p = p_k$ is then substituted.

## 8. Transform Tables

The expansion theorem has great practical significance. Together with other properties of the Laplace transformation described in Section 5 it makes it possible to compile transform tables which greatly simplify and speed up the process of finding an original function from its transform. These tables are used extensively in the Laplace transformation method.

When *The Handbook of Operational Calculus* (Ref. 19) is being used for practical purposes it must be borne in mind that it gives the transforms in the Carson form. This also applies to the tables given by Krug, Lur'ye and Efros and Danilevskii (Refs. 34, 38 and 61). The transform tables given

by Gardner and Barnes (Ref. 14) and Kontorovich (Ref. 28) are written in the Laplace form.

In the typical examples dealt with above the transforms were given in the Laplace form. Functions of this kind, and certain others which occur in practice, are set out in the summary table given in Appendix IV.

## 9. Solution of Differential Equations for Electrical Circuits by the Laplace Transformation

Let us first consider a single-loop circuit consisting of a resistance $r$, an inductance $L$ and a capacitance $C$ connected in series (Fig. 9). We shall suppose that at an instant $t = 0$ this circuit is energized by an e.m.f. source $u(t)$. It is required to find the current $i(t)$ in the circuit.

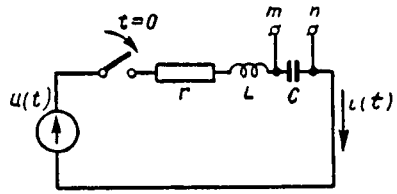



FIG. 9. Switching in an e.m.f. $u(t)$ in an $rLC$ network.

Using Kirchhoff's voltage law and taking into account the direction of the given e.m.f. $u(t)$ and the reference direction of the current $i(t)$, we can write the following linear integro-differential equation:

$$u(t) = ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t i(t)\,\mathrm{d}t + u(0). \tag{13.73}$$

It is assumed that the capacitor is initially charged to the voltage $u(0)$. This voltage can be defined as the potential difference between point $m$ and point $n$, when $t = 0$, i.e. $u(0) = u_m(0) - u_n(0)$. It can be either positive or negative.

Regarding the given e.m.f. $u(t)$ and the unknown current $i(t)$ as original functions, we can assume that their transforms are $U(p)$ and $I(p)$, i.e. $u(t) \sim U(p)$ and $i(t) \sim I(p)$.

In view of the fact that $u(0)$ is a constant

$$u(0) \sim \frac{u(0)}{p}.$$

As a result of the linearity of the Laplace transformation and the differentiation and integration theorems, the following transform equation

$$U(p) = rI(p) + pLI(p) - Li(0) + \frac{I(p)}{pC} + \frac{u(0)}{p} \tag{13.74}$$

corresponds to equation (73).

Here $i(0) = 0$ since, at the instant $t = 0$, the circuit was open and consequently there was no current in the inductance $L$.

It is evident from comparing the integro-differential equations (73) with the transform equation (74) that, owing to the fact that the operations of differentiating and integrating the original correspond to multiplying and dividing the transform by $p$, the transform equation will be algebraic, i.e. simpler than the original integro-differential equation. Its solution with respect to the transform $I(p)$ yields

$$I(p) = \frac{U(p) - \dfrac{u(0)}{p}}{r + pL + \dfrac{1}{pC}}. \tag{13.75}$$

The unknown function $i(t)$, the original, can be found from transform (75) by one of the above-mentioned methods, namely: (a) by an inverse Laplace transformation, (b) by the expansion theorem (or other transform rules), or (c) by using prepared tables.

The following is thus the general order for solving a problem:

(a) a differential, or integro-differential, equation describing the transient under examination is composed;

(b) the transforms of the given and unknown time functions are introduced;

(c) a transform equation corresponding to the original differential, or integro-differential, equation is written down;

(d) the transform equation is solved algebraically with respect to the transform of the unknown function;

(e) the original of the unknown function is found from the resulting transform equation.

If the capacitance is not charged at the instant $t = 0$, then

$$I(p) = \frac{U(p)}{r + pL + \dfrac{1}{pC}}. \tag{13.76}$$

The expression in the denominator is the generalized impedance of the network for the complex frequency $p = c + j\omega$ (Chapter VI, Section 13)

$$Z(p) = r + pL + \frac{1}{pC}; \tag{13.77}$$

it has the same structure as the net impedance (if $j\omega$ is substituted for $p$)

$$Z(j\omega) = r + j\omega L + \frac{1}{j\omega C}.$$

In the same way as the current sinor is multiplied by $j\omega L$ or divided by $j\omega C$ in the steady state, the current transform is either multiplied by $pL$ or divided by $pC$ in zero initial conditions.

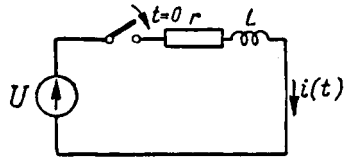



FIG. 10. Example 20.

Thus the expressions $pL$ and $1/pC$ are *transform analogues* of the impedances $j\omega L$ and $1/j\omega C$.

EXAMPLE 20. At an instant $t = 0$ a network consisting of an $r$ and $L$ in series is excited by a constant voltage source $U$ (Fig. 10). To find the current $i(t)$.

When $t \geqq 0$, the process under examination is described by the differential equation

$$U = ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t}.$$

Supposing $i(t) \sim I(p)$ we obtain the transform equation

$$\frac{U}{p} = rI(p) + pLI(p),$$

from which

$$I(p) = \frac{U}{p(r + pL)}.$$

Inverting the transform equation back into the time domain, we obtain

$$i(t) = \frac{U}{r}\left(1 - e^{-\frac{r}{L}t}\right).$$

EXAMPLE 21. At an instant $t = 0$ an $rL$ loop is short-circuited (Fig. 11). To determine the loop current $i(t)$ in this short-circuited loop.

When $t \geqq 0$, $0 = ri(t) + L\,\mathrm{d}i(t)/\mathrm{d}t$.

Supposing $i(t) \sim I(p)$, we get

$$0 = rI(p) + pLI(p) - Li(0),$$

from which

$$I(p) = \frac{Li(0)}{r + pL}.$$

Converting to the original we get

$$i(t) = i(0)\, e^{-\frac{r}{L}t}.$$

Here $i(0)$ is the instantaneous value of the current in the inductance when $t = 0$

$$i(0) = \frac{U_m}{\sqrt{[(r_0 + r)^2 + (\omega L)^2]}} \sin(\psi - \varphi);$$

$$\varphi = \operatorname{arc\,tan} \frac{\omega L}{r_0 + r}.$$

As we remarked in Chapter XI, Section 3 the transient response in a short-circuited $rL$ network is conditioned by the amount of energy stored in the magnetic field of the inductance up to the instant of switching.

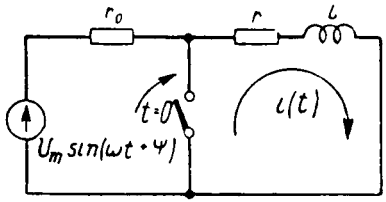



FIG. 11. Example 21.

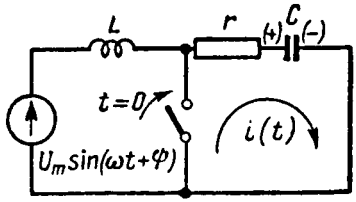



FIG. 12. Example 22.

EXAMPLE 22. At an instant $t = 0$ an $rC$ network is short-circuited (Fig. 12). To determine the loop current $i(t)$ in the short-circuited loop.

When $t \geqq 0$

$$0 = ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + u_C(0).$$

Supposing $i(t) \sim I(p)$, we find

$$0 = rI(p) + \frac{I(p)}{pC} + \frac{u_C(0)}{p},$$

from which

$$I(p) = \frac{-u_C(0)}{p\left(r + \frac{1}{pC}\right)}.$$

Converting to the original, we get

$$i(t) = -\frac{u_C(0)}{r}\, e^{-\frac{t}{rC}}.$$

Here $u_C(0)$ is the initial value of the voltage across the capacitance when $t = 0$, i.e.

$$u_C(0) = \frac{U_m}{\omega C \sqrt{\left[r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}} \sin\left(\psi - \varphi - \frac{\pi}{2}\right),$$

$$\varphi = \operatorname{arc\,tan} \frac{\omega L - \frac{1}{\omega C}}{r}.$$

The transient in a short-circuited network is conditioned by the amount of energy stored in the electrical field of the capacitance up to the instant of switching.

In the given case the minus sign indicates that when the value $u_C(0)$ is positive, i.e. when the polarity of the capacitance is as shown in Fig. 12, the capacitance discharges in the direction opposite to the reference direction.

EXAMPLE 23. At an instant $t = 0$, a network consisting of $r$, $L$ and $C$ elements in series is connected to a constant voltage source $U$. It is required to find the laws of the change in the current and the voltage across in the capacitance on the assumption that the capacitance is uncharged for $t < 0$ and $r < 2\sqrt{(L/C)}$.

When $t \geqq 0$

$$U = ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_0^t i\,\mathrm{d}t,$$

from which

$$I(p) = \frac{U}{p\left(r + pL + \frac{1}{pC}\right)}.$$

Denoting $c = r/2L$ and $\omega_0^2 = 1/LC$ we can rewrite the transform equation in this form:

$$I(p) = \frac{U}{L(p^2 + 2cp + \omega_0^2)} = \frac{U}{L[(p + c)^2 + \omega_0^2]},$$

where $\omega_c^2 = \omega_0^2 - c^2$.

The unknown original corresponds to the resulting current transform

$$i(t) = \frac{U}{\omega_c L} e^{-ct} \sin \omega_c t.$$

The voltage across the capacitance can be determined from the formula

$$u_C(t) = U\left[1 - e^{-ct}\left(\frac{c}{\omega_c}\sin \omega_c t + \cos \omega_c t\right)\right].$$

When

$$c \ll \omega_c, \quad u_C(t) \approx U[1 - e^{-ct} \cos \omega_c t],$$

i.e. the voltage across the capacitance is oscillatory and attenuating. In the first half-period when $t = \pi/\omega_c$, $\cos \omega_c t = \cos \pi = -1$, and hence

$$u_C = U\left(1 + e^{-c\frac{\pi}{\omega_c}}\right).$$

For a small value of $c$

$$u_C \approx 2U.$$

Let us now consider a multi-loop network. If we take the loop currents as the unknowns, the problem of finding the currents is reduced to solving a system of $n$ equations (according to the number of independent loops).

For the general case, in accordance with Kirchhoff's voltage law, we get a set of $n$ integro-differential equations for the instantaneous values of the loop currents:

$$u_k(t) = \sum_{m=1}^{n} \left[ r_{km} i_m(t) + L_{km} \frac{\mathrm{d}i_m(t)}{\mathrm{d}t} + \frac{1}{C_{km}} \int_0^t i_m(t)\, \mathrm{d}t \right] + u_{Ck}(0). \quad (13.78)$$

$$(k = 1, 2, \ldots n).$$

Here $u_k(t)$ is the loop e.m.f. and $u_{Ck}$ is the algebraic sum of the initial voltages across the capacitances in the $k$-th loop.

Parameters with subscripts $k = m$ belong to the individual elements of the $k$ loops, and parameters with subscripts $k \neq m$ belong to the common elements of the $k$ and $m$ loops. As in a single-loop network, the initial voltage across the capacitance is positive if the polarity of the capacitor is the opposite of the positive direction of the loop current $i_k(t)$ (the reference direction of the loop is that of the loop current). The rule of the signs of the voltages contained in (78) agrees with that given above in Chapter I, Section 6 and Chapter IV, Section 2.

As was mentioned above, the energy of the magnetic field which is stored in the inductances until the instant at which the process under examination begins, and the energy of the electrical field which is stored in the capacitances, have an effect on the subsequent transient response. The presence of this energy is taken into account by the initial conditions for the current in the inductances and the voltages across the capacitances when $t = 0$. The initial voltages of the capacitances $u_{Ck}(0)$ are contained in equations (78). The initial currents in the inductances are taken into account when writing the transform equations.

Let us suppose $i_m(t) \sim I_m(p)$. From (78) we get Laplace-transformed equations with initial conditions

$$U_k(p) = \sum_{m=1}^{n} \left[ r_{km} I_m(p) + L_{km} p I_m(p) - L_{km} i_m(0) + \frac{I_m(p)}{pC_{km}} \right] + \frac{u_{Ck}(0)}{p}$$

$$(k = 1, 2, \ldots n). \quad (13.79)$$

Transferring the terms containing the initial conditions to the left-hand side of the equations, we obtain

$$U_k(p) + \sum_{m=1}^{n} L_{km} i_m(0) - \frac{u_{Ck}(0)}{p} = \sum_{m=1}^{n} \left[ r_{km} + pL_{km} + \frac{1}{pC_{km}} \right] I_m(p). \quad (13.80)$$

Thus the analysis of transients with non-zero initial conditions (i.e. taking the previous conditions into account) can be reduced by artificial means to solving a problem for zero initial conditions, but with additional e.m.f.s equal, in the transform equations, to $Li(0)$ for the inductances and $u_C(0)/p$ for the capacitances of the given circuit, respectively.

Here $i(0)$ is the initial current in the inductance $L$ and $u_C(0)$ is the initial voltage across the capacitance $C$. The right-hand side of the equations can

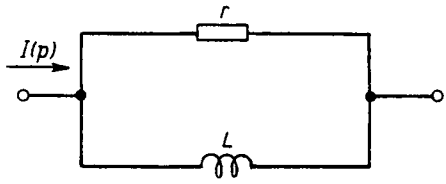



FIG. 13. Distribution of current transforms through branches.

then be written as for zero initial conditions, i.e. when $i(0) = 0$ and $u_C(0) = 0$.

In view of the fact that the structure of the right-hand side of equations (80) is analogous to that of the steady-state equations in complex form (if $j\omega$ is substituted for $p$), all the rules for the complex method of calculation retain their validity in the Laplace-transformed form for zero initial conditions. In particular, the distribution of the currents between the

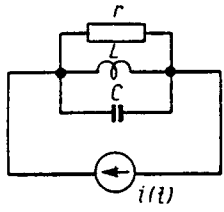



FIG. 14. Network with two nodes and a current source.

separate branches and the voltage drops across various sections can be found in transform form from the complex method rules. For example, if a current $I(p)$ is flowing through the two-terminal network shown in Fig. 13, the current in the resistance $r$ is, for zero initial conditions,

$$\frac{pL}{r + pL} I(p).$$

The differential and integro-differential equations, together with their corresponding transform equations, can also be based on Kirchhoff's current law.

Thus, for the circuit shown in Fig. 14, we get

$$i(t) = \frac{1}{r}u(t) + C\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{1}{L}\int_0^t u(t)\,\mathrm{d}t + i_L(0),$$

where $u(t)$ is the voltage between the nodes;
$i(t)$ is the source current;
$i_L(0)$ is the current through the inductance when $t = 0$.

Supposing that $u(t) \sim U(p)$ and $i(t) \sim I(p)$, we find

$$I(p) = \frac{U(p)}{r} + pCU(p) - CU_C(0) + \frac{U(p)}{pL} + \frac{I_L(0)}{p}.$$

Here $u_C(0)$ is the initial voltage across the capacitance.

## 10. Solutions in Real and Complex Form

If the e.m.f. transform in equations (80) corresponds to real e.m.f. values, then the current expressions are transforms of real functions of $i(t)$. Correspondingly, the non-zero initial conditions are also determined from instantaneous values of the currents $i_L(0)$ in the inductances and the voltages $u_C(0)$ across the capacitances. The possibility of obtaining the real values of unknown electrical quantities directly, e.g. $i(t)$, is decidedly convenient. But such calculation procedure frequently demands a great deal of time and effort compared with the complex form of calculation.

We mentioned in Chapter VI, Section 13 that the complex form of writing equations and their corresponding electrical quantities (current, voltage, charge) is not only applicable to steady-state analysis but also to transient analysis. Let us introduce transforms of complex e.m.f.s and currents in the complex form of calculating a transient, using the complex initial values of the currents $I_L(0)$ in the inductances and the voltages $U_C(0)$ across the capacitances.

The solution of transform equations with respect to the transforms of unknown electrical values permits the use of the original functions which are complex quantities. The required functions can be determined from the imaginary part of the latter.

When there are nonsinusoidal voltages (constant or exponential) among the sinusoidal voltages in a network, the problem can be solved in complex form by the method described above, if the nonsinusoidal voltages are first multiplied by $j$.

EXAMPLE 24. To determine the current $i(t)$ produced in an $rL$-network (Fig. 15) when it is excited by a sinusoidal e.m.f.

$$u(t) = U_m \sin(\omega_0 t + \psi).$$

In accordance with the above

$$U_m e^{j(\omega_0 t + \psi)} = r\dot{I}(t) + L\frac{d\dot{I}(t)}{dt}.$$

If $\dot{I}(t) \sim I(p)$, then

$$U_m e^{j\psi}\frac{1}{p - j\omega_0} = rI(p) + pLI(p),$$

from which

$$I(p) = \frac{U_m e^{j\psi}}{(p - j\omega_0)(r + pL)}.$$

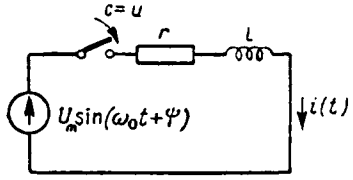



FIG. 15. Example 24.

Converting to the original, we get

$$\dot{I}(t) = \frac{U_m e^{j(\omega_0 t + \psi)}}{r + j\omega_0 L} - \frac{U_m e^{j\psi}}{r + j\omega_0 L} e^{-\frac{r}{L}t}.$$

The unknown current is determined from the imaginary part of this equation, i.e.

$$i(t) = \frac{U_m \sin(\omega_0 t + \psi - \varphi)}{\sqrt{[r^2 + (\omega_0 L)^2]}} - \frac{U_m \sin(\psi - \varphi)}{\sqrt{[r^2 + (\omega_0 L)^2]}} e^{-\frac{r}{L}t},$$

where

$$\varphi = \arctan\frac{\omega_0 L}{r}.$$

## 11. Calculation by the Equivalent Source Method Taking Initial Conditions into Account

If any branch is opened or closed in an electrical circuit with non-zero initial conditions (i.e. when there are currents in the inductances or voltages across the capacitances), then the transient produced by a change in the circuit can be calculated by the *equivalent source method.* When the branch is opened it is convenient to use the voltage source theorem, and when it is closed, the current source theorem (Chapter IV, Section 9).

Figure 16 illustrates the case of a branch of a network opening at an instant $t = 0$. As in Chapter IV, Section 9 the letter $A$ denotes an active network and the letter $P$ a passive. Here $u_0(t)$ is the voltage before the branch is closed, and by convention continuing for $t > 0$.

When $t < 0$ the circuit conditions can either be steady or transient. The calculation procedure is the same in either case. Connecting two e.m.f.s equal in value $u_0(t)$ and opposite in direction (Fig. 16*b*) to the closed branch and using the superposition method, we obtain the two circuit diagrams given in Fig. 16*c* and *d*, the first of which has a current distribution exactly identical to that of the original circuit, Fig. 16*a*. Thus, *when a branch is closed the transient response can be found by superimposing the current distribution of the previous conditions (when the branch was open) which*

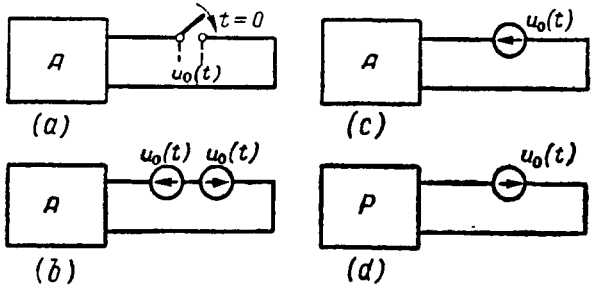



FIG. 16. Closing a branch of a network.

*conventionally last for $t > 0$, on to the current distribution which results from connecting an e.m.f. source $u_0(t)$ to the passive network.*

In the transform calculation method, the transform $U_0(p)$ is substituted for the function $u_0(t)$. Then, if conditions were steady when $t < 0$, the voltage can be either constant or alternating (sinusoidal or nonsinusoidal).

The transform $U_0(p) = U/p$ is introduced into the calculation when $u_0(t) = U = \text{const}$, and the transform

$$U_m e^{j\psi} \frac{1}{p - j\omega_0}$$

is introduced for a sinusoidal voltage $u_0(t)$ written in complex form $U_m e^{j(\omega_0 t + \psi)}$.

The current in the closing branch can then be found from the transform

$$I(p) = \frac{U_0(p)}{Z(p)}, \tag{13.81}$$

where $Z(p)$ is the impedance of the entire passive network regarded from the switch points. With application to Fig. 16 it is equal to the input impedance function of the passive two-terminal network.

EXAMPLE 25. To find the current $i(t)$ in the branch with resistance $r_2$, closing when $t = 0$ (Fig. 17).

The voltage at the open contact is $U$. Consequently, the transform of $U_0(p)$ is $U/p$.

The input impedance function of the passive circuit, regarded from the switch points, equals

$$Z(p) = r_2 + \frac{r_1 pL}{r_1 + pL}.$$

Consequently, from (81)

$$I(p) = \frac{U}{p\left(r_2 + \dfrac{r_1 pL}{r_1 + pL}\right)} = \frac{U(r_1 + pL)}{p[r_1 r_2 + (r_1 + r_2)pL]}.$$

Converting to the original

$$i(t) = \frac{U}{r_2} + \frac{U\left[r_1 - \dfrac{r_1 r_2}{r_1 + r_2}\right] e^{-\frac{r_1 r_2}{(r_1 + r_2)L}t}}{-\dfrac{r_1 r_2}{(r_1 + r_2)L}(r_1 + r_2)L}$$

$$= \frac{U}{r_2} - \frac{r_1 U}{r_2(r_1 + r_2)} e^{-\frac{r_1 r_2}{(r_1 + r_2)L}t}.$$

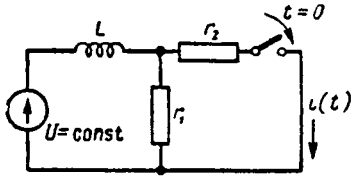



FIG. 17. Example 25.

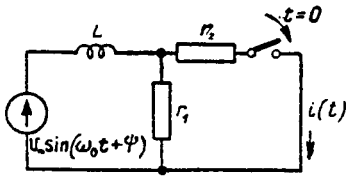



FIG. 18. Example 26.

EXAMPLE 26. To find the current $i(t)$ in the branch with resistance $r_2$, closing when $t = 0$ (Fig. 18), if

$$u(t) = U_m \sin(\omega_0 t + \psi).$$

The transform

$$U_0(p) = \frac{r_1 U_m e^{j\psi}}{r_1 + j\omega_0 L} \cdot \frac{1}{p - j\omega_0}$$

represents the voltage in complex form at the open contact, i.e. $[r_1/(r_1 + j\omega_0)]\, U_m e^{j(\omega_0 t + \psi)}$.

The input impedance $Z(p)$ is in this case the same as in the previous example. Consequently, from (81)

$$I(p) = \frac{r_1 U_m e^{j\psi}}{(r_1 + j\omega_0 L)(p - j\omega_0)\left(r_2 + \dfrac{r_1 pL}{r_1 + pL}\right)}$$

or

$$I(p) = \frac{r_1 U_m e^{j\psi}(r_1 + pL)}{(r_1 + j\omega_0 L)(p - j\omega_0)[r_1 r_2 + (r_1 + r_2)pL]}.$$

The original current in complex form is

$$\dot{I}(t) = \frac{r_1 U_m e^{j(\omega_0 t + \psi)}}{r_1 r_2 + j(r_1 + r_2)\omega_0 L} - \frac{r_1^2 U_m e^{j\psi} e^{-\frac{r_1 r_2}{(r_1 + r_2)L}t}}{(r_1 + j\omega_0 L)(r_1 + r_2)[r_1 r_2 + j(r_1 + r_2)\omega_0 L]}.$$

The unknown function $i(t)$ is determined from the imaginary part of the resultant expression

$$i(t) = \frac{r_1 U_m \sin(\omega_0 t + \psi - \varphi_1)}{\sqrt{[(r_1 r_2)^2 + (r_1 + r_2)^2 (\omega_0 L)^2]}} - \frac{r_1^3 U_m \sin(\psi - \varphi_1 - \varphi_2) e^{-\frac{r_1 r_2}{(r_1 + r_2)L} t}}{(r_1 + r_2)\sqrt{[r_1^2 + (\omega_0 L)^2]}\sqrt{[(r_1 r_2)^2 + (r_1 + r_2)^2 (\omega_0 L)^2]}},$$

where

$$\varphi_1 = \arctan \frac{(r_1 + r_2)\omega_0 L}{r_1 r_2},$$

$$\varphi_2 = \arctan \frac{\omega_0 L}{r_1}.$$

Figure 19 shows a branch of a network opening at an instant $t = 0$. Here $i_0(t)$ is the branch current before the branch opens, conventionally continuing for $t > 0$.

By connecting two current sources $i_0(t)$ acting in opposite directions (Fig. 19*b*) in parallel at the breaking point of the circuit and using the superposition method, we obtain the two circuits shown in Fig. 19*c* and *d*, in the first of which the current distribution is the same as that in the original circuit (Fig. 19*a*).

Thus, *when a branch is open-circuited a transient occurs as a result of the superposition of the current distribution of the previous state (when the*

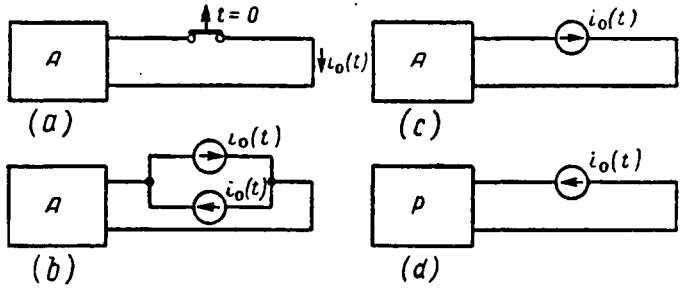



FIG. 19. Opening a branch of a network.

*branch was closed), which conventionally continues for* $t > 0$, *on the current distribution obtained by connecting the current source* $i_0(t)$ *to a passive network.*

The voltages in the transient in question can also be found by the superposition method, i.e. the algebraic addition of the voltage drops occurring in identical sections of the circuits shown in Fig. 19*a* and *d*.

In the transformation method of calculation, the current $i_0(t)$ is replaced by the transform $I_0(p)$. The voltage at the breaking point is determined from the transform

$$U(p) = Z(p) I_0(p). \qquad (13.82)$$

Here $Z(p)$ is the input impedance of the passive two-terminal network shown in Fig. 19*d*.

When a transient is the result of a combination of sequential switching operations, it is defined as a switching transient. If the switching transient under examination is caused by varying the electrical circuit when $t = t_1$, then for a change in the variable $\tau = t - t_1$ the time zero reference shifts to $t_1$ (see Examples 28 and 29).

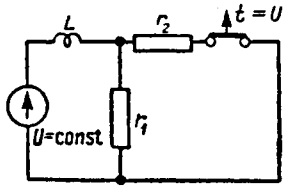



FIG. 20. Example 27.

EXAMPLE 27. In the circuit shown in Fig. 20 the branch with resistance $r_2$ is opened when $t = 0$. To determine the voltage at the point where the circuit is broken.

A current $U/r_2$ passes through resistance $r_2$ when $t < 0$.

Consequently, the current transform is $I(p) = U/pr_2$.

The input impedance function of the passive network reckoned from the breaking point is $Z(p) = r_2 + r_1pL/(r_1 + pL)$.

Having obtained the transform of the unknown voltage from (82)

$$U(p) = \frac{U}{r_2 p}\left(r_2 + \frac{r_1 pL}{r_1 + pL}\right) = \frac{U[r_1 r_2 + (r_1 + r_2)\, pL]}{r_2 p(r_1 + pL)},$$

we can find the original

$$u(t) = U + \frac{r_1 U e^{-\frac{r_1}{L}t}}{r_2}.$$

EXAMPLE 28 (Ref. 14). When $t = 0$ the contact $K_1$ in the Fig. 21 circuit closes, and when $t = t_2$ contact $K_2$ closes.

It is required to find the current in the loop of contact $K_2$.

When $t < t_1$, i.e. before contact $K_2$ closes the voltage on it is determined by the voltage drop across resistance $r_2$:

$$u_0(t) = \frac{Ur_2}{r_1 + r_2}(1 - e^{-at}),$$

where

$$a = \frac{r_1 + r_2}{L}.$$

The unknown current produced by this voltage is cancelled at the instant $t_1$ as a result of contact $K_2$ closing.

Let us change the variable to $t = \tau + t_1$. Consequently,

$$u_0(\tau) = \frac{Ur_2}{r_1 + r_2}(1 - e^{-a\tau}e^{-at_1}).$$

Let $u_0(Q) \sim U_0(p)$, i.e.

$$U_0(p) = \frac{Ur_2}{r_1 + r_2}\left[\frac{1}{p} - \frac{e^{-at_1}}{p + a}\right] = \frac{Ur_2[a + p(1 - e^{-at_1})]}{(r_1 + r_2)\,p(p + a)}.$$

The transform of the required current can be found from (81)

$$I(p) = \frac{U_0(p)}{Z(p)},$$

where

$$Z(p) = \frac{r_2(r_1 + pL)}{r_1 + r_2 + pL}.$$

Substituting the expressions $U_0(p)$ and $Z(p)$ in $I(p)$ we get

$$I(p) = \frac{U[a + p(1 - e^{-at_1})]}{(r_1 + r_2)\,p(p + b)}.$$

Here

$$b = \frac{r_1}{L}.$$

Converting to the original we find

$$i(\tau) = \frac{U}{r_1} - U\left(\frac{1}{r_1} - \frac{1 - e^{-at_1}}{r_1 + r_2}\right)e^{-b\tau}\,(\tau > 0)$$

or converting to $t$, $(t > t_1)$,

$$i(t) = \frac{U}{r_1} - U\left(\frac{1}{r_1} - \frac{1 - e^{-at_1}}{r_1 + r_2}\right)e^{-b(t - t_1)}.$$

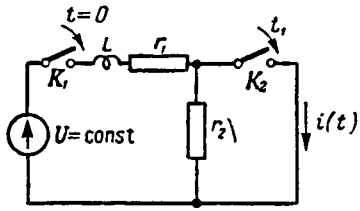



FIG. 21. Example 28.

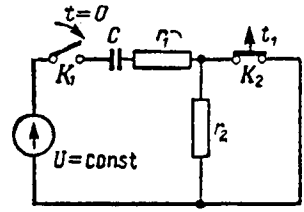



FIG. 22. Example 29.

EXAMPLE 29 (Ref. 14). When $t = 0$ the contact $K_1$ in the circuit shown in Fig. 22 closes and when $t = t_1$ contact $K_2$ opens. It is required to determine the voltage across the terminals of $K_2$ after it has opened.

When $t < t_1$, i.e. before $K_2$ has opened, the current passing through it is

$$i_0(t) = \frac{U}{r_1}e^{-at},$$

where

$$a = \frac{1}{r_1 C}.$$

Let us change the variable to $t = \tau + t_1$. Consequently,

$$i_0(\tau) = \frac{U}{r_1}e^{-at_1}e^{-a\tau}.$$

Supposing that $i_0(\tau) \sim I_0(p)$, we get

$$I_0(p) = \frac{U}{r_1} e^{-at_1} \frac{1}{p+a}.$$

Let us denote the input impedance of the passive network regarded from a terminal of the contact $K_2$ by $Z(p)$, i.e.

$$Z(p) = \frac{r_2\left(r_1 + \frac{1}{pC}\right)}{r_1 + r_2 + \frac{1}{pC}} = \frac{r_1 r_2}{r_1 + r_2} \cdot \frac{p+a}{p+b},$$

where

$$b = \frac{1}{(r_1 + r_2)C}.$$

The transform of the unknown voltage across the terminals of $K_2$ can be found from formula (82)

$$U(p) = Z(p)\, I_0(p) = \frac{Ur_2 e^{-at_1}}{(r_1 + r_2)(p+b)}.$$

The original of the unknown function is

$$u(\tau) = \frac{Ur_2}{r_1 + r_2} e^{-(at_1 + b\tau)} \quad (\tau > 0);$$

or

$$u(t) = \frac{Ur_2}{r_1 + r_2} e^{-(at_1 - bt_1)} e^{-bt} \quad (t > t_1).$$

## 12. Switching-in Formulae

Switching-in formulae are often used in solving problems with zero initial conditions. As we showed above, switching an e.m.f. source into a passive network produces a current $I(p) = U(p)/Z(p)$.

The impedance function $Z(p)$ can either represent the input impedance ($I(p)$ is the input current in this case) or the transfer impedance with respect to another loop, the current of which is unknown.

If the e.m.f. to which the network is connected is given in the form of an exponential function $Ue^{at}$, then $U(p) = U/(p - a)$ and, correspondingly,

$$I(p) = \frac{U}{(p-a) \cdot Z(p)}. \tag{13.83}$$

Using the expansion theorem (Section 7), we get the switching-in formula for an *exponential e.m.f.*

$$I(p) \sim \frac{Ue^{at}}{Z(a)} + \sum_{k=1}^{n} \frac{Ue^{p_k t}}{(p_k - a)\, Z'(p_k)}. \tag{13.84}$$

In practice the most frequently occurring cases are those of switching in a d.c. or a sinusoidal e.m.f.

When a constant e.m.f. is switched in, $a = 0$ and consequently

$$i(t) = \frac{U}{Z(0)} + \sum_{k=1}^{n} \frac{Ue^{p_k t}}{p_k Z'(p_k)}. \tag{13.85}$$

When a *sinusoidal e.m.f.* (in complex form) is switched in, $a = j\omega_0$, the complex amplitude $U_m e^{j\psi}$ being substituted for $U$

$$\dot{I}(t) = \frac{U_m e^{j(\omega_0 t + \psi)}}{Z(j\omega_0)} + \sum_{k=1}^{n} \frac{U_m e^{j\psi} e^{p_k t}}{(p_k - j\omega_0) Z'(p_k)}. \tag{13.86}$$

It is not difficult to confirm that the switching-in formulae (84)–(86) remain valid when $Z(p)$ is a fractional function in the form $Z(p) = A(p)/B(p)$ although it is then more convenient to use (70).

The use of switching-in formulae for networks with mutual inductances is complicated by the difficulty of calculating $Z(p)$.

In such cases the method explained in Section 9 is usually used.

Zero initial conditions were assumed in educing the switching-in formulae. The first term in the switching-in formula is the forced current, while the remainder of the formula expresses the natural current. In the case of non-zero initial conditions the switching-in formulae can be applied with the aid of the equivalent source method (Section 11).

## 13. Transient Analysis Based on the Superposition Principle (Duhamel's Integral)

The method of analysing a transient by the superposition principle (Duhamel's integral) is based on a preliminary examination of the transient in so-called indicial response, i.e. the response of a system to a unit-step function (Fig. 23*a*). Then, with the aid of one of the forms of Duhamel's integral the transient in the same network under the effect of an arbitrary e.m.f. $u(t)$ is calculated (Fig. 23*b*).

In the first case the current $i_1(t)$ which occurs under the effect of the unit step function, corresponds to the transform

$$I_1(p) = \frac{1}{pZ(p)}.$$

Function $i_1(t) \sim 1/pZ(p)$ is referred to in the literature as the *indicial admittance.*

In the second case, the current $i(t)$ which is produced under the effect of the e.m.f. $u(t)$ corresponds to the transform

$$I(p) = \frac{U(p)}{Z(p)} = pU(p)\frac{1}{pZ(p)}. \tag{13.87}$$

Consequently, the problem is reduced to finding the original function from the transform (87), the cofactors of which are $p\ U(p) \sim u(t)$ and $1/pZ(p) \sim i_1(t)$.

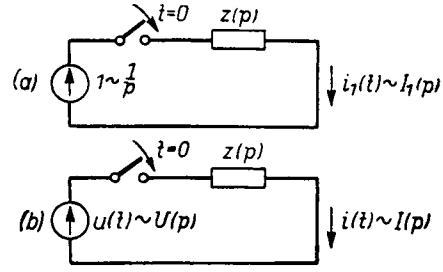



FIG. 23. Switching in an e.m.f. source: (*a*) unit e.m.f.; (*b*) arbitrary e.m.f.

In this case the corollary of the convolution theorem, i.e. formula (47) or Duhamel's formula (48) are applicable:

$$\left.\begin{aligned} i(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t u(\tau)\, i_1(t-\tau)\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t u(t-\tau)\, i_1(\tau)\,\mathrm{d}\tau \\ &= u(t)\, i_1(0) + \int_0^t u(\tau)\, i'_{1(t)}(t-\tau)\,\mathrm{d}\tau \\ &= u(0)\, i_1(t) + \int_0^t u'_{(t)}\,(t-\tau)\, i_1(\tau)\,\mathrm{d}\tau. \end{aligned}\right\} \tag{13.88}$$

Thus, *knowing the indicial response of a network, it is possible to find the response of the same network to an arbitrary excitation function by using one of the integral formulae* (88).

It should be noted that these formulae are not the only forms of Duhamel's integrals: there are other forms. Thus supposing that $\tau = t - \theta$ and bearing in mind that $\mathrm{d}\tau = -\mathrm{d}\theta$, and $\mathrm{d}t/\mathrm{d}\theta = 1$, we get

$$i(t) = u(t)\, i_1(0) - \int^0 u(t-\theta)\, i'_{(\theta)}(\theta)\,\mathrm{d}\theta,$$

from which, after substituting $\tau$ for $\theta$ under the integral sign, we find

$$i(t) = u(t)\, i_1(0) + \int_0^t u\,(t-\tau)\, i'_{(\tau)}(\tau)\,\mathrm{d}\tau. \tag{13.89}$$

Similarly

$$i(t) = u(0)\, i_1(t) - \int_t^0 u'_{(\theta)}(\theta)\, i_1(t - \theta)\, \mathrm{d}\theta$$

$$= u(0)\, i_1(t) + \int_0^t u'_{(\tau)}(\tau)\, i_1(t - \tau)\, \mathrm{d}\tau. \qquad (13.90)$$

In the case of a discontinuous excitation function, that form of the Duhamel integral is preferable which does not need the differentiation

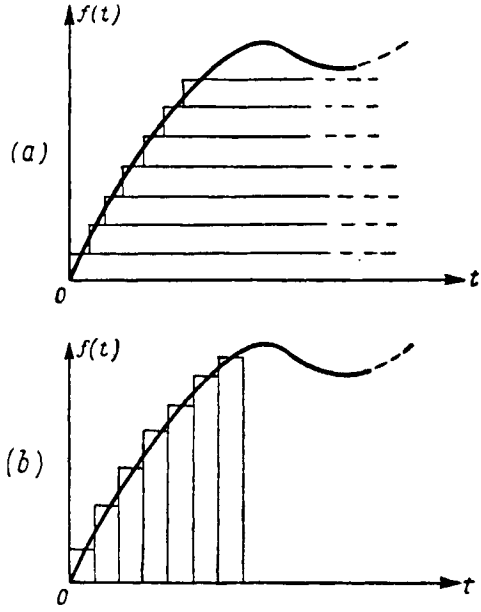



FIG. 24. Approximation of a given curve by an infinite number of step functions or pulses (superposition method).

of the function of concern. Otherwise, owing to the discontinuity, the circuit response should be taken into account additionally.

Formulae (88)–(90) are called *superposition formulae* since they can be educed by the superposition method: regarding function $u(t)$ as an infinite sum of step functions (with infinitely small steps, Fig. 24*a*), it is possible to find the unknown current $i(t)$ in the form of (89) and (90), by the summation operation of the step functions. The superposition method can also be applied when the given function is represented in the form of the sum of rectangular pulses (Fig. 24*b*).

In accordance with (87) the transform of the required current is equal to the product of the two functions $1/Z(p)$ and $U(p)$. Regarding function

$1/Z(p)$ as a transform corresponding to a certain original function $y(t)$ and bearing in mind that $U(p) \sim u(t)$, it is possible to use the formula for multiplying transforms (formula (45), real convolution theorem). As a result we obtain formula (60) given in Chapter XII, Section 7:

$$i(t) = \int_0^t u(\tau)\, y(t-\tau)\, d\tau = \int_0^t u(t-\tau)\, y(\tau)\, d\tau. \qquad (13.91)$$

The expression $1/Z(p)$ can be treated as a transform of the current occurring in the given network under the effect of an e.m.f. with a transform equal to unity. As will be shown in the following section, such an e.m.f. is a unit impulse.

Consequently, the original $y(t)$ is identical to the so-called impulse response of the network, i.e. the response unit impulse. The superposition method is desirable in those cases when the response to the unit step function or unit impulse is known (or can be easily determined), the excitation waveform being of a piecewise continuous type.

EXAMPLE 30. To determine the current in a network consisting of a resistance $r$ and inductance $L$ connected in series, if at the instant $t = 0$ a sinusoidal e.m.f. is switched into the network

$$u(t) = U_m \sin(\omega_0 t + \varphi).$$

The indicial response of the given network is

$$i_1(t) = \frac{1}{r}\left(1 - e^{-\frac{r}{L}t}\right).$$

Consequently, the response of the same network to the function $\dot{U}(t) = U_m e^{j(\omega_0 t + \varphi)}$ can be found from formula (88)

$$\dot{I}(t) = \dot{U}(t)\, i_1(0) + \int_0^t \dot{U}(\tau)\, i'_{1(t)}(t-\tau)\, d\tau$$

$$= \frac{1}{r}\int_0^t U_m e^{j(\omega_0 \tau + \varphi)}\, \frac{r}{L} e^{-\frac{r}{L}(t-\tau)}\, d\tau = \frac{U_m e^{j\varphi}}{r + j\omega_0 L}\left[e^{j\omega_0 t} - e^{-\frac{r}{L}t}\right].$$

The use of formula (91) produces the same result.

The unknown current $i(t)$ is determined from the imaginary part of the resulting complex expression

$$i(t) = \frac{U_m}{\sqrt{[r^2 + (\omega_0 L)^2]}}\left[\sin(\omega_0 t + \varphi - \varphi) - \sin(\varphi - \varphi)\, e^{-\frac{r}{L}t}\right],$$

where

$$\varphi = \arctan\frac{\omega_0 L}{r}.$$

## 14. Initial Conditions Replaced by Pulse Functions

Let us examine function $f(t)$ which is given for $t > 0$ and equals zero when $t < 0$. When approaching from the right, i.e. from the positive values of $t$, let the initial value of the function equal $f(0)$. According to the differentiation theorem

$$\frac{df(t)}{dt} \sim pF(p) - f(0). \tag{13.92}$$

It is assumed here that the derivative is taken at any point on the curve $f(t)$, starting from point $f(0+)$.

Let us introduce an original $\delta_1(t)$ with a transform equal to unity, i.e. let us suppose that

$$\delta_1(t) \sim 1, \tag{13.93}$$

from which, on the strength of the linearity of the Laplace transformation

$$f(0)\delta_1(t) \sim f(0). \tag{13.94}$$

Adding together the left and right sides of (92) and (94) respectively, we get

$$\frac{df(t)}{dt} + f(0)\delta_1(t) \sim pF(p). \tag{13.95}$$

As will be shown later, the function $\delta_1(t)$ which is known as a *unit impulse* is a derivative of the unit step function.

The differentiation of the function, starting from the zero point $t = 0$ corresponds to multiplying the transform $F(p)$ by $p$. This proposition is equally valid for the second derivative and derivatives of a higher order. Thus, in accordance with the differentiation theorem

$$\frac{d^2f(t)}{dt^2} \sim p^2F(p) - pf(0) - f'(0). \tag{13.96}$$

Let us introduce an original $\delta_2(t)$ with a transform equal to $p$, i.e. let us suppose

$$\delta_2(t) \sim p \tag{13.97}$$

and, correspondingly, on the strength of the linearity of the Laplace transformation,

$$f(0)\delta_2(t) \sim pf(0). \tag{13.98}$$

In addition, from (96)

$$f'(0)\delta_1(t) \sim f'(0). \tag{13.99}$$

Adding together separately the right and left parts of (96), (98) and (99) we get

$$\frac{d^2f(t)}{dt^2} + f(0)\delta_2(t) + f'(0)\delta_1(t) \sim p^2F(p). \tag{13.100}$$

The function $\delta_2(t)$ which is known as a *unit doublet* is the second derivative of the unit step function.

Let us examine the pulse functions $\delta_1(t)$ and $\delta_2(t)$ in greater detail.

The unit impulse $\delta_1(t)$ is defined as the limit of the rectangular pulses $f_1(t)$ when $\tau \to 0$ (Fig. 25*a*). Here $\tau$ is the length of the pulse, and $1/\tau$ the magnitude of the pulse. The area covered by $f_1(t)$ is unity.

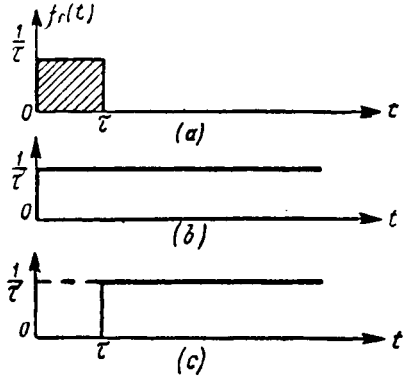



FIG. 25. Unit impulse as the limit of a rectangular pulse.

A rectangular pulse $f_1(t)$ can be represented as the difference between two step functions (Fig. 25*b* and *c*) with transforms $1/\tau p$ and $(1/\tau p)\, e^{-p\tau}$. Therefore, using the linear property of the Laplace transformation, we get

$$f_1(t) \sim \frac{1}{\tau p}(1 - e^{-p\tau}). \tag{13.101}$$

When $\tau \to 0$ the transform (101) assumes an indeterminate form 0/0. Discovering its indeterminate nature from L'Hôpital's rule, we find

$$\lim_{\tau \to 0} f_1(t) \sim \lim_{\tau \to 0} \frac{pe^{-p\tau}}{p} = 1.$$

Consequently, *a function with a transform equal to unity is the limit of a rectangular pulse $\tau$ in length and $1/\tau$ in height when $\tau \to 0$*. When the length of $\tau$ decreases, the area always remains equal to unity. Although this limit is physically impracticable we can theoretically regard a certain original $\delta_1(t)$ as corresponding to the function $F_1(p) = 1$.

Thus, *a unit impulse* $\delta_1(t)$ *denotes an infinitely large pulse of infinitely short length.*

Figure 26*a* gives a graph of the function $\int_0^t f_1(t)\,dt$ which becomes $\int_0^t \delta_1(t)\,dt$ when $\tau \to 0$. As Fig. 26*b* shows, the integral $\int_0^t \delta_1(t)\,dt$ is identical to the unit step function.

In its turn, $\delta_1(t) = \frac{d}{dt}\int_0^t \delta_1(t)\,dt$, i.e. a pulse function of the first order is equal to the derivative of the unit step function. When $t = 0$ the derivative is equal to infinity and when $t \lessgtr 0$ it equals zero.

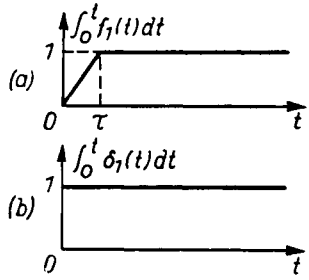



FIG. 26. Unit step function as the limit of an integral of a rectangular pulse.

With application to electrical networks, function $\delta_1(t)$ can be given as the charging current of a capacitor connected to an e.m.f. source in ideal conditions, i.e. when the resistance of the network $r = 0$. In fact, supposing that a circuit with a voltage source $u(t) = 1\text{V}$ and a capacitance $C$ (Fig. 27) is closed when $t = 0$, we obtain the expression

$$I(p) = \frac{1}{p} : \frac{1}{pC} = C,$$

for the current transform, from which it follows that $I(p) = 1$ when $C = 1$.

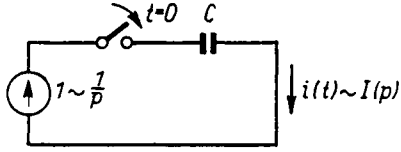



FIG. 27. Representation of a unit impulse (charging current of capacitance).

Let us now go on to examine the function $f_2(t)$ given in Fig. 28. From the real shifting theorem

$$f_2(t) \sim \frac{1}{p\tau^2}(1 - 2e^{-p\tau} + e^{-2p\tau}).$$

When $\tau \to 0$, by applying L'Hôpital's rule twice, we find

$$\lim_{\tau \to 0} f_2(t) \sim \lim_{\tau \to 0} \frac{1}{2p} (-2p^2 e^{-p\tau} + 4p^2 e^{-2p\tau}) = p.$$

We can thus assume, theoretically, *that a function with a transform p denotes the limit of two rectangular pulses $\tau$ in length and $1/\tau^2$ in height, which have different signs and are consecutive when $\tau \to 0$.* This function is the conventional original $\delta_2(t)$, a unit doublet.

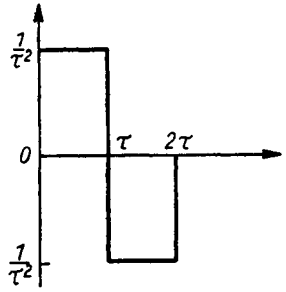



FIG. 28. Unit doublet as the limit of two rectangular pulses.

The conventional originals of the positive powers of the operator $p$ can be introduced analogously, i.e.

$$\delta_n(t) \sim p^{n-1}.$$

They become zero when $t \lesseqgtr 0$ and infinite when $t = 0$.

It should be noted that if the given function $f(t)$ equals zero when $t = 0+$, i.e. $f(0+) = 0$, then regardless of any finite discontinuity which

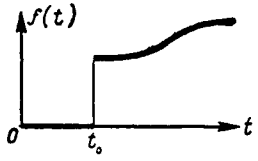



FIG. 29. Function satisfying the condition $f(0+) = 0$

the given function may have when $t > 0$ (Fig. 29), according to the differentiation theorem (25), differentiating function $f(t)$ from the instant $t = 0$ corresponds to multiplying transform $F(p)$ by $p$

$$\frac{df(t)}{dt} \sim pF(p).$$

Therefore with application to Fig. 29, for example, the derivative $df(t)/dt$ also contains a unit impulse at the point $t = t_0$.

The finding of transforms by using pulse functions in non-zero initial conditions is illustrated by the examples given below.

· EXAMPLE 31. To find the transform of the first derivative of a function in the form

$$f(t) = \begin{cases} \cos \omega_0 t & \text{when } t > 0, \\ 0 & \text{when } t < 0. \end{cases}$$

The given time function (Fig. 30*a*) has a transform $p/(p_2 + \omega_0^2)$. The initial value of the function $f(0) = 1$. The time derivative of the function $f(t)$ (Fig. 30*b*) is

$$\frac{df(t)}{dt} = -\omega_0 \sin \omega_0 t.$$

A unit impulse $\delta_1(t)$ obtained when differentiating with $t = 0$ is shown conventionally in Fig. 30*b*.

From (95)

$$-\omega_0 \sin \omega_0 t + \delta_1(t) \sim \frac{p^2}{p^2 + \omega_0^2},$$

from which

$$-\omega_0 \sin \omega_0 t \sim \frac{p^2}{p^2 + \omega_0^2} - 1 = -\frac{\omega_0^2}{p^2 + \omega_0^2}.$$

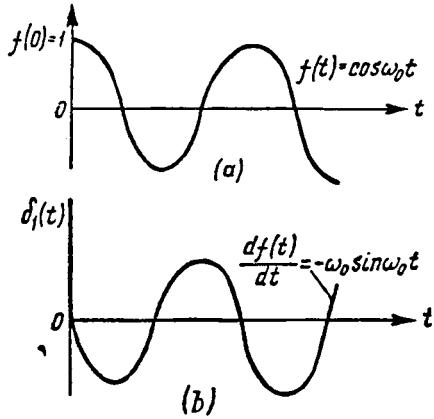



FIG. 30. Example 31.

EXAMPLE 32. To find the transform of the second derivative of a function in the form

$$f(t) = \begin{cases} e^{at} & \text{when } t > 0, \\ 0 & \text{when } t < 0. \end{cases}$$

The given time function has a transform $1/(p - a)$. The initial value of the function $f(0) = 1$. The first derivative of the function $f(t)$ equals $f'(t) = ae^{at}$, consequently $f'(0) = a$. The second derivative

$$\frac{d^2 f(t)}{dt^2} = a^2 e^{at}.$$

From (100)

$$a^2e^{at} + \delta_2(t) + a\delta_1(t) \sim \frac{p^2}{p-a},$$

from which

$$a^2e^{at} \sim \frac{p^2}{p-a} - p - a = \frac{a^2}{p-a}.$$

EXAMPLE 33. To find the transform of the first derivative of a function in the form

$$f(t) = \begin{cases} Ae^{-a(t-t_0)} & \text{when } t > t_0, \\ 0 & \text{when } t < t_0. \end{cases}$$

According to the real shifting theorem the given time function (Fig. 31) has a transform $[A/(p+a)]\,e^{-pt_0}$. The first derivative of the function $f(t)$ equals

$$\frac{df(t)}{dt} = -Aae^{-a(t-t_0)}.$$

The value of the function when $t = t_0+$ is $f(t_0) = A$.

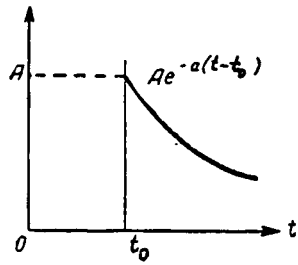



FIG. 31. Example 33.

On the strength of the above

$$-aAe^{-a(t-t_0)} + A\delta_1(t-t_0) \sim \frac{pA}{p+a}e^{-pt_0},$$

from which

$$-aAe^{-a(t-t_0)} \sim \frac{pA}{p+a}e^{-pt_0} - Ae^{-pt_0} = -\frac{aA}{p+a}e^{-pt_0}.$$

## 15. Laplace Transforms of Jump Functions and Sections of Periodic Functions

As we have shown in a number of examples (Section 13 *et al.*), the real shifting theorem is used in order to obtain the Laplace transform of jump functions.

Thus the jump function $f(t)$ given in Fig. 32 can be regarded as the sum of three intermittent functions having, respectively, the transforms

$$\frac{1}{p}A_0; \quad \frac{1}{p}(A_1 - A_0)\,e^{-pt_1}; \quad \frac{1}{p}(A_2 - A_1)\,e^{-pt_2}.$$

Consequently, the transform of function $f(t)$ is equal to

$$F(p) = \frac{1}{p}\left[A_0 + (A_1 - A_0)\, e^{-pt_1} + (A_2 - A_1)\, e^{-pt_2}\right].$$

If the function $f(t)$ contains not a finite number of steps as shown in Fig. 32 but an infinite number, the transform is expressed by an infinite series. Let us consider as an example a function with an infinite number of steps when $A_0 = 1$, $A_1 = 2$, $A_2 = 3$, ...; $t_1 = t_0$, $t_2 = 2t_0$, ... (Fig. 33).

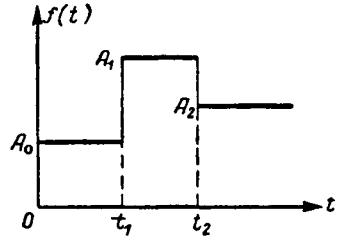



FIG. 32. Jump function.

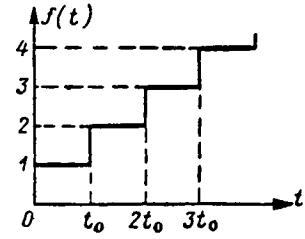



FIG. 33. Staircase function.

The transform of this type of function is expressed by a geometric progression

$$F(p) = \frac{1}{p}\left(1 + e^{-pt_0} + e^{-2pt_0} + e^{-3pt_0} + \cdots\right),$$

the sum of which is equal to

$$F(p) = \frac{1}{p(1 - e^{-pt_0})} = \frac{e^{\frac{pt_0}{2}}}{p\left(e^{\frac{pt_0}{2}} - e^{-\frac{pt_0}{2}}\right)}$$

$$= \frac{1}{2p}\left[\frac{2e^{\frac{pt_0}{2}}}{e^{\frac{pt_0}{2}} - e^{-\frac{pt_0}{2}}} - 1 + 1\right] = \frac{1}{2p}\left[1 + \cosh\frac{pt_0}{2}\right]. \tag{13.102}$$

If a function with an infinite number of steps (Fig. 34) is characterized by the conditions $A_0 = 1$, $A_1 = -1$, $A_2 = 1$, ...; $t_1 = t_0$, $t_2 = 2t_0$, ..., the geometric progression has the form

$$F(p) = \frac{1}{p}\left(1 - 2e^{-pt_0} + 2e^{-2pt_0} - \cdots\right)$$

$$= \frac{2}{p}\left(1 - e^{-pt_0} + e^{-2pt_0} - \cdots\right) - \frac{1}{p}$$

$$= \frac{2}{p(1 + e^{-pt_0})} - \frac{1}{p} = \frac{1 - e^{-pt_0}}{p(1 + e^{-pt_0})} = \frac{1}{p}\tanh\frac{pt_0}{2}. \tag{13.103}$$

The function shown in Fig. 34 is periodic when $t > 0$ and equals zero when $t < 0$. A general formula relating the transform $F(p)$ of the given function $f(t)$ (Fig. 35*a*) to transform $F_0(p)$ of function $f_0(t)$, coinciding with $f(t)$ throughout the first period and equal to zero outside it (Fig. 35*b*), may be given for periodic functions $f(t)$ of period $T$.

In fact, from the delay theorem

$$F(p) = F_0(p)\,(1 + e^{-pT} + e^{-2pT} + \cdots) = \frac{F_0(p)}{1 - e^{-pT}}. \qquad (13.104)$$

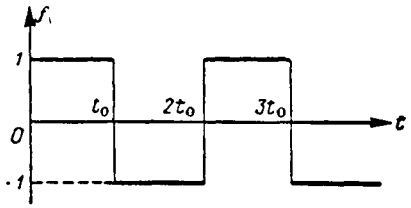



FIG. 34. Square function.

This formula can also be used to find $F_0(p)$ from the given transform $F(p)$

$$F_0(p) = F(p)\,(1 - e^{-pT}). \qquad (13.105)$$

The transform of function $f_1(t)$ which, in the time interval from 0 to $nT$ ($n$ being an integer), coincides with the periodic function $f(t)$ and outside this interval equals zero, can be found from the formula

$$F_1(p) = F(p)\,(1 - e^{-pnT}).$$

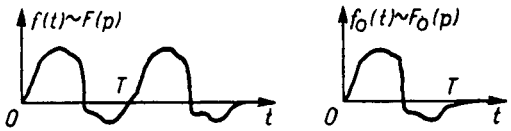



FIG. 35. Finding the transform of the function $f(t)$ from the transform of the function $f_0(t)$.

If the periodic function $f(t)$ possesses half-wave symmetry, i.e.

$$f(t) = -f\left(t + \frac{T}{2}\right),$$

as, for example, happens in Fig. 36*a*, then the transform $F(p)$ of the given function $f(t)$ can be expressed by the transform $\Psi(p)$ of the function $\psi(t)$ which coincides with $f(t)$ throughout the first half of the period and is equal to zero outside this interval (Fig. 36*b*).

Using the delay theorem,

$$F(p) = \Psi(p)\left(1 - e^{-\frac{pT}{2}} + e^{-pT} - e^{-\frac{3}{2}pT} + \cdots\right)$$

$$= \frac{\Psi(p)}{1 + e^{-\frac{pT}{2}}}. \tag{13.106}$$

In its turn,

$$\Psi(p) = F(p)\left(1 + e^{-\frac{pT}{2}}\right). \tag{13.107}$$

EXAMPLE 34. To find the transform of a half-period of a sine curve (Fig. 36*b*).

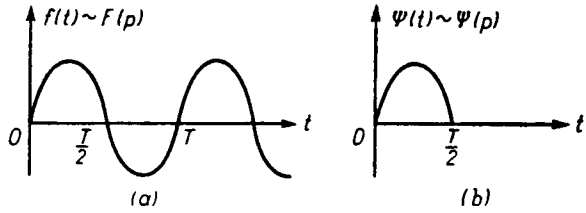



FIG. 36. Case of a function possessing half-wave symmetry (Example 34).

We shall use formula (107). In this case

$$F(p) = \frac{\omega_0}{p^2 + \omega_0^2}, \quad \omega_0 T = 2\pi;$$

consequently,

$$\Psi(p) = \frac{\omega_0}{p^2 + \omega_0^2}\left(1 + e^{-\frac{p\pi}{\omega_0}}\right). \tag{13.108}$$

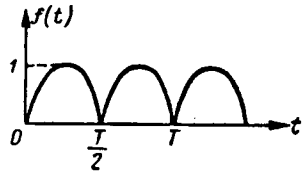



FIG. 37. Example 35.

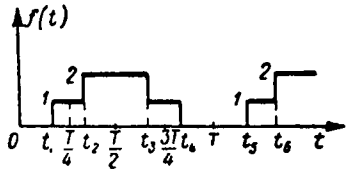



FIG. 38. Example 36.

EXAMPLE 35 .To find the transform of the full-wave rectified sine wave, i.e. the function

$$f(t) = |\sin \omega_0 t| \quad \text{(Fig. 37).}$$

Taking formula (104) into account and selecting a period equal to $T/2$, we get

$$F(p) = \frac{F_0(p)}{1 - e^{-\frac{pT}{2}}} = \frac{F_0(p)}{1 - e^{-\frac{p\pi}{\omega_0}}}.$$

In this case $F_0(p)$ is determined from expression (108); consequently,

$$F(p) = \frac{\omega_0\left(1 + e^{-\frac{p\pi}{\omega_0}}\right)}{(p^2 + \omega_0^2)\left(1 - e^{-\frac{p\pi}{\omega_0}}\right)} = \frac{\omega_0}{p^2 + \omega_0^2} \coth \frac{p\pi}{2\omega}. \tag{13.109}$$

EXAMPLE 36. To find the transform of the periodic jump function $f(t)$ with a period $T$, shown in Fig. 38.

$$t_2 - \frac{T}{4} = \frac{T}{4} - t_1, \quad t_3 - \frac{T}{2} = \frac{T}{2} - t_2, \quad t_4 - \frac{3T}{4} = \frac{3T}{4} - t_3.$$

We shall use formula (104); then

$$F_0(p) = \frac{1}{p}(e^{-pt_1} + e^{-pt_2} - e^{-pt_3} - e^{-pt_4})$$

$$= \frac{1}{p}\left(e^{-pt_1} + e^{pt_1 - \frac{pT}{2}} - e^{-pt_1 - \frac{pT}{2}} - e^{pt_1 - pT}\right);$$

$$F(p) = \frac{e^{-pt_1}\left(1 - e^{-\frac{pT}{2}}\right)}{p(1 - e^{-pT})} + \frac{e^{pt_1} e^{-p\frac{T}{2}}\left(1 - e^{-p\frac{T}{2}}\right)}{p(1 - e^{-pT})}$$

$$= \frac{e^{-pt_1}}{p\left(1 + e^{-\frac{pT}{2}}\right)} + \frac{e^{pt_1} e^{-p\frac{T}{2}}}{p\left(1 + e^{-p\frac{T}{2}}\right)} = \frac{e^{p\left(\frac{T}{4} - t_1\right)} + e^{-p\left(\frac{T}{4} - t_1\right)}}{p\left(e^{p\frac{T}{4}} + e^{-p\frac{T}{4}}\right)}$$

$$= \frac{\cosh p\left(\frac{T}{4} - t_1\right)}{p \cosh p\frac{T}{4}}. \tag{13.110}$$

EXAMPLE 37. To find the transform of the periodic jump function $f(t)$ with a period $T$, shown in Fig. 39.

We shall use formula (106), from which

$$\Psi(p) = \frac{1}{p}\left[e^{-pt_1} - e^{-p\left(\frac{T}{2} - t_1\right)}\right],$$

$$F(p) = \frac{e^{-pt_1} - e^{-p\frac{T}{2}} e^{pt_1}}{p\left(1 + e^{-p\frac{T}{2}}\right)} = \frac{e^{p\left(\frac{T}{4} - t_1\right)} - e^{-p\left(\frac{T}{4} - t_1\right)}}{p\left(e^{p\frac{T}{4}} + e^{-p\frac{T}{4}}\right)} = \frac{\sinh p\left(\frac{T}{4} - t_1\right)}{p \cosh p\frac{T}{4}}. \tag{13.111}$$

EXAMPLE 38. To find the transform of a periodic jump function with a period $T/2$ (Fig. 40):

$$t_2 - \frac{T}{4} = \frac{T}{4} - t_1; \quad \frac{T}{2} - t_2 = t_1$$

and

$$F_0(p) = \frac{1}{p}\left[e^{-pt_1} - e^{-p\left(\frac{T}{2} - t_1\right)}\right].$$

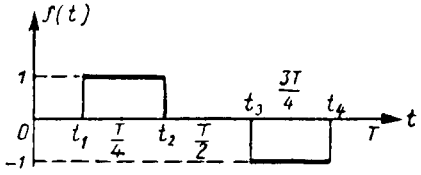



FIG. 39. Example 37.

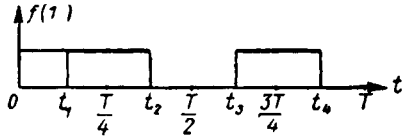



FIG 40. Example 38.

Taking (104) into account and regarding the period as being equal to $T/2$, we get

$$F(p) = \frac{e^{-pt_1} - e^{-p\left(\frac{T}{2} - t_1\right)}}{p\left(1 - e^{-p\frac{T}{2}}\right)} = \frac{e^{p\left(\frac{T}{4} - t_1\right)} - e^{-p\left(\frac{T}{4} - t_1\right)}}{p\left(e^{p\frac{T}{4}} - e^{-p\frac{T}{4}}\right)} = \frac{\sinh p\left(\frac{T}{4} - t_1\right)}{p \sinh p\frac{T}{4}}$$

(13.112)

The transforms of step and periodic functions are used in studying pulse networks and solving the various equations to which circuit problems can be reduced.

One mathematical system developed from the Laplace transformation, i.e. the discrete Laplace transformation, has great potentialities. This convenient form was developed by Ya. Z. Tsypkin (Ref. 58) with application to networks with pulse feedback (intermittent adjustment and control systems) and with relay feedback (relay control systems).

## 16. Transients in Transmission Lines

The differential equations for a uniform two-wire line were educed in Chapter IX, Section 2. Owing to the presence of the two independent variables ($x$ and $t$), these equations were obtained in partial derivatives

$$\left.\begin{aligned} -\frac{\partial u}{\partial x} &= ri + L\frac{\partial i}{\partial t}, \\ -\frac{\partial i}{\partial x} &= gu + C\frac{\partial u}{\partial t}. \end{aligned}\right\} \qquad (13.113)$$

The minus signs in the left-hand side of equations (113) correspond to the case when the distance $x$ is calculated from the beginning of the line.

In order to solve equations (113) with the aid of the Laplace transformation, we introduce transforms of the unknown electrical quantities (in time)

$$\left.\begin{aligned} u(x, t) &\sim U(x, p) = U, \\ i(x, t) &\sim I(x, p) = I. \end{aligned}\right\} \tag{13.114}$$

A set of Laplace-transformed equations

$$\left.\begin{aligned} -\frac{\mathrm{d}U}{\mathrm{d}x} &= (r + pL)\, I - Li(x, 0), \\ -\frac{\mathrm{d}I}{\mathrm{d}x} &= (g + pC)\, U - Cu(x, 0). \end{aligned}\right\} \tag{13.115}$$

then corresponds to the set of differential equations in the partial derivatives (113).

Here $i(x, 0)$ and $u(x, 0)$ are the initial (time) values of current and voltage at point $x$.

In view of the fact that $p$ does not depend on $x$ and that there are no derivatives of $p$ in equations (115), the partial $x$ derivatives can be replaced by ordinary derivatives when converting from equations (113) to (115).

Eliminating the current transform I from set (115) we obtain an equation with respect to $U$

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - (r + pL)(g + pC)\, U = L\frac{\mathrm{d}i(x, 0)}{\mathrm{d}x} - C(r + pL)\, u(x, 0). \tag{13.116}$$

Similarly, eliminating the voltage transform $U$ from (115) we obtain an equation with respect to $I$

$$\frac{\mathrm{d}^2I}{\mathrm{d}x^2} - (r + pL)(g + pC)\, I = C\frac{\mathrm{d}u(x, 0)}{\mathrm{d}x} - L(g + pC)\, i(x, 0). \tag{13.117}$$

These equations can be solved as ordinary differential equations of the second order.

If equation (116) is solved with respect to $U$, the current transform can most easily be found from the first equation of (115):

$$I = -\frac{1}{r + pL}\left[\frac{\mathrm{d}U}{\mathrm{d}x} - Li(x, 0)\right]. \tag{13.118}$$

Let us denote

$$\gamma = \sqrt{[(r + pL)(g + pC)]} \tag{13.119}$$

and call this expression the *operator propagation factor**.

With zero initial conditions the equations (116) and (117) are simplified

$$\left.\begin{aligned} \frac{d^2U}{dx^2} &= \gamma^2 U, \\ \frac{d^2I}{dx^2} &= \gamma^2 I. \end{aligned}\right\} \tag{13.120}$$

The resemblance of the resulting equations (120) to the differential equations of a uniform line in the steady state merits attention.

Correspondingly, the solutions obtained in Chapter IX, Section 2 are also valid here if by $U$ and $I$ we understand the transforms of voltage and current, and by $Z_c = \sqrt{[(r + pL)/(g + pC)]}$ we understand the line's *operator image impedance*†. Thus

$$U(x, p) = U_1 \cosh \gamma x - Z_c I_1 \sinh \gamma x,$$

$$I(x, p) = I_1 \cosh \gamma x - \frac{U_1}{Z_c} \sinh \gamma x.$$

Supposing that $x = l$ and denoting $U(l, p) = U_2$, $I(l, p) = I_2$ we get transform equations in which the voltage and current at the receiving end of the line are expressed by the voltage and current at the sending end of the line

$$\left.\begin{aligned} U_2 &= U_1 \cosh \gamma l - Z_c I_1 \sinh \gamma l, \\ I_2 &= I_1 \cosh \gamma l - \frac{U_1}{Z_c} \sinh \gamma l. \end{aligned}\right\} \tag{13.121}$$

If $x$ is reckoned from the receiving end of the line then, denoting $U(0, p) = U_2$ and $I(0, p) = I_2$, we get

$$\left.\begin{aligned} U(x, p) &= U_2 \cosh \gamma x + Z_c I_2 \sinh \gamma x, \\ I(x, p) &= I_2 \cosh \gamma x + \frac{U_2}{Z_c} \sinh \gamma x. \end{aligned}\right\} \tag{13.122}$$

When $x = l$, $U(l, p) = U_1$ and $I(l, p) = I_1$ transform equations are obtained in which the voltage and current transforms at the sending end of

* When $j\omega$ is substituted for $p$, a propagation factor $\gamma(j\omega)$ is obtained for the sinusoidal steady-state.

† When $j\omega$ is substituted for $p$, an image impedance $Z_c(j\omega)$ is obtained for the sinusoidal steady-state.

the line are determined by the voltage and current transforms at the receiving end of the line

$$\left.\begin{aligned} U_1 &= U_2 \cosh \gamma l + Z_c I_2 \sinh \gamma l, \\ I_1 &= I_2 \cosh \gamma l + \frac{U_2}{Z_c} \cdot \sinh \gamma l. \end{aligned}\right\} \tag{13.123}$$

If the impedance of the load is denoted by $Z_2$, then, substituting $I_2 = U_2/Z_2$ in (123) we get

$$U_1 = U_2 \left(\cosh \gamma l + \frac{Z_c}{Z_2} \sinh \gamma l\right),$$

from which

$$U_2 = \frac{U_1}{\cosh \gamma l + \dfrac{Z_c}{Z_2} \sinh \gamma l}, \quad I_2 = \frac{U_1}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}.$$

In conjunction with this, from (122)

$$U(x, p) = U_1 \frac{Z_2 \cosh \gamma x + Z_c \sinh \gamma x}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}, \tag{13.124}$$

$$I(x, p) = U_1 \frac{\cosh \gamma x + \dfrac{Z_2}{Z_c} \sinh \gamma x}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}. \tag{13.125}$$

These equations express the transforms of the voltage and current at an arbitrary point $x$ in accordance with the transforms of the voltage $U_1$ at the sending end of the line.

In addition, formulae (124) and (125) make it easy to obtain analogous relationships for the case when $x$ is read off from the sending end of the line. In order to do this it is only necessary to substitute $l - x$ for $x$ in the right-hand side of (124) and (125):

$$U(x, p) = U_1 \frac{Z_2 \cosh \gamma(l - x) + Z_c \sinh \gamma(l - x)}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}, \tag{13.126}$$

$$I(x, p) = U_1 \frac{\cosh \gamma(l - x) + \dfrac{Z_2}{Z_c} \sinh \gamma(l - x)}{Z_2 \cosh \gamma l + Z_c \sinh \gamma l}. \tag{13.127}$$

The conversion of the transforms $U(x, p)$ and $I(x, p)$, transcendental functions of $p$, into the unknown originals $u(x, t)$ and $i(x, t)$ usually presents considerable difficulties. We generally adopt some simplifying assumptions when solving these problems.

Two cases are examined below: first the switching of a constant voltage source on a dissipationless line ($r = 0, g = 0$) and then on a non-inductive and leakage-free cable ($L = 0, g = 0$) to a constant voltage source. We shall take $U_1(p)$ as being equal to $1/p$.

*Dissipationless Line*

Let us examine the transient on switching a unit voltage on a line when the line is open at the end ($z_2 = \infty$) and short-circuited at the end ($z_2 = 0$).

In the case of a line open at the end, by substituting $Z_2 = \infty$ in (126) and (127) we get

$$U(x, p) = \frac{\cosh \gamma(l - x)}{p \cosh \gamma l}, \quad I(x, p) = \frac{\sinh \gamma(l - x)}{pZ_c \cosh \gamma l}.$$

The current at the sending end of the line

$$I(0, p) = \frac{1}{pZ_c} \tanh \gamma l.$$

When $r = 0, g = 0$,

$$\gamma = p\sqrt{(LC)}, \quad Z_c = \sqrt{\left(\frac{L}{C}\right)}.$$

As is generally known, the propagation velocity of an electromagnetic wave in a dissipationless line is $v\ 1/\sqrt{(LC)}$. Therefore, $l/v = l\sqrt{(LC)} = \tau_0$, where $\tau_0$ is the time taken for the propagation of an electromagnetic wave from the beginning to the end of the line.

Consequently

$$I(x, p) = \sqrt{\left(\frac{C}{L}\right)} \frac{\sinh p(\tau_0 - \tau)}{p \cosh p\tau_0},$$

$$U(x, p) = \frac{\cosh p(\tau_0 - \tau)}{p \cosh p\tau_0},$$

$$I(0, p) = \frac{1}{p} \sqrt{\left(\frac{C}{L}\right)} \tanh p\tau_0.$$

Here

$$\tau = \sqrt{(LC)}x.$$

Comparing the resulting equations with (111), (110) and (103) we can confirm that function $i(x, t)$ has the form shown in Fig. 39 $T/4 = \tau_0$, $t_1 = \tau$; function $u(x, t)$ has the form shown in Fig. 38 $T/4 = \tau_0$, $t_1 = \tau$; and function $i(0, t)$ has the form shown in Fig. 34 $t_0/2 = \tau_0$. The pulse height of the current is, respectively, $\sqrt{(C/L)}$ and $-\sqrt{(C/L)}$.

In the case of a line short-circuited at the end, by substituting $Z_2 = 0$ in (126), we get

$$U(x, p) = \frac{\sinh \gamma(l - x)}{p \sinh \gamma l} = \frac{\sinh p(\tau_0 - \tau)}{p \sinh p\tau_0}.$$

By comparing the given equation with (112) we can conclude that function $u(x, t)$ has the form shown in Fig. 40 $T/4 = \tau_0$, $t_1 = \tau$.

In its turn

$$I(x, p) = \frac{\cosh \gamma(l - x)}{pZ_c \sinh \gamma l} = \frac{1}{p}\sqrt{\left(\frac{C}{L}\right)}\frac{\cosh p(\tau_0 - \tau)}{\sinh p\tau_0}.$$

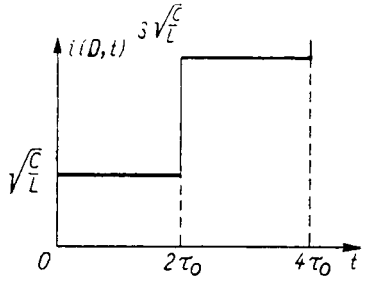



FIG. 41. Change in current at the input of a dissipationless line, short-circuited at the end.

When $x = 0$, from a comparison of the given equation with (102) we can conclude the $i(0, t)$ has the form of the jump function shown in Fig. 41. The given function is obtained from the formula

$$i(0, t) = \sqrt{\left(\frac{C}{L}\right)}[2f(t) - 1]$$

taking function $f(t)$ of Fig. 33 into account when $t_0/2 = \tau_0$.

*Non-inductive Cable-line without Leakage*

Let us consider the case of a short circuit. Taking $Z_2 = 0$ into account, from formulae (126) and (127) we get

$$U(x, p) = \frac{\sinh \gamma(l - x)}{p \sinh \gamma l},$$

$$I(x, p) = \sqrt{\left(\frac{pC}{r}\right)} \cdot \frac{\cosh \gamma(l - x)}{p \sinh \gamma l}.$$

Here

$$\gamma = \sqrt{(prC)}, \quad Z_c = \sqrt{\left(\frac{r}{pC}\right)}.$$

Using the expansion theorem (71)

$$u(x, t) = \left|\frac{\sinh \gamma(l - x)}{\sinh \gamma l}\right|_{p=0} + \sum_{k=1}^{\infty} \frac{\sinh \gamma_k(l - x)\, e^{p_k t}}{p_k \left[\frac{\mathrm{d}}{\mathrm{d}p} \sinh \gamma l\right]_{p=p_k}}.$$

The roots $p_k$ can be found from the equation $\sinh \gamma l = 0$. From this $\gamma l = jk\pi\,(k = 1, 2, 3 \ldots)$ and, consequently,

$$\gamma_k = \sqrt{(p_k rC)} = j\frac{k\pi}{l}; \quad p_k = -\frac{k^2\pi^2}{rCl^2};$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}p} \sinh \gamma l\right]_{p=p_k} = \left[\cosh \gamma l \frac{\mathrm{d}(\gamma l)}{\mathrm{d}p}\right]_{p=p_k}$$

$$= \left[\frac{\cosh \gamma l \sqrt{(rC)}\, l}{2\sqrt{p}}\right]_{p=p_k} = \frac{rCl^2 \cos k\pi}{2jk\pi}.$$

Further,

$$\lim_{p\to 0} \frac{\sinh \gamma(l - x)}{\sinh \gamma l} = \lim_{p\to 0} \frac{\cosh \gamma(l - x) \frac{\sqrt{(rC)}}{2\sqrt{p}} (l - x)}{\cosh \gamma l \cdot \frac{\sqrt{(rC)}}{2\sqrt{p}}\, l} = 1 - \frac{x}{l};$$

$$\sinh \gamma_k(l - x) = j \sin \frac{k\pi}{l}(l - x) = j \sin k\pi \left(1 - \frac{x}{l}\right)$$

$$= -j \cos k\pi \sin \frac{k\pi x}{l}.$$

Consequently,

$$u(x, t) = 1 - \frac{x}{l} + \sum_{k=1}^{\infty} \frac{j \cos k\pi \sin \frac{k\pi x}{l}\, e^{-\frac{k^2\pi^2}{rCl^2} t}}{\pi^2 k^2 \frac{\cos k\pi}{2jk\pi}}$$

$$= 1 - \frac{x}{l} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{k\pi x}{l}\, e^{-\frac{k^2\pi^2}{rCl^2} t}.$$

Similarly, by using the expansion theorem it is possible to obtain an equation for the current

$$i(x, t) = \frac{1}{rl}\left[1 + 2 \sum_{k=1}^{\infty} \cos \frac{k\pi x}{l}\, e^{-\frac{k^2\pi^2}{rCl^2} t}\right].$$

## 17. Spectrum Concepts and Laplace Transforms

Owing to the close affinity between Laplace and Fourier transformations, substituting $j\omega$ for $p$ in a Laplace transform immediately produces a frequency spectrum (Section 3)

$$F(j\omega) = [F(p)]_{p=j\omega}.$$

This is the simplest and most convenient way of obtaining a spectrum since it permits the use of tables of Laplace transforms.

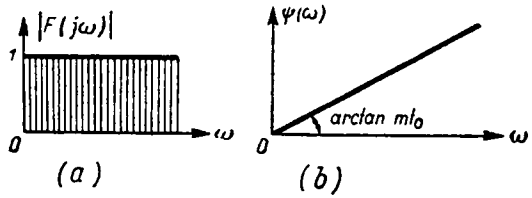



FIG. 42. Amplitude and phase spectra of a unit impulse.

Thus, for example, starting from the formulae $\sin \omega_0 t \sim \omega_0/(p^2 + \omega_0^2)$ and $\cos \omega_0 t \sim p/(p^2 + \omega_0^2)$ we get $F(j\omega) = \omega_0/(\omega_0^2 - \omega^2)$ for a sine curve and $F(j\omega) = \omega e^{j\frac{\pi}{2}}/(\omega_0^2 - \omega^2)$ for a cosine curve.

These spectrum equations have already been encountered in Chapter XII, Section 6.

Let us stop to consider the spectra of the *pulse* functions described in Section 13 of this chapter.

A unit impulse $\delta_1(t)$ applied at the instant $t = 0$ has a transform $F(p) = 1$. Correspondingly, the spectrum is $F(j\omega) = 1$. This denotes that function $\delta_1(t)$ possesses a continuous spectrum extending over the entire range of frequencies with a constant spectral density (Fig. 42*a*).

If a unit impulse is applied at the instant $t = t_0$ then $\delta_1(t - t_0) \sim e^{-pt_0}$ and, correspondingly, $F(j\omega) = e^{-j\omega t_0}$, from which $|F(j\omega)| = 1, \psi(\omega) = \omega t_0$. Consequently, the modulus of the spectrum of a delayed unit impulse $\delta_1(t - t_0)$ equal to unity is also given by Fig. 42*a*. The phase response in this case is a straight line forming, with the abscissa axis, an angle equal to arc tan $mt_0$ (Fig. 42*b*). Here $m$ is the scale factor. When $t_0 = 0$ the phase response coincides with the abscissa axis.

A unit doublet $\delta_2(t)$ has a transform $F(p) = p$. Correspondingly, the spectrum is $F(j\omega) = j\omega = \omega e^{j\frac{\pi}{2}}$. Consequently, in this case the modulus of the spectrum of a unit doublet is equal to $|F(j\omega)| = |\omega|$ (Fig. 43*a*, when $m = 1$) and the phase factor is $\psi(\omega) = -\pi/2$ (Fig. 43*b*).

Analogously, when $\delta_n(t) \sim p^{n-1}$

$$F(j\omega) = (j\omega)^{n-1} = \omega^{n-1} e^{j\frac{\pi}{2}(n-1)}.$$

The best visual representation of the spectra of pulse functions is obtained by regarding them as the limit transitions from the spectra of pulses of finite length.

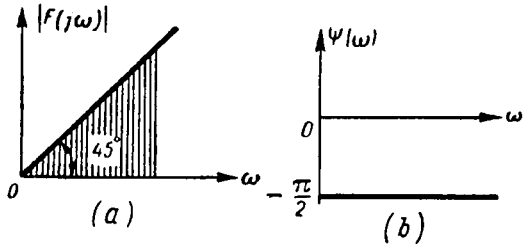



FIG. 43. Amplitude and phase spectra of a unit doublet.

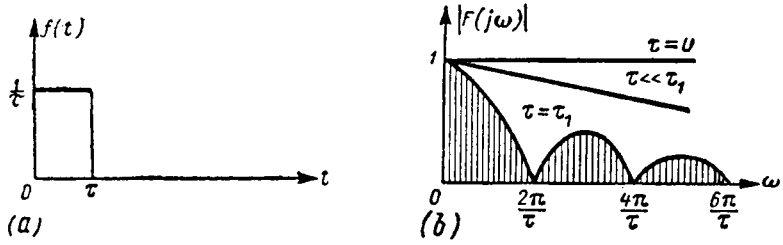



FIG. 44. Spectrum of a unit impulse as the limit of the spectrum of a rectangular pulse.

Figure 44*a* shows a rectangular pulse $1/\tau$ in magnitude and of duration $\tau$. From the transform of the pulse $(1/p\tau)(1 - e^{-p\tau})$ it is easy to find the spectrum

$$F(j\omega) = \frac{1}{j\omega\tau}(1 - e^{-j\omega\tau}) = \frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j\omega\tau e^{j\frac{\omega\tau}{2}}} = \frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}e^{j\frac{\omega\tau}{2}}}.$$

The modulus of this equation is

$$|F(j\omega)| = \left|\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right|.$$

Three graphs of $|F(j\omega)|$ corresponding to different values of $\tau$ are shown in Fig. 44. The shaded graph corresponds to a certain value $\tau = \tau_1$. As $\tau$ decreases the interval between the zeros of function $F(j\omega)$ increases and when $\tau$ is considerably smaller than $\tau_1$ the graph acquires the form of an inclined line. Finally, when $\tau \to 0$ the value $F(j\omega)$ tends to unity, i.e. the amplitude spectrum of Fig. 42*a* is obtained.

Let us suppose that an impulsive e.m.f. $u(t) = \delta_1(t)$ is applied to a network when $t = 0$. The spectrum of the current passing the network is equal to the ratio of the spectrum of the impulsive e.m.f. $F(j\omega) = 1$ to the impedance of the given network $Z(j\omega)$, i.e.

$$I(j\omega) = \frac{1}{Z(j\omega)} = Y(j\omega),$$

from which

$$|I(j\omega)| = |Y(j\omega)|.$$

Thus the current spectrum is represented by the frequency characteristic of the network's admittance (Chapter XII, Section 7).

If a unit e.m.f. $u(t) = 1$ is applied to the network when $t = 0$, the current spectrum in the circuit can be determined from the ratio of the spectrum of the unit step function $F(j\omega) = 1/j\omega$ to the network's impedance $Z(j\omega)$

$$I(j\omega) = \frac{1}{j\omega Z(j\omega)} = -j\frac{1}{\omega} Y(j\omega),$$

from which

$$|I(j\omega)| = \left|\frac{Y(j\omega)}{\omega}\right|.$$

Figures 45 and 46 show the moduli of current spectra thus obtained for cases of applying an impulsive and a unit e.m.f. to different types of network, and indicate the resulting currents (Refs. 76 and 92).

*The rules and theorems applicable to transforms can be used successfully in spectrum analysis.*

For example, let us say it is required to establish the relationship between the spectra of a single and a repeated signal.

If $F_0(p)$ is the transform of a single signal $f_0(t)$ and $F(p)$ is the transform of a periodic function $f(t)$ obtained by the repetition of $f_0(t)$ with a period $T$, then according to (105)

$$F(p) = \frac{F_0(p)}{1 - e^{-pT}}.$$

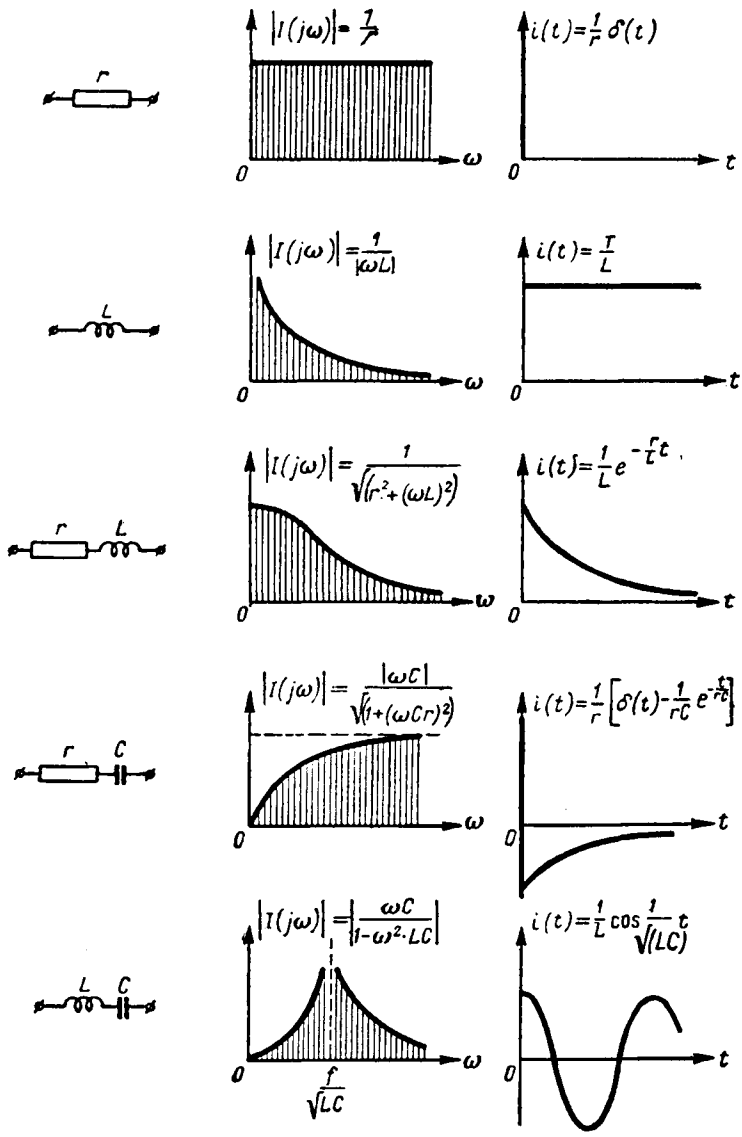



FIG. 45. Currents and their amplitude spectra when a unit impulse is applied to circuits (impulsive response).

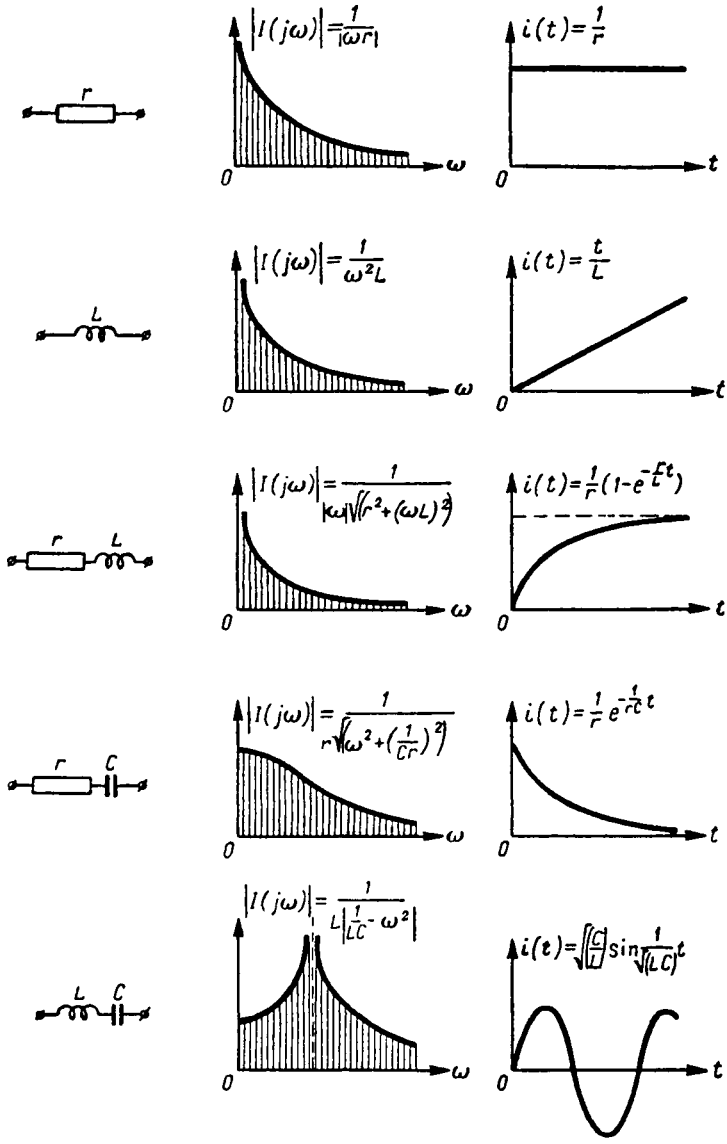



FIG. 46. Currents and their amplitude spectra when a unit step function is applied to circuits (indicial response).

Thus the spectrum of the periodic function

$$F(j\omega) = \frac{F_0(j\omega)}{1 - e^{-j\omega T}} = \frac{e^{j\frac{\omega T}{2}} F_0(j\omega)}{e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}}} = \frac{e^{j\frac{\omega T}{2}} F_0(j\omega)}{2j \sin \frac{\omega T}{2}}.$$

The modulus of the spectrum is

$$|F(j\omega)| = \left| \frac{F_0(j\omega)}{2 \sin \frac{\omega T}{2}} \right|.$$

On the basis of the properties of the Laplace transformation examined above the rules and theorems for spectra, given in the table in Appendix II, can easily be established.

Thus, for example, formula (31) for differentiating a time function by a direct Fourier transformation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \sim j\omega F(j\omega) - f(0),$$

corresponds to the theorem on differentiating an original function with respect to time

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \sim pF(p) - f(0),$$

and the switching formula (44*a*) from Chapter XII for finding a time function from a spectrum

$$\frac{F_1(j\omega)}{F_2(j\omega)} \sim \sum_{k=1}^{n} \frac{F_1(j\omega_k)\, e^{j\omega_k t}}{\left[\frac{\mathrm{d}}{\mathrm{d}j\omega} F_2(j\omega)\right]_{\omega=\omega_k}} = j \sum_{k=1}^{n} \frac{F_1(j\omega_k)\, e^{j\omega_k t}}{\left[\frac{\mathrm{d}}{\mathrm{d}\omega} F_2(j\omega)\right]_{\omega=\omega_k}}$$

corresponds to the expansion theorem

$$\frac{F_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{F_1(p_k)\, e^{p_k t}}{\left[\frac{\mathrm{d}}{\mathrm{d}p} F_2(p)\right]_{p=p_k}}.$$

When $F_2(j\omega) = j\omega F_3(j\omega)$, the switching formula (44*b*) given in Chapter XII is obtained

$$\frac{F_1(j\omega)}{j\omega F_3(j\omega)} \sim \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n-1} \frac{F_1(j\omega_k)\, e^{j\omega_k t}}{\omega_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega} F_3(j\omega)\right]_{\omega=\omega_k}}.$$

It should be noted in conclusion that we have been using the one-sided Laplace transformation, i.e. taking $f(t) = 0$ when $t < 0$. The one-sided Fourier transformation is a particular form of this transformation, when $p = j\omega$. If the function $F(p)$ has no poles on the imaginary axis or to the right of it, the Laplace and Fourier integrals in one-sided form are identical.

Thus, *the Fourier transformation may be regarded as a particular type of Laplace transformation.*

The essential advantage of the Laplace transformation, like the advantage of the generalized form of the Fourier integral (Chapter XII, Section 6) over the common form of the Fourier transformation, is that it can be extended to functions which fail to be absolutely convergent – e.g. increasing exponential, periodic and other functions.

EXAMPLE 39. To determine the spectrum of the function

$$f_1(t) = \begin{cases} \cos \omega_0 t & \text{when } 0 < t < \tau, \\ 0 & \text{when } 0 > t > \tau, \end{cases}$$

using the Laplace transformation.

Here $\tau = nT = n\,2\pi/\omega_0$, where $T$ is the period of the cosine curve and $n$ is the number of periods during the time $\tau$.

If the periodic function is given in the form of a cosine curve then

$$F(p) = \frac{p}{p^2 + \omega_0^2}.$$

The transform of a signal containing $n$ periods of the cosine curve is obtained from $F(p)$ by using the formula

$$F_1(p) = F(p)\left(1 - e^{-pn\frac{2\pi}{\omega_0}}\right) = \frac{p}{p^2 + \omega_0^2}\left(1 - e^{-pn\frac{2\pi}{\omega_0}}\right).$$

The spectrum of this signal

$$F_1(j\omega) = [F_1(p)]_{p=j\omega} = \frac{j\omega}{\omega_0^2 - \omega^2}\left(1 - e^{-j\frac{\omega}{\omega_0}n2\pi}\right)$$

$$= \frac{2\omega}{\omega_0^2 - \omega^2} \sin\left(\frac{\omega}{\omega_0} n\pi\right) e^{-j\left(\frac{\omega}{\omega_0}n\pi - \pi\right)}.$$

The resulting formula agrees with the result of Example 1 in Chapter XII.

## 18. Transfer Function

An *operator transfer function* is the same as the transfer function for a complex frequency $p = c + j\omega$ examined in Chapter VII, Section 20. In the case of non-zero initial conditions $u_C(0) = 0$ and $i_L(0) = 0$ it is analogous to a transfer function in the steady state, the only difference being that $p$ is substituted for $j\omega$ and the voltages and currents are replaced by their transforms.

For example, in the case of an $rC$ network (Chapter VII, Table 4, circuit 3) the voltage-ratio transfer function is

$$K_U(p) = \frac{U_2(p)}{U_1(p)} = \frac{\dfrac{1}{pC}}{r + \dfrac{1}{pC}} = \frac{1}{prC + 1} = \frac{1}{p\tau + 1},$$

where $\tau = r \times C$ is the time constant of the $rC$ network.

The voltage at the output of the network can be found from the transform

$$U_2(p) = U_1(p)\, K_U(p).$$

For example, if a constant voltage $U$ is applied to the network under examination, i.e.

$$U_1(p) = \frac{U}{p},$$

then

$$U_2(p) = \frac{U}{p(p\tau + 1)} = \frac{U}{\tau} \frac{1}{p\left(p + \dfrac{1}{\tau}\right)},$$

from which

$$u_2(t) = U\left(1 - e^{\frac{t}{\tau}}\right).$$

For more complex circuits the transfer function can be found from formula (76) as

$$K_U(p) = \frac{1}{A_{11}(p) + \dfrac{A_{12}(p)}{Z_2(p)}},$$

where $A_{11}(p)$, $A_{12}(p)$ and $Z_2(p)$ are obtained from $A_{11}(j\omega)$, $A_{12}(j\omega)$ and $Z_2(j\omega)$ by substituting $p$ for $j\omega$.

When four-terminal networks are connected in cascade with matched characteristic impedances, the resulting transfer function is equal to the product of the transfer functions of the component four-terminal networks. For example, if

$$K_1 = \frac{U_2}{U_1}, \quad K_2 = \frac{U_3}{U_2}, \quad K_3 = \frac{U_4}{U_3},$$

where $K_1$, $K_2$ and $K_3$ are the voltage-ratio transfer functions of four-terminal networks each connected separately to a matched load, then when

these same networks are in cascade and connected to the output of the matched load, the over-all transfer function will be

$$K_U = \frac{U_4}{U_1} = \frac{U_2}{U_1}\frac{U_3}{U_2}\frac{U_4}{U_3} = K_1K_2K_3.$$

The same formula is valid for amplifiers connected in cascade when, as a result of the small value of the current at the input of each amplifier cascade, we understand by $K_1$, $K_2$, $K_3$ ... the transfer functions of the corresponding amplification stages in open-circuit conditions. It is thus assumed that each consecutive stage of amplification has no effect on the operation of the previous stage, and therefore the transfer functions characterizing the separate operation of each stage in open-circuit conditions are also valid for cascade connexion.

In practice it is convenient to use a special table of transfer functions compiled for various types of $rC$ network and illustrating the frequency characteristics (Refs. 36 and 94).

EXAMPLE 40. To determine the voltage at the output of an amplifier which is excited by a d.c. voltage $U$ at its input, if the transfer function for the steady state is

$$K = \frac{j\omega\tau_1}{1 + j\omega\tau_1} \cdot \frac{1}{1 + j\omega\tau_2}; \quad \tau_1 = 1000\tau_2.$$

The operator transfer function is

$$K(p) = \frac{p\tau_1}{(1 + p\tau_1)(1 + 0{\cdot}001p\tau_1)}.$$

The transform of the unknown voltage

$$U_2(p) = \frac{U}{p}K(p) = \frac{U}{0{\cdot}001\tau_1\left(p + \frac{1}{\tau_1}\right)\left(p + \frac{1}{0{\cdot}001\tau_1}\right)},$$

from which the voltage at the output of the amplifier is

$$u_2(t) = \frac{U}{0{\cdot}001\tau_1} \cdot \frac{e^{-\frac{t}{0{\cdot}001\tau_1}} - e^{-\frac{t}{\tau_1}}}{\frac{1}{\tau_1} - \frac{1}{0{\cdot}001\tau_1}}$$

or approximately

$$u_2(t) \approx U\left(e^{-\frac{t}{\tau_1}} - e^{-\frac{t}{0{\cdot}001\tau_1}}\right).$$

APPENDIX I

# ELEMENTS OF COMPLEX VARIABLE THEORY

## 1. Concept of the Complex Variable Function

If, in accordance with a certain law, a value of a complex number $w = u + jv$ corresponds to every value which the complex number* $z = x + jy$ can assume, then $w$ is a *function of an independent complex variable* $z$: $w = f(z)$.

A certain pair of real numbers $u$ and $v$ then corresponds exactly to each pair of real numbers $x$ and $y$. Consequently to give a function of a complex variable means the same as to give two real functions $u = u(x, y)$ and $v = v(x, y)$ of the real arguments $x$ and $y$. For example, in the case $w = z^2$ we get: $w^2 = x^2 - y^2 + j\,2xy$ from which $u = x^2 - y^2$, $v = 2xy$.

Function $w$ is called a *single-valued* function if only one value of $w$ corresponds to each value of $z$ and a *multi-valued* function if more than one value of $w$ corresponds to certain values of $z$. In particular, a function is called *n-valued* if the same number ($n$) of values of $w$ corresponds to each $z$, and *infinitely valued* if an infinite number of values of $w$ correspond to each $z$ (with the exception, perhaps, of the finite number of $z$ to which another number of values of $w$ corresponds).

If a single value of $w$ corresponds to different values of $z$, the former is known as a *multiple* value.

Giving $w$ as a function of $z$ can be reduced geometrically to establishing the relationship between the numbers of $z$ and $w$ points in the complex $z$- and $w$-planes: a certain locus of the points denoting $w$ corresponds to the locus of the points denoting $z$, the $z$- and $w$-planes then being regarded either individually or together.

A point $z_0$ possessing the property that a function $w$ takes on multiple values if $z$ varies along a path enclosing $z_0$ is called the *branch point* or the critical point of the function $w$.

Examples of various complex variable functions – single-valued, multi-valued and with branch points – are given below.

* We shall retain the notations $z$ and $w$ for complex numbers as in complex variable function theory.

## 2. Elementary Complex Variable Functions

In this section we shall use power series to examine certain elementary functions.

All the rules of algebraic operations are valid for elementary complex variable functions.

Let us now consider some examples of elementary complex variable functions.

*The linear function* $w = az + b$, where $a$ and $b$ are arbitrary numbers, generally complex.

A linear function is a single-valued complex variable function.

*The power function* $w = z^a$, where $a$ is an arbitrary real number.

When $a$ is an integer, the function $z^a$ is single-valued. Substituting $z = |z|e^{j\varphi}$ yields

$$w = |z|^a e^{ja\varphi} = |z|^a (\cos a\varphi + j \sin a\varphi).$$

When the point $z$ circumscribes the origin, argument $\varphi$ increases by $\pm 2\pi$ (the plus sign corresponds to the positive direction of anti-clockwise plotting). The point $w$ then circumscribes the point $w = 0$ $a$ times, i.e. the argument of the complex number $w$ increases by $\pm 2a\pi$.

When $a$ is not an integer, the function $z^a$ is multi-valued: the points $w = |z|^a e^{j(a\varphi + 2ka\pi)}$ lying on a circumference of radius $|z|^a$ correspond to the point $z = |z|e^{j\varphi}$.

For example, when $a = 1/n$ ($n$ being a real integer), the radical $w = \sqrt[n]{z}$ is an $n$-valued function since $w = \sqrt[n]{|z|}e^{j\frac{(\varphi + 2k\pi)}{n}}$. Assigning to $k$ the values $0, 1, 2, \ldots, n - 1$ we get $n$ different values of the function $w$ with the same modulus and argument which forms an arithmetic progression with the difference $2\pi/k$.

*The exponential function* $w = e^z = 1 + z + z^2/2! + \cdots = \sum_{n=0}^{\infty} z^n/n!$, where $e$ is the natural logarithmic base.

An exponential function is fixed and single-valued for all $z$. When the constant number $j\,2\pi$ is added to $z$, the value of the exponential function does not vary

$$e^{z+j2\pi} = e^z e^{j2\pi} = e^z,$$

since

$$e^{j2\pi} = 1.$$

Consequently, an exponential function has an imaginary period $j2\pi$.

*The logarithmic function* $w = \ln z$.

A logarithmic function is multi-valued since the argument of the number $z$ has an infinite number of values $\varphi + 2k\pi$ ($k$ being an integer). It is possible to show that

$$\ln z = \ln |z| + j(\varphi + 2k\pi).$$

The point $z = 0$ is the branch point for a logarithmic function: when point $z$ circumscribes the origin, the imaginary part of $\ln z$ increases by $\pm 2\pi$.

*Trigonometric functions:*

$$\sin z = \frac{e^{jz} - e^{-jz}}{2j} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

$$\cos z = \frac{e^{jz} + e^{-jz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\tan z = \frac{\sin z}{\cos z},$$

$$\cot z = \frac{\cos z}{\sin z}.$$

For trigonometric functions determined in this way in a complex plane the following trigonometric formulae are valid:

$$\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$

$$\cos (z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2,$$

$$\sin 2z = 2 \sin z \cos z,$$

$$\sin^2 z + \cos^2 z = 1, \quad e^{jz} = \cos z + j \sin z, \quad \text{etc.}$$

Substituting $z_1 = z$ and $z_2 = 2\pi$ in the formulae of the sine and cosine of the sum or the difference

$$\sin (z \pm 2\pi) = \sin z \cos 2\pi \pm \cos z \sin 2\pi = \sin z,$$

$$\cos (z \pm 2\pi) = \cos z \cos 2\pi \mp \sin z \sin 2\pi = \cos z,$$

confirms that the number $2\pi$ is a period of the sine and cosine in a complex domain; $\sin z$ and $\cos z$ become zero, respectively, when $z = k\pi$ and $z = \pi/2 + k\pi$ where $k$ is an integer.

It should be noted that in contrast to the trigonometric functions of a real variable, the functions $\sin z$ and $\cos z$ cease to be limited: the moduli $|\sin z|$ and $|\cos z|$ may be greater than unity.

*Inverse trigonometric functions:* $w = \arcsin z$,

$$w = \arccos z, \quad w = \arctan z, \quad w = \operatorname{arc\,cot} z.$$

Since trigonometric functions can be expressed by an exponential function, inverse trigonometric functions may be expressed by logarithms.

By definition we get $z = \sin w = (e^{jw} - e^{-jw})/2j$, from which $e^{2jw} - 2jze^{jw} - 1 = 0$, i.e.

$$e^{jw} = jz \pm \sqrt{(1 - z^2)}$$

and

$$w = \arcsin z = \frac{1}{j} \ln \left[jz \pm \sqrt{(1 - z^2)}\right].$$

Similarly we can show that

$$\arccos z = \frac{1}{j} \ln \left[z \pm \sqrt{(z^2 - 1)}\right],$$

$$\arctan z = \frac{j}{2} \ln \frac{1 - jz}{1 + jz}.$$

Inverse trigonometric functions of a complex variable are multi-valued functions.

*Hyperbolic functions:*

$$\sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!},$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

The following relationships obtain between trigonometric and hyperbolic functions:

$$\sin jz = j \sinh z, \quad \cos jz = \cosh z, \quad \tan jz = j \tanh z,$$

$$\cot jz = -j \coth z.$$

Any relationship existing between the trigonometric functions of $\sin z$ and $\cos z$ can be converted in the corresponding relationship between the hyperbolic functions of $\sinh z$ and $\cosh z$ by substituting $j \sinh z$ for $\sin z$ and $\cosh z$ for $\cos z$.

*Inverse hyperbolic functions:*

$$w = \operatorname{arc\,sinh} z, \quad w = \operatorname{arc\,cosh} z, \quad w = \operatorname{arc\,tanh} z, \quad w = \operatorname{arc\,coth} z.$$

It follows from the expression $z = (e^w - e^{-w})/2$ that $e^{2w} - 2ze^w - 1 = 0$ from which $e^w = z \pm \sqrt{(z^2 + 1)}$ and $w = \operatorname{Arc\,sinh} z = \ln [z \pm \sqrt{(z^2 + 1)}]^2$.

Similarly, we can show that

$$\operatorname{Arc\,cosh} z = \ln[z \pm \sqrt{(z^2 - 1)}]; \quad \operatorname{Arc\,tanh} z = \frac{1}{2} \ln \frac{1 + z}{1 - z}.$$

### 3. Continuity of a Complex Variable Function

The function $w = f(z)$ is said to be *continuous at the point* $z_0$ if $\lim_{z \to z_0} f(z)$ exists and equals $f(z_0)$. This definition is equivalent to stating that the function $w = f(z)$ is continuous at the point $z_0$ if for all the points lying in the vicinity of $z_0$ the corresponding values of $w$ lie in the vicinity of the point $w_0 = f(z_0)$.

As opposed to the real domain where the variable can approach the limit only from the right and left, a complex variable $z$ in a complex region can approach the complex number $z_0$ from all directions in the plane.

The continuity of function $f(z)$ at point $z_0 = x_0 + jy_0$ means that its real and imaginary part $u(x, y)$ and $v(x, y)$, regarded as functions of real variables $x$ and $y$, are continuous at point $(x_0, y_0)$. This makes it possible to apply the basic properties of real variable functions which are continuous at the point to complex variable functions. Thus the sum, difference and product of a finite number of complex variable functions continuous at point $z_0$ will also be a function continuous at that point; the quotient of two functions continuous at point $z_0$ will be a function continuous at that point unless the divisor becomes zero when $z = z_0$.

A complex variable function is *continuous* in the $D$ region of a complex plane if it is continuous at each point $D$. The sum, the difference and the product of a finite number of complex variable functions continuous in the $D$ region are functions which are continuous in the same region. The quotient of two such functions is also continuous at each point of the $D$ region, except those values of $z$ at which the divisor becomes zero.

For example, the function $w = \sum_{k=0}^{n} a_k z^k$, which is a polynomial* with given complex factors $a^k$, is continuous in the entire plane of the complex variable $z$ since it is composed of functions which satisfy this condition.

Analogously, the rational function $w = \sum_{k=0}^{n} a_k z^k / \sum_{k=0}^{m} b_k z^k$ is continuous at each point in the plane of the complex variable $z$, except for those values at which the denominator is zero.

### 4. Differentiation of a Complex Variable Function

The function $w = f(z)$ can be differentiated at the point $z$ of a certain region if the relationship $\Delta w/\Delta_z = [f(z + \Delta z) - f(z)]/\Delta_z$ tends to a fixed limit when $z$ tends to zero in any way, $z$ being constant. The limit of this

* A polynomial is any function which can be constructed from an independent variable and constant numbers by a finite number of addition and multiplication operations.

relationship is called the *derivative* of function $f(z)$ at point $z$ and is denoted by $\mathrm{d}w/\mathrm{d}z$, $\mathrm{d}f/\mathrm{d}z$ or $f'(z)$.

The *differential* of function $w$ is equal to the product of its derivative with the differential of the independent variable $\mathrm{d}w = f'(z)\,\mathrm{d}z$.

All the rules of differential calculus are valid for differentiated complex variable functions:

$$(cf)' = cf'; \quad (f_1 \pm f_2)' = f_1' \pm f_2';$$

$$(f_1 \cdot f_2)' = f_1 f_2' + f_1' f_2;$$

$$\left(\frac{f_1}{f_2}\right)' = \frac{f_2 f_1' - f_1 f_2'}{f_2^2}, \text{ etc.}$$

The derivatives of complex variable functions can be calculated from the known formulae for the real variable:

$$\frac{\mathrm{d}z^n}{\mathrm{d}z} = nz^{n-1}, \quad \frac{\mathrm{d}e^z}{\mathrm{d}z} = e^z, \quad \frac{\mathrm{d}\ln z}{\mathrm{d}z} = \frac{1}{z},$$

$$\frac{\mathrm{d}\sin z}{\mathrm{d}z} = \cos z, \quad \frac{\mathrm{d}\cos z}{\mathrm{d}z} = -\sin z,$$

$$\frac{\mathrm{d}\sinh z}{\mathrm{d}z} = \cosh z, \quad \frac{\mathrm{d}\cosh z}{\mathrm{d}z} = \sinh z,$$

$$\frac{\mathrm{d}\sqrt[n]{z}}{\mathrm{d}z} = \frac{\sqrt[n]{z}}{nz}, \quad \frac{\mathrm{d}\,\mathrm{arc}\sin z}{\mathrm{d}z} = \frac{1}{\sqrt{(1-z^2)}}, \text{ etc.}$$

In order that the function $w = f(z) = u(x, y) + jv(x, y)$, which is determined in a certain region, should be differentiable as a complex variable function at point $z$ of this region, it is only necessary for functions $u(x, y)$ and $v(x, y)$ to be differentiable at the same point (as the functions of two real variables) and, in addition, for the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

to obtain.

These equations are known as the *Cauchy–Riemann* (or D'Alembert–Euler) *conditions.*

The elementary complex variable functions examined in Section 2 are differentiable.

## 5. Concept of a Function which is Analytic in a Region

The function $w = f(z)$ is called *analytic* (regular, holomorphic) in the $D$ region of a complex plane if it is single valued at each point in this region and has a finite derivative $f'(z)$ within $D$. As a result of the existence of this derivative, function $f(z)$ is continuous in $D$.

For a complex variable function to be analytic in the $D$ region it is only necessary for the real and imaginary parts of this function to have continuous derivatives of the first order, satisfying the Cauchy–Riemann conditions, at each point in the region.

We can use the function

$$w = e^z = e^x e^{jy} = e^x(\cos y + j \sin y)$$

as an example of a function which is analytic in the entire complex plane.

The proof is as follows:

$$u = e^x \cos y, \quad v = e^x \sin y,$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x \sin y.$$

The function $w = e^z$ is analytic in the entire complex plane except at an infinitely remote point.

We can use the function $f(z) = \overset{*}{z}$ as an example of a function with no derivatives at any point in the complex plane, with

$$\overset{*}{z} = x - jy.$$

Here

$$u = x, \quad v = -y,$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1,$$

from which

$$\frac{\partial u}{\partial x} \neq \frac{\partial u}{\partial y}.$$

It can be shown that functions $u$ and $v$ have continuous derivatives of the second order. Differentiating the Cauchy–Riemann equations with respect to $x$ and $y$ we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y},$$

from which

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Functions $u$ and $v$ which satisfy these so-called Laplace equations are called *harmonic functions.*

## 6. Integral of a Complex Variable Function

Let $f(z)$ be a certain continuous complex variable function defined at all points of a curve $L$, either joining the two points or closed. We shall divide curve $L$ into intervals $\Delta z_k$ and denote the values of the function in these intervals by $f(z_k)$.

By the *contour (or line) integral* of function $f(z)$ with respect to contour $L$ we shall understand the limit of the sum of the products of $f(z_k)\,\Delta z_k$ for an unlimited increase in the number of intervals and a decrease in the intervals themselves:

$$\int_L f(z)\,dz = \lim_{\substack{\Delta z \to 0 \\ n \to \infty}} f\sum_{k=1}^{n} (z_k)\,\Delta z_k.$$

Taking

$$f(z_k) = u_k + jv_k, \quad \Delta z_k = \Delta x_k + j\Delta y_k,$$

we find

$$\int_L f(z)\,dz = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ n \to \infty}} \sum_{k=1}^{n} (u_k + jv_k)\,(\Delta x_k + j\Delta y_k)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ n \to \infty}} \sum_{k=1}^{n} [(u_k \Delta x_k - v_k \Delta y_k) + j(v_k \Delta x_k + u_k \Delta y_k)]$$

$$= \int_L (u\,dx - v\,dy) + j\int_L (v\,dx + u\,dy).$$

Thus, the contour integral of a complex variable function is determined by the sum of the line integrals of the real variable function. Consequently, a contour integral possesses the same properties as an ordinary line integral, i.e. a constant factor can be taken away from the integral sign; the integral of the sum is equal to the sum of the integrals of the component terms; when the tracing direction of the integration contour changes, the

value of the integral only changes its sign; if the integration contour is divided into several parts, the value of the integral throughout the contour is equal to the sum of the integrals of the individual sections.

The path of the integration can either have two ends or be closed. In the latter case the closed-contour integral is sometimes denoted by the symbol $\oint$.

On the basis of the fact that the modulus of the sum is less than or equal to the sum of the moduli of the component terms, it can be proved that

$$\left|\int_L f(z)\,dz\right| \leqq \int_L |f(z)|\,|dz|.$$

Let us denote the length of the contour $L$ by $l$. If the modulus of the integrand on contour $L$ is not greater than the number $M$, i.e. $|f(z)| \leqq M$, then

$$\left|\int_L f(z)\,dz\right| \leqq Ml.$$

## 7. Simply- and Multiply-connected Regions. Cauchy's Integral Theorem

If a certain region of a complex plane is bounded by a single closed curved line without any multiple points, it is known as a *simply-connected* region (Fig. 1). In other words the $D$ region is simply-connected if every closed contour contained in $D$ bounds a region also contained within $D$.

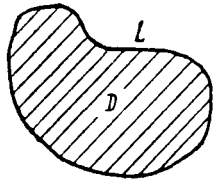



FIG. 1.

FIG. 2.

On the basis of the Riemann–Green theorem, if

$$u(x, y), \quad v(x, y), \quad \frac{du}{dy} \quad \text{and} \quad \frac{dv}{dx}$$

are continuous single-valued functions in a simply-connected $D$ region, then

$$\oint_L (u\,dx + v\,dy) = \iint_D \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy.$$

Consequently,

$$\oint_L f(z)\,dz = \oint_L (u\,dx - v\,dy) + j\oint_L (v\,dx + u\,dy)$$

$$= \iint_D \left(-\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy + j\iint_D \left(-\frac{dv}{dy} + \frac{du}{dx}\right) dx\,dy.$$

If the function $f(z)$ is analytic in the $D$ region, i.e.

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx},$$

then

$$\oint_L f(z)\,dz = 0.$$

Thus, Cauchy's theorem states: if a function is analytic in a simply-connected region, its integral equals zero with respect to any closed contour which, together with all its internal points, lies within this region.

Let us suppose two points $z_1$ and $z_2$ fixed in the region within which the function $f(z)$ is analytic (Fig. 2).

Taking the direction of the path of the lines $L_1$ and $L_2$ connecting these points from $z_1$ to $z_2$, using Cauchy's theorem we get

$$\int_{L_1} f(z)\,dz = \int_{L_2} f(z)\,dz.$$

Thus, in the region in which the function is analytic, the value of the line integral along an unclosed contour does not depend on the integration path but only on the end points $z_1$ and $z_2$:

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

If we fix one of these points ($z_1$) and regard the other as variable ($z$), the integral will be a function of $z$, i.e.

$$\int_{z_1}^{z} f(z)dz = F(z).$$

Then

$$F'(z) = f(z).$$

The integral of an analytic function is also an analytic function, the differential being equal to the integrand.

Let us now suppose that the $D$ region in which the function is analytic is a *multiply-connected* region, i.e. its boundary consists of several closed

curved lines (Fig. 3). By introducing a cross-cut in such a way that the internal contour is connected to the outer, the multiply-connected region becomes a simply-connected region. The closed contour and direction of traversal are shown in Fig. 4. Cauchy's theorem can be applied to this region, and in accordance with this theorem the integral around the closed contour equals zero. Then, as Fig. 4 shows, the integration is carried out along the cross-cut twice in opposite directions, as a result of which

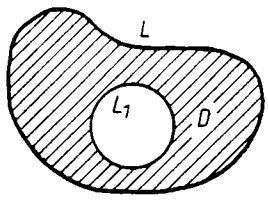



FIG. 3.

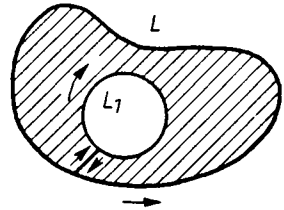



FIG. 4.

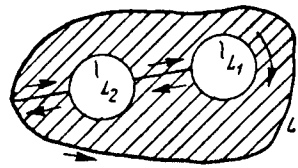



FIG. 5.

the integrals are mutually cancelled by the division. We are left with the integrals along the inner and outer contours:

$$\oint_L f(z)\,dz + \oint_{-L_1} f(z)\,dz = 0.$$

The integrals along these contours are taken in opposite directions. If we change the tracing direction of integrating along the inner contour, i.e. take the same (counter-clockwise) direction for both contours, we get

$$\oint_L f(z)\,dz = \oint_{L_1} f(z)\,dz.$$

This result can easily be applied to the case of $n$ closed contours located within a region bounded by an external contour (Fig. 5):

$$\oint_L f(z)\,dz = \sum_{k=1}^{n} \oint_{L_k} f(z)\,dz.$$

Thus, if a function is analytic within a closed multiply-connected region, the integral along the external contour is equal to the sum of the integrals

along all the internal contours on the condition that integration is carried out throughout all the contours in a positive direction, i.e. counterclockwise (the contour regions always being on the left).

## Cauchy's Integral Formula

Cauchy's theorem expresses the value of an analytic function at any point $z_0$ within the region by its value on the boundary of the region

$$f(z_0) = \frac{1}{2\pi j} \oint \frac{f(z)}{z - z_0} \, dz.$$

For a fixed value of $z_0$ the function $f(z)/(z - z_0)$ is analytic at all $z$ points in the $D$ region except, perhaps*, the point $z = z_0$, since the denominator of the fraction becomes zero at this point.

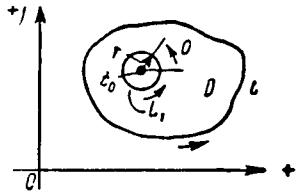



FIG. 6.

Describing a circle about point $z_0$ with a circumference $L_1$, of an arbitrarily small radius $r$, and lying entirely within the $D$ region (Fig. 6), on the basis of the above we get

$$\oint_L \frac{f(z)}{z - z_0} \, dz = \oint_{L_1} \frac{f(z)}{z - z_0} \, dz.$$

Adding and subtracting the integral

$$\oint_{L_1} \frac{f(z_0)}{z - z_0} \, dz,$$

in the right-hand side, we get

$$\oint_L \frac{f(z)}{z - z_0} \, dz = \int_{L_1} \frac{f(z_0)}{z - z_0} \, dz + \int_{L_1} \frac{f(z) - f(z_0)}{z - z_0} \, dz.$$

On the circumference $L_1$ $z = z_0 + re^{j\theta}$, $dz = rje^{j\theta} \, d\theta$, where $\theta$ varies from 0 to $2\pi$.

* If $f(z)$ contains the factor $z - z_0$, the function $f(z)/(z - z_0)$ is also analytic at point $z_0$.

Consequently,

$$\oint_{L_1} \frac{f(z_0)}{z - z_0}\,dz = f(z_0)\int_0^{2\pi} \frac{rje^{j\theta}\,d\theta}{re^{j\theta}} = 2\pi j f(z_0).$$

On the assumption that function $f(z)$ is differentiable, the expression $[f(z) - f(z_0)]/(z - z_0)$ is limited by a certain constant $M$. On the other hand the length of the integration path is $2\pi r$. Consequently,

$$\left|\int_{L_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| < M2\pi r.$$

In view of the arbitrary character of $r$, the above-mentioned inequality can occur only when the left-hand side equals zero. Thus Cauchy's formula has been proved.

For example, let it be required to calculate the integral

$$\oint_L \frac{z^2 - 2z - 1}{z + j}\,dz,$$

where $L$ is a circle of radius $|z| = 2$.

According to Cauchy's formula

$$\oint \frac{z^2 - 2z - 1}{z + j}\,dz = 2\pi j(-1 + 2j - 1) = -4\pi(1 + j).$$

Substituting $z$ for $z_0$ in Cauchy's formula and denoting the integration variable by $\zeta$ we can rewrite the formula as follows:

$$f(z) = \frac{1}{2\pi j}\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

The right-hand side of the resulting formula permits consecutive differentiation with respect to $z$ under the integral sign. Hence it follows that a function which is analytic in the $D$ region possesses derivatives of any order in $D$ and that all these derivatives are also functions analytic within $D$:

$$f'(z) = \frac{1}{2\pi j}\oint_L \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta,$$

$$f''(z) = \frac{1 \cdot 2}{2\pi j}\oint_L \frac{f(\zeta)}{(\zeta - z)^3}\,d\zeta.$$

.....

$$f^n(z) = \frac{n!}{2\pi j}\oint_L \frac{f(\zeta)}{(\zeta - z)_{n+1}}\,d\zeta.$$

In these formulae $L$ denotes are arbitrary, closed, even* contour located within $D$.

It is easy to see that $\oint_L dz/(z - z_0)^n = 0$, if $n \neq 1$ is an integer.

## 9. Taylor's Series

Cauchy's formula can be used to expand an analytic function $f(z)$ into a power series with respect to $z - z_0$. Here $z_0$ is the centre of a circle $L$ which lies wholly within the $D$ region in which the function $f(z)$ is analytic (Fig. 7).

In accordance with Cauchy's formula

$$f(z) = \frac{1}{2\pi j} \oint \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Here $\zeta$ is an arbitrary point on $L$, and $z$ is a point inside $L$.

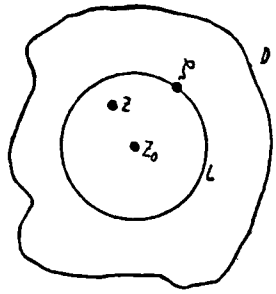



FIG. 7.

Let us convert the expression $1/(\zeta - z)$ thus:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{(\zeta - z_0)\left(1 - \dfrac{z - z_0}{\zeta - z_0}\right)}.$$

Supposing $(z - z_0)/(\zeta - z_0) = m$ for the sake of simplicity, the expression $1/(1 - m)$ may be regarded as the sum of an infinitely decreasing geometrical progression, since the modulus $|m| < 1$:

$$\frac{1}{1 - m} = 1 + m + m^2 + \cdots$$

Consequently

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \cdots$$

* A line is called *even* if it has a continuously changing tangent.

Multiplying this series by $f(\zeta)$ and integrating term by term round the circumference $L$ (which is permissible on account of the uniform convergence of the series*), we find

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots$$

This series is called a *Taylor series* of the function $f(z)$ with its centre at the point $z_0$. This series converges on $f(z)$ in a complete circle of radius less than $r$. When $z_0 = 0$, a *MacLauren series* is obtained.

## 10. Zeros of an Analytic Function

The zero of a function which is analytic in the $D$ region is the point $z_0$ in this region at which the given function becomes zero: $f(z_0) = 0$.

Expanding $f(z)$ into a Taylor series about $z = z_0$ we get

$$f(z) = (z - z_0) f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2!} + \cdots$$

If $f^{(k)}(z_0) =$ when $k > m$ and $f^{(m)}(z_0) \neq 0$, then

$$f(z) = (z - z_0)^m \frac{f^{(m)}(z_0)}{m!} + (z - z_0)^{m+1} \frac{f^{(m+1)}(z_0)}{(m + 1)!} + \cdots$$

and the number $m$ is called the *order of multiplicity of the zero* $z_0$ of the analytic function. The zero is called *simple* when $k = 1$, and *multiple* when $k > 1$. The point $z_0$ is a zero of the $k$-th order of the function $f(z)$.

Thus a simple zero of the function is characterized by the fact that it is not a zero for the derivative $df(z)/dz$ and a $k$-fold zero is characterized by the fact that it is also a zero of the derivative to $(k - 1)$-th order, but is not a zero of a derivative of the $k$-th order

$$f^{(k)}(z_0) \neq 0.$$

## 11. Laurent's Series

It was shown above that the function $f(t)$ can be expanded into a power series with respect to $z - z_0$ about any point $z$ inside a circle $L$ (centred at $z_0$) at which the given function is analytic.

* The series converges uniformly to function $S(z)$ in the $D$ region if at any point $z$ in the $D$ region for each positive number $\varepsilon$, however small, a natural number $N(\varepsilon)$ can be found so that the inequality $|S(z) - S_n(z)| < \varepsilon$ obtains for all values of $n \geqq N$, $S_n(z)$ being the sum of the first $n$ terms of the given series.

Let us now show how a function can be expanded in a type of power series about even points at which it is not analytic.

Let $f(z)$ be analytic within a ring which is formed by the concentric circles $L_1$ and $L_2$ centred at point $z_0$. We shall denote the radii of the circles by $r_1$ and $r_2$ respectively (Fig. 8). Let us introduce a series, formed according to positive and negative powers of $z - z_0$, which converges to function $f(z)$ at any point $z$ in the ring, i.e. when

$$r_2 < |z - z_0| < r_1.$$

Using Cauchy's formula, we get

$$f(z) = \frac{1}{2\pi j} \oint_{L_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi j} \oint_{L_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta,$$

where the minus sign in front of the second integral takes into account the fact that $L_2$ is traversed in the positive direction.

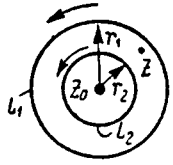



FIG. 8.

In the first integral $\zeta$ denotes a point on the circle $L_1$, and

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)\left(1 - \dfrac{z - z_0}{\zeta - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

This series converges uniformly for all points $\zeta$ on $L_1$ since

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < 1.$$

In the second integral $\zeta$ denotes a point on the circle $L_2$, and

$$\frac{1}{\zeta - z} = -\frac{1}{(z - z_0)\left(1 - \dfrac{\zeta - z_0}{z - z_0}\right)} = -\sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}.$$

This series converges uniformly for all points $\zeta$ on $L_2$ since

$$\left|\frac{\zeta - z_0}{z - z_0}\right| < 1.$$

Integrating these series term by term (which is possible since they converge uniformly with respect to $\zeta$), we find

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi j} \oint_{L_1} \frac{f(\zeta)\,(z - z_0)^n}{(\zeta - z_0)^{n+1}}\,d\zeta + \sum_{n=0}^{\infty} \frac{1}{2\pi j} \oint_{L_2} \frac{f(\zeta)\,(\zeta - z_0)^n}{(z - z_0)^{n+1}}\,d\zeta.$$

Denoting

$$c_n = \frac{1}{2\pi j} \oint_{L_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta \quad (n = 0, 1, 2, \ldots),$$

$$b_n = \frac{1}{2\pi j} \oint_{L_2} f(\zeta)\,(\zeta - z_0)^{n-1}\,d\zeta \quad (n = 1, 2, \ldots),$$

we get

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}.$$

Taking into account that $b_n = c_{-n}$,

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n + \sum_{n=1}^{\infty} c_{-n}(z - z_0)^{-n}$$

or

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n.$$

The resulting series is called a *Laurent series*. It consists of two parts. The first part is an ordinary power series with respect to $z - z_0$ converging for all points inside $L_1$. It is called the *regular part* of the Laurent series.

The second part is a power series with respect to $1/(z - z_0)$, i.e. to a series with negative powers, which can be regarded as an ordinary power series, if we suppose $1/(z - z_0) = z'$. This series, called the *principal part* of the Laurent series, converges for all points $z'$ for which $|z'| < 1/r_2$ or, the same thing, $|z - z_0| > r_2$, i.e. for all points $z$ located outside $L_2$. Thus $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is a function which is analytic inside $L_1$ and $f_2(z)$ is a function which is analytic outside $L_2$. Inside the ring formed by the circles $L_1$ and $L_2$ both the functions $f_1(z)$ and $f_2(z)$ are analytic.

If there is a finite number of turns with negative powers in a Laurent expansion, then $r_2$ must equal 0* and if there is a finite number of terms with positive powers then $r_1$ must equal $\infty$.

If function $f(z)$ is analytic at all points inside $L_1$, including point $z_0$, then $(\zeta - z_0)^{n-1} f(\zeta)$ is also an analytic function.

In this case

$$b_n = c_{-n} = \frac{1}{2\pi j} \oint_{L_2} (\zeta - z_0)^{n-1} f(\zeta)\,d\zeta = 0,$$

i.e. the Laurent series becomes a Taylor series.

* It is supposed that there are no singular points of function $f(z)$ other than $z = z_0$.

## 12. Isolated Singular Points. Behaviour of a Function in the Neighbourhood of an Isolated Singular Point

Points at which the function $f(z)$ ceases to be analytic are called *singular points* of the function.

The point $z_0$ is called the *isolated* singular point of the function $f(z)$ if the neighbourhood $|z - z_0| < R$ of this point (excluding $z_0$) exists, in which $f(z)$ is analytic.

There are three types of isolated singular point which are distinguished according to the behaviour of the function $f(z)$ in their neighbourhood.

1. Point $z_0$ is called a *removable singular point* if there is a finite $\lim\limits_{z \to z_0} f(z)$.

For $z_0$ to be a removable singular point of $f(z)$ it is only necessary that a Laurent expansion in the vicinity of point $z_0$ should not contain the principal part of $f(z) = \sum\limits_{n=0}^{\infty} c_n (z - z_n)^n$. The following are examples of functions with a removable singular point at the origin: $(\sin z)/z, (1 - e^z)/z, (1 - \cos z)/z^2$, etc.

2. Point $z_0$ is called a *pole* if $|f(z)|$ increases indefinitely as $z$ approaches $z_0$. A function becomes infinity at a pole: $\lim\limits_{z \to z_0} |f(z)| = \infty$.

For point $z_0$ to be a pole of function $f(z)$ it is only necessary that the main part of the Laurent expansion around $z_0$ should contain a finite number of terms

$$f(z) = c_{-m}(z - z_0)^{-m} + \cdots + c_{-1}(z - z_0)^{-1} + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

The number of the highest negative term of the expansion ($m$) is called the order of the pole.

If $m = 1$ the pole is called a simple pole or a pole of the first order. When $m > 1$ it is called a multiple pole or a pole of the $m$-th order.

For example, the function $f(z) = 1/(z + a)(z - a)^2$ has a simple pole at the point $z = -a$ and a pole of the second order at the point $z = a$.

Multiplying both parts of the Laurent expansion by $(z - z_0)^m$ and approximating $z$ to $z_0$ we get

$$\lim_{z \to z_0} (z - z_0)^m f(z) = c_{-m}.$$

Let us establish the relationship between the zero and the pole of a function. Let function $f(z)$ have a zero of the $m$-th order at point $z_0$.

In accordance with Section 10 the function $f(z)$ can be given as a power series in a certain region around such a point $z_0$:

$$f(z) = (z - z_0)^m \left[ \frac{f^{(m)}(z_0)}{m!} + (z - z_0) \frac{f^{(m+1)}(z_0)}{(m+1)!} + \cdots \right]$$
$$= (z - z_0)^m \varphi(z),$$

where $\varphi(z)$ is an analytic function at point $z_0$ and is not equal to zero at that point.

The reciprocal $1/f(z)$ is expressed thus:

$$\frac{1}{f(z)} = (z - z_0)^{-m} \frac{1}{\varphi(z)} = (z - z_0)^{-m} \psi(z),$$

$\psi(z) = \varphi(z)$ being an analytic function at point $z_0$ and not equal to zero at that point.

Taking into account that

$$\psi(z) = \psi(z_0) + (z - z_0)\psi'(z_0) + (z - z_0)^2 \frac{\psi''(z_0)}{2!} + \cdots,$$

we obtain

$$\frac{1}{f(z)} = (z - z_0)^{-m} \psi(z_0) + (z - z_0)^{-(m-1)} \psi'(z_0) + \cdots,$$

where $\psi(z_0) \neq 0$.

Consequently, if point $z_0$ is a zero of the $m$-th order for function $f(z)$, then the same point is a pole of the $m$-th order for function $1/f(z)$. And conversely: if point $z_0$ is a pole of the $m$-th order for function $f(z)$, the same point is a zero of the $m$-th order for function $1/f(z)$.

For example, the function $f(z) = 1/(e^{z^2} + 1)$ has innumerable poles at the points

$$z = \pm \sqrt{[j\pi(2k + 1)]} \quad (k = 0, \pm 1, \pm 2, \ldots),$$

all the poles, therefore, are simple since function $1/f(z) = e^{z^2} + 1$ contains zeros of the first order. The derivative $(\mathrm{d}/\mathrm{d}z)\,[1/f(z)] = 2ze^{z^2}$ at the points does not equal zero.

3. If $\lim_{z \to z_0} f(z)$ does not exist, $z_0$ is called an *essentially singular point.*

For point $z_0$ to be an essentially singular point of function $f(z)$ it is only necessary for the principal part of the Laurent expansion around point $z_0$ to contain an infinite number of terms.

A function is indeterminate about an essentially singular point, i.e. it tends to any previously given limit (finite or infinite) in various directions.

The following are examples of functions with an essentially singular point at the origin:

$$e^{\frac{1}{z}}, \quad \sin\frac{1}{z}, \quad \cos\frac{1}{z}.$$

Singularities are important characteristics of functions. According to kinds, numbers and locations of singularities one can specify the two simplest types of single-valued function: i.e. entire functions and rational functions.

A function is called *entire* if it has no singular points in the finite part of the plane. A function of this kind is a power series $\sum_{n=0}^{\infty} c_n z^n$ converging throughout the plane.

Polynomials, power functions, sin $z$, etc. are examples of entire functions.

A function is called *rational* if it has only a finite number of finite-plane poles and, at most, one pole at infinity. A rational function can be expressed as a ratio of polynomials.

## 13. The Point at Infinity. Behaviour of a Function at Infinity

Up till now we have been considering finite points in the plane of a complex variable.

One may consider the entire infinite region as a single point in the complex plane. Infinity is referred to as the point at infinity. By the *neighbourhood* of this point we mean the exterior of a circle of radius $R$, i.e. a ring having an inner radius $R$ and an infinitely large outer radius.

The plane of a complex variable together with its point at infinity is called the *total complex plane*, as distinct from a plane without this point, which is called an *open complex plane*.

The concepts of simply- and multiply-connected regions (Section 7) are also applicable to unlimited regions. For example, the vicinity of the point $z = \infty$, including the latter, is a simply-connected region, but the same vicinity without $z = \infty$ is a double-connected region.

Let function $f(z)$ be analytic in the vicinity of the point at infinity but not, perhaps, at the point itself, i.e. it is expanded into a Laurent series by integral powers of $z$ about the infinitely remote point. The definition of a singular point given in the previous section also applies to this type of function: i.e. the point $z = \infty$ is a removable singular point, pole or essentially singular point of function $f(z)$ depending on whether $\lim_{z\to\infty} f(z)$ is finite, infinite or non-existent.

If we take $z = 1/\zeta$ in the given function $f(z)$, then function $f(1/\zeta)$ will be analytic about the point $\zeta = 0$. The latter will be for function $f(1/\zeta)$ a singular point of the same type as $z = \infty$ for $f(z)$ since $\lim_{z\to\infty} f(z) = \lim_{\zeta\to 0} f(1/\zeta)$. Function $f(z)$ can be expanded in the vicinity of $z = \infty$ by substituting $\zeta = 1/z$ in the expansion $f(1/\zeta)$ about $\zeta = 0$. The regular part of the Laurent expansion will then be replaced by the principal part, and vice versa.

It follows from this that when there is a removable singularity at infinity, the expansion of function $f(z)$ into a Laurent series in the vicinity of this singularity contains no positive powers of $z$. In the case of a pole it contains a finite number of them and in the case of an essential singularity it contains an infinite number.

Thus in the case of a removable singularity the Laurent series has the form

$$f(z) = \sum_{n=0}^{-\infty} c_n z^n.$$

As $z$ tends to infinity, $f(z)$ tends to the finite limit $c_0$: $\lim_{z\to\infty} f(z) = c_0$. In this case $f(z)$ is said to be analytic at infinity (at point $z = \infty$), and $f(\infty) = c_0$.

If function $f(z)$ is analytic in the total $z$-plane then it is constant (the Cauchy–Liouville theorem – Section 17).

In the case of an infinitely remote pole

$$f(z) = \sum_{n=-\infty}^{m} c_n z^n = \sum_{n=1}^{m} c_n z^n + \varphi(z),$$

where $\varphi(z)$ is analytic at the point $z = \infty$.

As $z$ tends to infinity, $f(z)$ tends to infinity $\lim_{z\to\infty} f(z) = \infty$.

The highest positive term of the expansion $(m)$ agrees with the order of the pole and the total of the terms $\sum_{n-1}^{m} c_n z^n$ is called the infinite part at this pole.

Finally, in the case of an infinitely remote singularity

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n.$$

In any arbitrary vicinity of the infinitely remote singularity, function $f(z)$ can assume values as near as is required to any previously given number.

It follows from this that the total of the positive powers of the expansion into a Laurent series about an infinitely remote point plays the same defining part, in the sense of the way in which the function behaves, as the total of the negative powers of a Laurent series plays in the vicinity of a finite point $z_0$ (Section 12).

In the case when the expansion of the function only contains terms with negative powers and $f(z) = \sum_{n=-\infty}^{-m} c_n z^n$, the function tends to zero $\lim_{z\to\infty} f(z) = 0$ as $z$ tends to infinity. Function $f(z)$ is then said to have an $m$-fold zero at point $z = \infty$ and function $f(1/z)$ an $m$-fold zero at point $z = 0$.

Let $f(z)$ be a rational function, i.e. the quotient of two polynomials:

$$f(z) = \frac{\varphi(z)}{\psi(z)}.$$

We shall assume that the roots (zeros) of the numerator and denominator are different, i.e. that the fraction cannot be reduced.

The behaviour of the function at infinity will depend on the $n$ and $m$ powers of the polynomials in the numerator and denominator. If the power of $\varphi(z)$ is greater than the power $\psi(z)$ by $n-m$ units, then $f(z)$ will tend to infinity when $z \to \infty$, i.e. the function will have a pole of the $(n-m)$-th order at infinity, and the number of poles of $f(z)$ in the total complex plane will be $n$. And if the power of $\varphi(z)$ is not greater than the power of $\psi(z)$ the function will be analytic at infinity, and the number of poles of $f(z)$ in the total plane will be $m$.

## 14. Residue of a Function with Respect to a Singular Point

Let the function $f(z)$ be analytic at all points in a region bounded by a contour $L$, except at the singular point $z_0$.

On the basis of Section 11 the coefficient for the first negative power of the Laurent expansion, i.e. when $n = -1$, is

$$c_{-1} = \frac{1}{2\pi j} \oint_L f(z)\, dz.$$

The quantity $c_{-1}$ is known as the residue of function $f(z)$ with respect to the singular point $z_0$.

The residue of a function can only differ from zero if $z_0$ is a pole or an essentially singular point.

On the other hand, according to Cauchy's theorem, the contour integral $\oint f(z)\, dz = 0$ and, consequently, the residue equals zero.

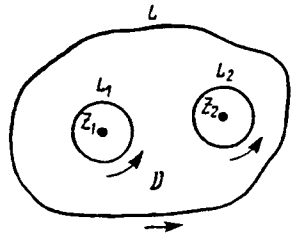



FIG. 9.

Let us now examine a function $f(z)$ which is analytic in the $D$ region and on the closed contour $L$, except at the singular points $z_1, z_2, \ldots, z_n$ which do not lie on $L$ (Fig. 9).

It is required to calculate the integral $\oint_L f(z)\, dz$.

In order to do this we shall enclose the singular points by contours $L_1, L_2, \ldots, L_n$.

The integral around the outer contour is equal to the sum of the integrals around the internal contours

$$\oint_L f(z)\,dz = \sum_{k=1}^{n} \oint_{L_k} f(z)\,dz = 2\pi j \sum_{k=1}^{n} \operatorname{Res}_{z_k} f(z).$$

Thus, the integral of $f(z)$ around $L$ divided by $2\pi j$ is equal to the sum of the residues of function $f(z)$ with respect to all the points $z_1, z_2, \ldots, z_n$. This is the basic residue theorem.

### 15. Calculating the Residue of a Function with Respect to a Pole

Let us suppose that $z_k$ is a pole of the first order, i.e. in the expansion there is only one negative power and, in addition, the first power $(z - z_k)$ is

$$f(z) = \frac{c_{-1}}{z - z_k} + c_0 + c_1(z - z_k) + \cdots$$

In this case, multiplying both sides of the equation by $(z - z_k)$ yields

$$(z - z_k)f(z) = c_{-1} + c_0(z - z_k) + c_1(z - z_k)^2 + \cdots$$

Proceeding to the limit, with $z$ tending to $z_k$, we find

$$\operatorname{Res}_{z_k} f(z) = c_{-1} = \lim_{z \to z_k} (z - z_k)f(z).$$

This formula makes it easy to determine the residue of a function with respect to a simple pole.

The point $z_k$ is a simple pole of the function $f(z) = \varphi(z)/\psi(z)$ if $\varphi(z)$ and $\psi(z)$ are analytic at point $z_k$, and $\varphi(z_k) \neq 0$, $\psi(z_k) = 0$, $\psi'(z_k) \neq 0$.

Using the formula educed above the residue of function $f(z)$ with respect to pole $z_k$ can in this case be determined as follows:

$$\operatorname{Res}_{z_k} f(z) = \lim_{z \to z_k} (z - z_k) \frac{\varphi(z)}{\psi(z)}.$$

Solving the indeterminate part we get $\operatorname{Res}_{z_k} f(z) = \varphi(z_k)/\psi'(z_k)$. For example, let it be required to determine the sum of the residues of the function $f(z) = e^{-z}/(z^2 - a^2)$. The given function has simple poles when $z = \pm a$.

$$\operatorname{Res}_a f(z) = \lim_{z \to a} (z - a) f(z) = \lim_{z \to a} \frac{e^{-z}}{z + a} = \frac{e^{-a}}{2a};$$

$$\operatorname{Res}_{-a} f(z) = \lim_{z \to -a} (z + a) f(z) = \lim_{z \to -a} \frac{e^{-z}}{z + a} = -\frac{e^{a}}{2a};$$

$$\frac{e^{-a}}{2a} - \frac{e^{a}}{2a} = -\frac{1}{a} \sinh a.$$

Let us now suppose that point $z_k$ is a pole of the $m$-th order, i.e. that the highest power $1/(z - z_k)$ in the Laurent expansion is $m$:

$$f(z) = c_{-m}(z - z_k)^{-m} + c_{-(m-1)}(z - z_k)^{-(m-1)} + \cdots + c_{-1}(z - z_k)^{-1} + c_0 + c_1(z - z_k) + \cdots$$

Multiplying both parts of this equation by $(z - z_k)^m$ and differentiating the resulting equation $(m - 1)$ times, we find

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} [(z - z_k)^m f(z)] = c_{-1}(m - 1)! + c_0 m!(z - z_k) + \cdots$$

Proceeding to the limit, with $z$ tending to $z_k$, we get

$$\operatorname{Res}_{z_k} f(z) = c_{-1} = \frac{1}{(m - 1)!} \lim_{z \to z_k} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} [(z - z_k)^m f(z)].$$

When $m = 1$ this formula becomes the previous formula for determining a residue with respect to a simple pole, if we agree to suppose that $0! = 1$.

Let us suppose that it is required to find the sum of the residues of the function $f(z) = 1/z(z - j)^2$, which has a simple pole when $z = 0$ and a second-order pole when $z = j$. Obviously,

$$\operatorname{Res}_0 f(z) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{1}{(z - j)^2} = -1,$$

$$\operatorname{Res}_j f(z) = \lim_{z \to j} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{z}\right) = \lim_{z \to j} \left(-\frac{1}{z^2}\right) = 1;$$

consequently, the sum of the residues is zero.

The concept of the residue can be extended to apply to the case of an infinitely remote point.

Let us suppose that an infinitely remote point is a singular point of function $f(z)$. We shall denote the closed contour lying entirely within the vicinity of this point (a circle of sufficiently large radius) by the letter $L$.

The Laurent expansion of function $f(z)$ about the infinitely remote point has the form

$$f(z) = c_{-1}z^{-1} + c_{-2}z^{-2} + \cdots + c_0 + c_1 z + c_2 z^2 + \cdots$$

Integrating term-by-term round $L$ in a clockwise direction (so that infinity remains on the left all the time), we get

$$-c_{-1}2\pi j = \oint_{-L} f(z)\,dz.$$

Consequently, the residue of the function with respect to infinity is equal to the coefficient for the first negative power of the Laurent expansion, but with the opposite sign.

The residue with respect to a removable singular point located at a finite distance is always zero. This cannot happen in the case of a removable point at infinity. For example, function $1/z$ has a removable singular point at infinity: the residue with respect to this point is $-1$.

If function $f(z)$ has a finite number of singular points in the total plane, then the sum of all its residues, including the residue at infinity, is zero. In fact, describing a circle $L$ of any radius with its centre at the origin in such a way that all the singular points of the function lie inside it (except the point at infinity), we get the sum of the residues with respect to all these singular points in the form of the integral $(1/2\pi j)\oint_L f(z)\,dz$. On the other hand, the residue of the same function with respect to infinity is described by the integral

$$\frac{1}{2\pi j}\oint_{-L} f(z)\,dz = -\frac{1}{2\pi j}\oint_{L} f(z)\,dz.$$

The sum of these two integrals, which gives the sum of all the residues, is zero.

## 16. Mean-value Theorem. Maximum and Minimum Modulus Principle

The *mean-value theorem* states that if a function $f(z)$ is continuous in a closed circle and analytic inside it, then its value at the centre of the circle is equal to the arithmetic mean on the circumference.

The proof of the theorem is based on Cauchy's integral formula. If a circle $|z - z_0| = R$ is selected as the integration path then, denoting $z - z_0 = Re^{j\varphi}$, we get $dz = jRe^{j\varphi}\,d\varphi$ from which

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \mathrm{R}e^{j\varphi})\,d\varphi.$$

Consequently $f(z_0)$ is the arithmetic mean on a circle of radius $R$ with its centre at $z_0$.

Using the mean-value theorem it is possible to prove the following principle: the modulus of a function which is analytic in the $D$ region and is not a constant, cannot reach its maximum or minimum at the internal points of $D$.

## 17. Cauchy's Inequality. Liouville's Theorem

If $f(z)$ is analytic in the circle $|z - z_0| = R$, the maximum of the modulus of $f(z)$ being $M$, then $|f^{(n)}(z_0)| \leqq M \cdot n/R^n!$ .

The latter expression, *Cauchy's inequality*, results from the formula of the $n$-th derivative

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint \frac{f(z)}{(z - z_0)^{n+1}} \, dz,$$

in accordance with which

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right| \leqq \frac{n! M 2\pi R}{2\pi R^{n+1}}.$$

The resulting equation makes it possible to prove *Liouville's theorem:* if a function $f(z)$ is analytic throughout a plane and limited in modulus, then it is constant.

Let $|f(z)| \leqq M$ throughout. For an arbitrary point $z$ in the plane and for any $R$, Cauchy's inequality yields, when $n = 1$,

$$|f'(z)| \leqq \frac{M}{R}.$$

The left-hand side of the inequality does not depend on $R$ which can be of any magnitude. From this $|f'(z)| = 0$, i.e. in the whole plane

$$f'(z) = \frac{du}{dx} + j\frac{dv}{dx} = 0 \quad \text{and} \quad \frac{du}{dx} = \frac{dv}{dx} = 0.$$

From the Cauchy–Riemann conditions

$$\frac{du}{dy} = \frac{dv}{dy} = 0,$$

i.e. $f(z)$ is constant.

It follows from the Liouville's theorem that any function which is not a constant should have at least one singularity in the total plane.

## 18. Expansion of a Rational Function into Simplest Fractions

Let a rational function $f(z) = \varphi(z)/\psi(z)$ have poles at the finite points $z_1, z_2, \ldots$ and at infinity. Function $f(z)$ can be expanded into a Laurent series in the vicinity of each pole. Let us denote its main part for each finite pole $z_k$ by $h_k(z) = c_{-1}^{(k)}/(z - z_k) + c_{-1}^{(k)}/(z - z_k)^2 + \cdots$ and for the infinity $g(z) = c_1 z + c_2 z^2 + \cdots$.

Subtracting the rational function $R(z) = h_1(z) + h_2(z) + \cdots + g(z)$ from the given function $f(z)$ we get a function $F(z) = f(z) - R(z)$ which has no singularities in the total plane and is, consequently, according to Liouville's theorem identically equal to a constant number $C$. Therefore

$$f(z) = R(z) + C.$$

Thus the rational function $f(z)$ can be given in the form of the sum of a finite number of rational functions, each of which has only one pole. The resulting expression is the expansion of the rational function into simple fractions.

## 19. Conformal Transformation

The function $w = f(z)$, analytic in a certain $D$ region, transforms a $z$ plane into a $w$ plane: a certain curve (a mapping curve or reflexion) in the $w$ plane corresponds exactly to each curve in the $z$ plane.

The mapping is called univalent if two different points on the second curve always correspond to two different points on the first.

Let us suppose that two curves in the $z$ plane intersect at a certain point at an angle $\varphi$. Since the point of intersection is common to both curves of the $z$ plane, the point of intersection of the said mapped curves in the $w$ plane will correspond to it, i.e. the reflected curves in the $w$ plane will also intersect.

We shall show that the angle of intersection of the mapped curves is equal to $\varphi$ on the condition that there is no singularity at the intersection point and the derivative $f'(z)$ does not equal zero.

The small increments starting from the intersection point along the curves in the $z$ plane can be expressed in exponential form

$$\Delta z_1 = |\Delta z_1| e^{j\psi_1}, \quad \Delta z_2 = |\Delta z_2| e^{j\psi_2}.$$

Then, if $\Delta z_1$ and $\Delta z_2$ are made infinitesimally small, the angle of intersection of the curves, i.e. the angle formed by the tangents at the intersection point, is determined as the limit of difference $\psi_1 - \psi_2$ when $\Delta z_1 \to 0$ and $\Delta z_2 \to 0$.

Similarly, in the $w$ plane

$$\Delta w_1 = |\Delta w_1| e^{j\xi_1}, \quad \Delta w_2 = |\Delta w_2| e^{j\xi_2}$$

and the intersection angle is $\lim\limits_{\substack{\Delta w_1 \to 0 \\ \Delta w_2 \to 0}} (\xi_1 - \xi_2)$.

Obviously,

$$\frac{\Delta w_1}{\Delta z_1} = \left|\frac{\Delta w_1}{\Delta z_1}\right| e^{j(\xi_1 - \psi_1)}, \quad \frac{\Delta w_2}{\Delta z_2} = \left|\frac{\Delta w_2}{\Delta z_2}\right| e^{j(\xi_2 - \psi_2)}.$$

In view of the fact that the function is analytic at the point of intersection, the derivative $dw/dz$ does not depend on the direction of approach selected (when $dw/dz \neq 0$), i.e.

$$\lim_{\substack{\Delta w_1 \to 0 \\ \Delta z_1 \to 0}} (\xi_1 - \psi_1) = \lim_{\substack{\Delta w_2 \to 0 \\ \Delta z_2 \to 0}} (\xi_2 - \psi_2),$$

from which

$$\lim_{\substack{\Delta z_1 \to 0 \\ \Delta z_2 \to 0}} (\psi_1 - \psi_2) = \lim_{\substack{\Delta w_1 \to 0 \\ \Delta w_2 \to 0}} (\xi_1 - \xi_2).$$

Thus the angles of intersection of the curves in the $z$ plane and the corresponding mapped curves in the $w$ plane are identical both in magnitude and sign. We can conclude from this that when a certain closed contour in the $z$ plane is traced in a positive direction, its corresponding curve in the $w$ plane is also mapped in a positive direction. In this case the univalent mapping $w = f(z)$ is said to preserve the orientation (property of preserving angles).

Besides the angles being preserved, when $f'(z) \neq 0$ a constant distortion of scale takes place.

A mapped curve given by the transformation $w = f(z)$ and possessing both these properties is called *conformal*. A mapped curve constructed from the function $w = f(z)$ and analytic in a certain $D$ region is conformal at all points $z$ at which $f'(z) \neq 0$.

As opposed to a conformal mapped curve $w = f(z)$ for which the positive direction of the contour is preserved (first-order mapping), a mapped curve of a given function which is conjugate with the values of function $f(z)$ is a conformal curve of the second order: plotting the contour in the $w$ plane in a negative direction corresponds to plotting the contour in the $z$ plane in a positive direction.

## 20. Linear-fractional Transformation

A transformation in the form

$$w = \frac{az + b}{cz + d},$$

where $a$, $b$, $c$ and $d$ are arbitrary complex numbers satisfying the condition

$ad - bc \neq 0$ (since in the opposite case the fraction is reduced and equals a constant number), is known as a *linear-fractional (or bilinear) transformation.*

In particular when $c = 0$ and $d \neq 0$ an entire linear function is obtained which is analytic at all finite points in the plane. It has a first-order pole at infinity.

The linear-fractional function ($c \neq 0$) is analytic at all points in the plane except point $-d/c$ where it can have a first-order pole. The function is analytic at infinity: its value here being $a/c$.

A linear-fractional function, like an entire linear function, maps the whole $z$ plane univalently on to the whole $w$ plane. It transforms one circumference into another, and here by the term "circumference" we mean not only a circumference in the strict sense of the word but also a straight line (the circumference of a circle of infinitely great radius).

In order to prove this property of a linear-fractional function, the function is written in the form

$$w = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + \dfrac{d}{c}}.$$

The transformation expressed by this function consists of the parallel shifting $w_1 = z + d/c$, transformation of the type $w_2 = [(bc - ad)/c^2]/w_1$, and the parallel shifting $w = w_2 + a/c$.

As a result of parallel shifting a straight line is transformed into a straight line, and a circle into a circle.

A transformation of the form $w_2 = e/w_1$ carries a circle into a circle or straight line, and carries a straight line into a straight line or a circle. In fact, the equation for the circle is written in the form

$$Aw_1\overset{*}{w}_1 + \overset{*}{B}w + B\overset{*}{w} + C = 0,$$

where $A$ and $C$ are real constants (in the case of a straight line $A = 0$); the transformation $w_2 = e/w_1$ gives

$$A\frac{e\overset{*}{e}}{w_2\overset{*}{w}_2} + \overset{*}{B}\frac{e}{w_2} + B\frac{\overset{*}{e}}{\overset{*}{w}_2} + C = 0$$

or

$$Ae\overset{*}{e} + \overset{*}{B}e\overset{*}{w}_2 + B\overset{*}{e}w_2 + Cw_2\overset{*}{w}_2 = 0.$$

The resulting equation maps a circle in the $w_2$ plane (when $C = 0$, a straight line is obtained).

Consequently, each of the three transformations expressed by a linear-fractional function maps a circle on a circle.

The general function $w$ can therefore be regarded as the result of executing a sequence of the above transformations, mapping a circle of the $z$ plane on a circle of the $w$ plane.

APPENDIX II

# BASIC PROPERTIES OF THE FOURIER TRANSFORMATION

| | |
|---|---|
| Frequency spectrum (Fourier transform) | $F(j\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t$ |
| Time function (Fourier integral) | $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t}\, \mathrm{d}\omega = \sum \operatorname{Res} F(p) e^{pt}$ |
| Interchangeability of independent variables | $f(\omega) \sim \frac{1}{2\pi} F(-jt)$ |
| Replacing a time function and spectrum by conjugate complex quantities | $\overset{*}{f}(t) \sim \overset{*}{F}(-j\omega)$ |
| Linearity of the Fourier transformation | $\sum_{i=1}^{n} a_i f_i(t) \sim \sum_{i=1}^{n} a_i F_i(j\omega)$ |
| Differentiation of a time function | Two-sided transformation $\frac{\mathrm{d}}{\mathrm{d}t} f(t) \sim j\omega F(j\omega)$<br>One-sided transformation $\frac{\mathrm{d}}{\mathrm{d}t} f(t) \sim j\omega F(j\omega) - f(0)$ |

Appendix II (*cont.*)

| | |
|---|---|
| Integration of a time function | Two-sided transformation $\int_{-\infty}^{t} f(t)\,\mathrm{d}t \sim \frac{F(j\omega)}{j\omega}$ One-sided transformation $\int_{0}^{t} f(t)\,\mathrm{d}t \sim \frac{F(j\omega)}{j\omega}$ |
| Differentiation of a spectrum | $\frac{\mathrm{d}}{\mathrm{d}\omega} F(j\omega) \sim (-jt) f(t)$ |
| Integration of a spectrum | $\int_{-\infty}^{\omega} F(j\omega)\,\mathrm{d}\omega \sim \frac{f(t)}{-jt}$ |
| Change of scale | $f(at) \sim \frac{1}{a} F\left(j\frac{\omega}{a}\right)$ |
| Shift of time function | $f(t \pm t_0) \sim e^{\pm j\omega t_0} F(j\omega)$ |
| Shift of spectrum | $f(t) \,.\, e^{\pm j\omega_0 t} \sim F[j(\omega \mp \omega_0)$ |
| Multiplication of spectra | Two-sided transformation $F_1(j\omega) \,.\, F_2(j\omega) \sim \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{+\infty} f_2(\tau) f_1(t-\tau)\,\mathrm{d}\tau$ |

Appendix II (*cont.*)

| | One-sided transformation |
|---|---|
| Multiplication of spectra | $F_1(j\omega) \,.\, F_2(j\omega) \sim \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau = \int_0^t f_2(\tau) f_1(t-\tau)\,\mathrm{d}\tau$ |
| Multiplication of time function | $f_1(t) \,.\, f_2(t) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(j\Omega) \,.\, F_2[j(\omega-\Omega)]\,\mathrm{d}\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(j\Omega) \,.\, F_1[j(\omega-\Omega)]\,\mathrm{d}\Omega$ |
| Multiplication of time function by cosine | $f(t) \,.\, \cos\omega_0 t \sim \frac{1}{2}\{F[j(\omega-\omega_0)] + F[j(\omega+\omega_0)]\}$ |
| Multiplication of time function by sine | $f(t) \,.\, \sin\omega_0 t \sim \frac{1}{2j}\{F[j(\omega-\omega_0)] - F[j(\omega+\omega_0)]\}$ |
| Integration of product of time functions | $\int_{-\infty}^{\infty} f_1(t) \,.\, \overset{*}{f_2}(t)\,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(j\omega) \,.\, \overset{*}{F_2}(j\omega)\,\mathrm{d}\omega$ |
| Parseval's equality | $\int_{-\infty}^{\infty} \lvert f(t)\rvert^2\,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert F(j\omega)\rvert^2\,\mathrm{d}\omega$ |

Appendix II (*cont.*)

| | |
|---|---|
| Finding the spectrum of a section of a periodic function with one-sided transformation | $F_1(j\omega) = (1 - e^{-j\omega nT})\, F(j\omega)$<br>($n$– an integer; $T$– period) |
| Finding a time function from a spectrum (switching-in formula) with one-sided transformation | $\frac{F_1(j\omega)}{F_2(j\omega)} \sim j \sum_{k=1}^{n} \frac{F_1(j\omega_k)\, e^{j\omega_k t}}{\left[\frac{\mathrm{d}}{\mathrm{d}\omega} F_2(j\omega)\right]_{\omega=\omega_k}}$<br>$\frac{F_1(j\omega)}{j\omega F_3(j\omega)} \sim \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n-1} \frac{F_1(j\omega_k)\, e^{j\omega_k t}}{\omega_k \left[\frac{\mathrm{d}}{\mathrm{d}\omega} F_3(j\omega)\right]_{\omega=\omega_k}}$ |

## APPENDIX III

# BASIC PROPERTIES OF THE LAPLACE TRANSFORMATION

| Transformation | Laplace | Carson |
|---|---|---|
| Direct transform | $F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt$ | $F(p) = p \int_0^\infty f(t)\, e^{-pt}\, dt$ |
| Inverse transform | $f(t) = \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} F(p)\, e^{pt}\, dp = \sum \text{Res}\, F(p)\, e^{pt}$ | $f(t) = \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} \frac{F(p)}{p}\, e^{pt}\, dp = \sum \text{Res}\, \frac{F(p)}{p}\, e^{pt}$ |
| Differentiation of original | $f'(t) \sim pF(p) - f(0)$<br>$f^{(n)}(t) \sim p^n F(p) - \sum_{k=1}^{n} f^{(k-1)}(0)\, p^{n-k}$ | $f'(t) \sim p[F(p) - f(0)]$<br>$f^n(t) \sim p^n F(p) - \sum_{k=1}^{n} f^{(k-1)}(0) \cdot p^{n-k+1}$ |
| Integration of original | $\int_0^t f(t)\, dt \sim \frac{F(p)}{p}$<br>$\int_0^t \cdots \int_0^t f(t)\, (dt)^n \sim \frac{F(p)}{p}$ | |
| Differentiation of transform | $F^{(n)}(p) \sim (-t)^n f(t)$ | $p \left[\frac{F(p)}{p}\right]^{(n)} \sim (-t)^n f(t)$ |

Appendix III (*cont.*)

| Transformation | Laplace | Carson |
|---|---|---|
| Integration of transform | $\int_p^\infty F(p)\,dp \sim \frac{f(t)}{t}$<br>$\int_0^\infty F(p)\,dp = \int_0^\infty \frac{f(t)}{t}\,dt$ | $p\int_p^\infty \frac{F(p)}{p}\,dp \sim \frac{f(t)}{t}$<br>$\int_0^\infty \frac{F(p)}{p}\,dp = \int_0^\infty \frac{f(t)}{t}\,dt$ |
| Differentiation by parameter | $\frac{\partial}{\partial x} f(t, x) \sim \frac{\partial}{\partial x} F(p, x)$ | |
| Integration by parameter | $\int_{x_0}^{x} f(t, x)\,dx \sim \int_{x_0}^{x} F(p, x)\,dx$ | |
| Similarity theorem (change in scale of independent variable) | $f(at) \sim \frac{1}{a} F\left(\frac{p}{a}\right)$ | $f(at) \sim F\left(\frac{p}{a}\right)$ |
| Real shifting theorem (shift in region of real variable) | $f(t - \tau) \sim e^{-p\tau} F(p)$ | |
| Complex shifting theorem (shift in region of complex variable) | $e^{\mp \lambda t} f(t) \sim F(p \pm \lambda)$ | $e^{\mp \lambda t} f(t) \sim \frac{p}{p \pm \lambda} F(p \pm \lambda)$ |

Appendix III (*cont.*)

| Transformation | Laplace | Carson |
|---|---|---|
| Limit relationships | $f(0) = \lim_{p\to\infty} pF(p)$<br>$f(\infty) = \lim_{p\to 0} pF(p)$ | $f(0) = F(\infty)$<br>$f(\infty) = F(0)$ |
| Convolution theorem (multiplication of transforms) | $F_1(p) \,.\, F_2(p) \sim \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau$<br>$= \int_0^t f_2(\tau) f_1(t-\tau)\,\mathrm{d}\tau$ | $\frac{1}{p} F_1(p) \,.\, F_2(p) \sim \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau$<br>$= \int_0^t f_2(\tau) \,.\, f_1(t-\tau)\,\mathrm{d}\tau$ |
| Corollary of convolution theorem (Duhamel's integral) | $pF_1(p) \,.\, F_2(p) \sim \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau$<br>$= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f_2(\tau) f_1(t-\tau)\,\mathrm{d}\tau$<br>$pF_1(p) \,.\, F_2(p) \sim \int_0^t f_1(\tau) f'_{2(t)}(t-\tau)\,\mathrm{d}\tau + f_1(t) \,.\, f_2(0)$<br>$= \int_0^t f_2(\tau) f'_{1(t)}(t-\tau)\,\mathrm{d}\tau + f_1(0) \,.\, f_2(t)$ | $F_1(p) \,.\, F_2(p) \sim \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau$<br>$= \frac{\mathrm{d}}{\mathrm{d}t} f_2(\tau) f_1(t-\tau)\,\mathrm{d}\tau$<br>$F_1(p) \,.\, F_2(p) \sim \int_0^t f_1(\tau) f'_{2(t)}(t-\tau)\,\mathrm{d}\tau + f_1(t) \,.\, f_2(0)$<br>$= \int_0^t f_2(\tau) f'_{1(t)}(t-\tau)\,\mathrm{d}\tau + f_1(0) \,.\, f_2(t)$ |

Appendix III (*cont.*)

| Transformation | Laplace | Carson |
|---|---|---|
| Multiplication of originals | $f_1(t) \,.\, f_2(t) \sim \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} F_1(q)\, F_2(p-q)\, \mathrm{d}q$ $= \sum \text{Res}\, F_1(q)\, F_2(p-q)$ | $f_1(t) \,.\, f_2(t) \sim \frac{p}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} \frac{F_1(q) \,.\, F_2(p-q)}{q(p-q)}\, \mathrm{d}q$ $= p \sum \text{Res} \frac{F_1(q)\, F_2(p-q)}{q(p-q)}$ |
| Multiplication of original by cosine or sine | $\cos \omega_0 \,.\, f(t) \sim \frac{1}{2} [F(p-j\omega_0) + F(p+j\omega_0)]$ $\sin \omega_0 t \,.\, f(t) \sim \frac{1}{2j} [F(p-j\omega_0) - F(p+j\omega_0)]$ | $\cos \omega_0 t \,.\, f(t) \sim \frac{p}{2} \left[\frac{F(p-j\omega_0)}{p-j\omega_0} + \frac{F(p+j\omega_0)}{p+j\omega_0}\right]$ $\sin \omega_0 t \,.\, f(t) \sim \frac{p}{2j} \left[\frac{F(p-j\omega_0)}{p-j\omega_0} - \frac{F(p+j\omega_0)}{p+j\omega_0}\right]$ |
| Integration of product of originals | $\int_0^\infty f_1(t) \,.\, f_2(t)\, \mathrm{d}t = \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} F_1(p)\, F_2(-p)\, \mathrm{d}p$ $= \sum \text{Res}\, F_1(p) \,.\, F_2(-p)$ | $\int_0^\infty f_1(t) \,.\, f_2(t)\, \mathrm{d}t = \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} \frac{F_1(p) \,.\, F_2(-p)}{-p^2}\, \mathrm{d}p$ $= \sum \text{Res} \frac{F_1(p) \,.\, F_2(-p)}{-p^2}$ |
| Analogue of Parseval's relation | $\int_0^\infty [f(t)]^2\, \mathrm{d}t = \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} F(p) \,.\, F(-p)\, \mathrm{d}p$ $= \sum \text{Res}\, F(p)\, F(-p)$ | $\int_0^\infty [f(t)]^2\, \mathrm{d}t = \frac{1}{2\pi j} \int_{c_0-j\infty}^{c_0+j\infty} \frac{F(p) \,.\, F(-p)}{-p^2}\, \mathrm{d}p$ $= \sum \text{Res} \frac{F(p) \,.\, F(-p)}{-p^2}$ |

Appendix III (*cont.*)

| Transformation | Laplace | Carson |
|---|---|---|
| Expansion theorem (simple roots) | $\frac{F_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{F_1(p_k)\, e^{p_k t}}{F_2'(p_k)}$<br>$\frac{F_1(p)}{pF_3(p)} \sim \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n-1} \frac{F_1(p_k)\, e^{p_k t}}{p_k F_3'(p_k)}$ | $\frac{pF_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{F_1(p_k)\, e^{p_k t}}{F_2'(p_k)}$<br>$\frac{F_1(p)}{F_3(p)} \sim \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n-1} \frac{F_1(p_k)\, e^{p_k t}}{p_k F_3'(p_k)}$ |
| Expansion theorem (multiple roots) | $\frac{F_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{1}{(m_k - 1)!} \times$<br>$\times \left[ \frac{d^{m_k - 1}}{dp^{m_k - 1}} \frac{F_1(p)\,(p - p_k)^{m_k} e^{pt}}{F_2(p)} \right]_{p = p_k}$ | $\frac{pF_1(p)}{F_2(p)} \sim \sum_{k=1}^{n} \frac{1}{(m_k - 1)!} \times$<br>$\times \left[ \frac{d^{m_k - 1}}{dp^{m_k - 1}} \frac{F_1(p)\,(p - p_k)^{m_k} e^{pt}}{F_2(p)} \right]_{p = p_k}$ |

## APPENDIX IV

# LAPLACE TRANSFORMS

| $F(p)$ | $f(t)$ |
|---|---|
| $\frac{1}{p}$ | $1$ |
| $\frac{1}{p^2}$ | $t$ |
| $\frac{1}{p^n}$ | $\frac{t^{n-1}}{(n-1)!}$ ($n$ is a positive integer) |
| $\frac{1}{p+a}$ | $e^{-at}$ |
| $\frac{1}{(p+a)^2}$ | $te^{-at}$ |
| $\frac{p}{(p+a)^2}$ | $(1-at)\,e^{-at}$ |
| $\frac{1}{(p+a)^3}$ | $\frac{1}{2}t^2e^{-at}$ |
| $\frac{p}{(p+a)^3}$ | $t\left(1-\frac{a}{2}t\right)e^{-at}$ |
| $\frac{1}{(p+a)^n}$ | $\frac{1}{(n-1)!}t^{n-1}\cdot e^{-at}$ ($n$ is a positive integer) |
| $\frac{p^n}{(p+a)^{n+1}}$ | $e^{-at}\sum_{k=0}^{n}\frac{n!\,(-a)^k}{(n-k)!\,(k!)^2}t^k$ ($n$ is a positive integer) |
| $\frac{1}{p(p+a)}$ | $\frac{1}{a}(1-e^{-at})$ |
| $\frac{1}{(p+a)(p+b)}$ | $\frac{1}{b-a}(e^{-at}-e^{-bt})$ |

Appendix IV (*cont.*)

| $F(p)$ | $f(t)$ |
|---|---|
| $\frac{p}{(p+a)(p+b)}$ | $\frac{1}{(b-a)}(be^{-bt} - ae^{-at})$ |
| $\frac{1}{p(p+a)(p+b)}$ | $\frac{1}{ab} + \frac{1}{(b-a)}\left(\frac{e^{-bt}}{b} - \frac{e^{-at}}{a}\right)$ |
| $\frac{1}{(p+a)(p+b)(p+c)}$ | $\frac{e^{-at}}{(b-a)(c-a)} + \frac{e^{-bt}}{(a-b)(c-b)} + \frac{e^{-ct}}{(a-b)(b-c)}$ |
| $\frac{p}{(p+a)(p+b)(p+c)}$ | $\frac{ae^{-at}}{(c-a)(a-b)} + \frac{be^{-bt}}{(a-b)(b-c)} + \frac{ce^{-ct}}{(b-c)(c-a)}$ |
| $\frac{p^2}{(p+a)(p+b)(p+c)}$ | $\frac{a^2e^{-at}}{(c-a)(b-a)} + \frac{b^2e^{-bt}}{(a-b)(c-b)} + \frac{c^2e^{-ct}}{(b-c)(a-c)}$ |
| $\frac{1}{(p+a)(p+b)^2}$ | $\frac{e^{-at} - e^{-bt}[1-(a-b)t]}{(a-b)^2}$ |
| $\frac{p}{(p+a)(p+b)^2}$ | $\frac{[a-b(a-b)t]e^{-bt} - ae^{-at}}{(a-b)^2}$ |
| $\frac{1}{(p+a)^2+b^2}$ | $\frac{e^{-at}}{b}\sin bt$ |
| $\frac{(p+a)}{(p+a)^2+b^2}$ | $e^{-at}\cos bt$ |
| $\frac{p}{(p+a)^2+b^2}$ | $\left(\cos bt - \frac{a}{b}\sin bt\right)e^{-at}$ |
| $\frac{(p+a)}{(p+a)^2-b^2}$ | $e^{-at}\cosh bt$ |
| $\frac{b}{(p+a)^2-b^2}$ | $e^{-at}\sinh bt$ |
| $\frac{p}{p^2+a^2}$ | $\cos at$ |
| $\frac{1}{p^2+a^2}$ | $\frac{1}{a}\sin at$ |

Appendix IV (*cont.*)

| $F(p)$ | $f(t)$ |
|---|---|
| $\frac{p\cos b - a\sin b}{p^2 + a^2}$ | $\cos(at + b)$ |
| $\frac{p\sin b + a\cos b}{p^2 + a^2}$ | $\sin(at + b)$ |
| $\frac{p}{p^2 - a^2}$ | $\cosh at$ |
| $\frac{1}{p^2 - a^2}$ | $\frac{1}{a}\sinh at$ |
| $\frac{1}{p(p^2 + a^2)}$ | $\frac{1}{a^2}(1 - \cos at)$ |
| $\frac{1}{p^2(p^2 + a^2)}$ | $\frac{1}{a^3}(at - \sin at)$ |
| $\frac{1}{(p^2 + a^2)^2}$ | $\frac{1}{2a^3}(\sin at - at\cos at)$ |
| $\frac{p}{(p^2 + a^2)^2}$ | $\frac{1}{2a}t\sin at$ |
| $\frac{p^2}{(p^2 + a^2)^2}$ | $\frac{1}{2a}(\sin at + at\cos at)$ |
| $\frac{p^2 - a^2}{(p^2 + a^2)^2}$ | $t\cos at$ |
| $\frac{1}{p(p^2 + a^2)^2}$ | $\frac{1}{a^4}(1 - \cos at) - \frac{1}{2a^3}t\sin at$ |
| $\frac{1}{p^4 - a^4}$ | $\frac{1}{2a^3}(\sinh at - \sin at)$ |
| $p^{-\frac{1}{2}}$ | $\frac{1}{\sqrt{(\pi t)}}$ |
| $p^{-\frac{3}{2}}$ | $2\sqrt{\left(\frac{t}{\pi}\right)}$ |

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